TEORIJSKA I PRIMENJENA MEHANIKA 6. pp. 115–122, 1980, UDK 539

DIFFRACTION OF ANTI PLANE SHEAR WAVES BY RIGID STRIP LYING AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF-SPACES

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1. Introduction

It is well known that the problems of diffraction of elastic waves by cracks or inclusions are of considerable interest in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevent. Although the study of diffraction of elastic waves by cracks (inclusions in homogeneous, isotropic elastic solids has been the subject of many investigations, it was only recently that Srivastava, Palaiya and Gupta [1], [2] have studied the interaction of longitudinal waves by a Griffit) penny shaped crack located at the interface of two dissimilar elastic half-spaces. Corresponding problems for shear wawes have been studied in [3] and [4].

In continuation to the above study, our object in this paper is to attempt the problem of diffraction of shear waves by a rigid strip situated at the interface of two dissimilar elastic half-spaces. Usind Fourier cosine transforms, the problem has been reduced to the solution of a Fredholm integral equation of the second kind. An iterative solution, valid for smaller wave frequencies, has been obtained. These solutions are used to calculate stress intensity factors at the edge of the strip.

2. Formulation of the problem

Let a rigid strip of infinite lenght and finite width be located at the interface of two bonded dissimilar elastic semi-infinite solids. Consider a rectangular cartesion co-ordinate system (x_1, x_2, x_3) such that the strip occupies the region $-a \leqslant x_1 \leqslant a$, $-\infty \leqslant x_2 \leqslant \infty$, $x_3 = 0$ at the interface of two half-spaces $x_3 > 0$ and $x_3 < 0$. It is convenient to normalize all lengths with respect to a, half-width of the strip. Writing $x_1/a = x$, $x_2/a = y$, $x_3/a = z$, the strip is defined by relation $-1 \leqslant x \leqslant 1$, $-\infty < y < \infty$, $z = 0$ at the interface of the half-spaces $z \leqslant 0$ and $z \geqslant 0$. An antiplane shear wave is assumed to be incident normally on the strip. The only non-vanishing displacement is the y – direction component $u_y = u_y(x, z) e^{-i\omega t}$. As a consequence, all stress components vanish identically except for the shear stress σ_{xy} and σ_{yz}. In the absence of body forces, the only equation of motion not identically satisfied by non-vanishing stress is

$$(u_{y,xx} + u_{y,zz} + k^2 u_y) e^{-i\omega t} = 0 \tag{2.1}$$

where $k = \omega a (\rho/\mu)^{\frac{1}{2}}$, μ is the Lame's constant, ρ the material density, ω the circular wawe frequency. It may be noted that k is the dimensionles wave frequency.

We have the problem of finding stress distribution subject to the following boundary conditions:

$$u_y(x, 0+) = u_y(x, 0-) = -u_0(x) e^{-i\omega t}, \quad |x| \leqslant 1 \tag{2.2}$$

$$\left.\begin{aligned} &u_y(x, 0+) = u_y(x, 0-) &(2.3)\\ &\sigma_{yz}(x, 0+) = \sigma_{yz}(x, 0-) &(2.4)\end{aligned}\right| \text{ for } |x| > |$$

In what follows, the factor $e^{-i\omega t}$ shall be supressed. Using Fourier cosine transforms, following are the solution of equation (2.1) suitable for the above problem:

$$u_y(x, z) = \begin{cases} \int_0^\infty A_1(\xi) \exp(-\beta_1 z) \cos \xi x \, d\xi, & z \geqslant 0 \\ \int_0^\infty A_2(\xi) \exp(\ \beta_2 z) \cos \xi x \, d\xi, & z \leqslant 0 \end{cases} \tag{2,5}$$

where

$$\beta_j = \begin{cases} (\xi^2 - k_j^2)^{\frac{1}{2}}, & k_j < \xi \\ -i(k_j^2 - \xi^2)^{\frac{1}{2}} & k_j > \xi \end{cases} \tag{2.6}$$

$$k_j = \omega a (\rho_j/\mu_j), \qquad j = 1,2$$

The suffix 1 and 2 correspond to the half-spaces $z \geqslant 0$ and $z \leqslant 0$ respectively. In equation (2.5) A_1 (ξ) and A_2 (ξ) are unknown functions which shall be determined on using the boundary conditions (2.1) — (2.3).

Corresponding to (2.5), following are the expressions for stress components:

$$\sigma_{yz}(x, z) = \begin{cases} -\mu_1 \int_0^\infty \beta_1 A_1(\xi) \exp(-\beta_1 z) \cos \xi x \, d\xi, & z \geqslant 0 \\ \mu^2 \int_0^\infty \beta_2 A_2(\xi) \exp(\ \beta_2 z) \cos \xi x \, d\xi, & z \leqslant 0 \end{cases} \tag{2.7}$$

3. Derivation of the integrai equation

From the boundary conditions (2,2) and (2.3) we observe that $u_y(x, 0+) = u_y(x, 0-)$ for an values of x, hence we get

$$A_1(\xi) = A_2(\xi) \tag{3.1}$$

The boundary conditions (2.2) and (2.4) lead to the following dual integral equations for the determination of unknown function $A_1(\xi)$:

$$\int_0^\infty A_1(\xi) \cos \xi x \, d\xi = -U_0(x), \quad |x| < | \tag{3.2}$$

$$\int_0^\infty (\mu_1 \beta_1 + \mu_2 \beta_2) A_1(\xi) \cos \xi x \, d\xi = 0, \quad |x| > | \tag{3.3}$$

Writing

$$(\mu_1 \beta_1 + \mu_2 \beta_2)(\mu_1 + \mu_2)^{-1} A_1(\xi) = \Psi(\xi) \tag{3.4}$$

The above integral equations can be written in a convenient form

$$\int_0^\infty \xi^{-1} [1 + H(\xi)] \Psi(\xi) \cos \xi x \, d\xi = -U_0(x), \quad |x| < | \tag{3.5}$$

$$\int_0^\infty \Psi(\xi) \cos \xi x \, d\xi = 0 \quad , \quad |x| < | \tag{3.6}$$

where

$$H(\xi) = \xi (\mu_1 + \mu_2)(\mu_1 \beta_1 + \mu_2 \beta_2)^{-1} - 1 \tag{3.7}$$

It may be observed that $H(\xi) \to 0$ as $\xi \to \infty$.

In order to solve the dual integral equation (3.5—3.6) let us assume

$$\Psi(\xi) = \xi \int_0^1 y \theta(y) \tau_1(\xi y) \, dy \tag{3.8}$$

$$= -\theta(1) \tau_0(\xi) + \int_0^1 \frac{d}{dy} \{y \theta(y)\} \tau_0(\xi y) \, d\xi$$

where the function $\theta(y)$ shall soon be determined. The equation (3.6) is automatically satisfied and (3.5) reduces to the Fredholm integral equation [5]:

$$\theta(y) + \int_0^1 \theta(t) \mathfrak{G}(t, y) \, dt = \frac{2}{\lambda} \frac{d}{dy} \int_0^y \frac{U_0(x)}{(y^2 - x^2)^{\frac{1}{2}}} dx \tag{3.9}$$

where

$$(3.10) \qquad \mathfrak{G}(t, y) = t \int_0^\infty \xi H(\xi) \tau_1(\xi t) \tau_1(\xi y)\, d\xi$$

The integrand in (3.10) has no poles, it has only branch points at $\xi = k_1$ and $\xi = k_2$. The infinite integral in equation (3.10) can be converted into integrals with finite limits by the procedure given in [6]. We shall give only the results here:

Let $H(\xi) = L(\xi, \beta_1, \beta_2)$, then

$$(3.11) \qquad \mathfrak{G}(t, y) = \frac{t}{2}\left[\int_0^{k_1} \xi\{L(\xi, -i\beta_1^1, -i\beta_2^1) - L(\xi, i\beta_1^1, i\beta_2^1)\} H_1^{(1)}(\xi t) \tau_1(\xi y) d\xi\right.$$

$$\left. + \int_{k_1}^{k_2} \xi\{L(\xi, \beta_1, -i\beta_2^1) - L(\xi, \beta_1, i\beta_2^1)\} H_1^{(1)}(\xi t) \tau_1(\xi y)\, d\xi\right], \qquad t > y$$

where $\beta_j^1 = (k_j^2 - \xi^2)^{\frac{1}{2}}$, $j = 1,2$ and $H_1^{(1)}(x)$, $J_1(x)$ are the Bessel's functions.

In the above derivations, we have taken $k_1 < k_2$. The value of this kernal for $k_1 > k_2$ is obtained by interchanging k_1 and k_2. For simplicity, however, we shall choose $k_1 < k_2$ in our further discussion. So also for $y > t$, t and y can be interchanged in (3.11).

On substituting the value of $L(\xi, \beta_1, \beta_2)$ in (3.11) from (3.7), we obtain

$$(3.12) \qquad \mathfrak{G}(t, y) = it(\mu_1 + \mu_2) k_1^2 \left[\int_0^1 \frac{\xi^2 H_1^{(1)}(k_1 \xi t) \tau_1(k_1 \xi y)}{\mu_1 (1 - \xi^2)^{\frac{1}{2}} + \mu_2 (\gamma^2 = \xi^2)^{\frac{1}{2}}} d\xi\right.$$

$$\left. + \mu_2 \int_1^\gamma \frac{\gamma \xi^2 (\gamma^2 - \xi^2)^{\frac{1}{2}} H_1^{(1)}(k_1 \zeta t) \tau_1(k_1 \xi y)\, d\xi}{\mu_1^2 (\xi^2 - 1) + \mu_2^2 (\gamma^2 - \xi^2)}\right], \qquad t > y$$

where $\gamma = k_2 \ | k_1 > |$

4. Solution of the integral equation

The dual integral equation (3.5) — (3,6) has been reduced to Fredholm integral equation (3.9) with the kernel $\mathfrak{G}(t, y)$ given by the equation (3.10). This form of the kernel is suitable for expansion in powers of k_1 (or k_2). Hence an iterative solution of (3.9), valid only for small values of the dimensionaless frequency k_1 (or k_2) can be obtained.

For small values of the argument, the Bessel function $J_1(x)$ and $H_1^{(1)}(x)$ may be expanded in ascending powers of x as

$$H_1^{(1)}(x) = \left[1 + \frac{2i}{\lambda} In\left(\frac{x}{2}\right)\right] \tau_1(x) + \sum_{n=0}^{\infty} ib_{2n+1} x^{2n-1}$$

$$\tau_1(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

where the coefficients a_{2n+1} and b_{2n+1} are given by Abramowitz and Stegun [7, p. 370]. Using the above expansions in (3.12), $\mathfrak{G}(t, y)$ may be expressed as

$$t^{-1} \mathfrak{G}(t, y) = k_1^2 \mathfrak{G}_1(t, y) + k_1^4 \mathfrak{G}_2(t, y) = k_1^4 \log k_1 \mathfrak{G}_3(t, y) + k_1^6 \mathfrak{G}_4(t, y)) + k_1^6 \log k_1 \mathfrak{G}_5(t, y) = 0(k_1^8) \tag{4.1}$$

where

$$\mathfrak{G}_1(t, y) = -a_1 b_1 yt^{-1} M_0$$

$$\mathfrak{G}_2(t, y) = -\left[a_3 b, \; y^3 t^{-1} = ia_1^2, \; yt + a_1 b_3 yt + \frac{2a_1^2}{\lambda} yt \log(t/2)\right] M_1 - 2\lambda^{-1} a_1^2 yt N_2$$

$$\mathfrak{G}_3(t, y) = -2\lambda^{-1} a_1^2 yt M_2,$$

$$\mathfrak{G}_4(t, y) = -\{-ia_1 a_3 yt(y^2 + t^2) + a_5 b_1 y^5 t^{-1} + a_3 b_3 y^3 t + a_1 b_5 yt^3 + 2\lambda^{-1} a_1 a_3 yt(y^2 + t^2) \log(t/2)\} M_4 - 2\lambda^{-1} a_1 a_3 yt(y^2 + t^2) N_4$$

$$\mathfrak{G}_5(t, y) = -2\lambda^{-1} a_1 a_3 yt(y^2 + t^2) M_4 \quad y > t \tag{4.2}$$

and

$$M_n = \int_0^1 \alpha(\xi) d\xi + \int_1^{\gamma} \beta(\xi) d\xi$$

$$N_n = \int_0^1 \alpha(\xi) \log \xi d\xi + \int_0^1 \beta(\xi) \log(\xi) d\xi \tag{4.3}$$

$$\alpha(\xi) = (1 + \mu) \xi^{n+2} \{\mu(1 - \xi^2)^{\frac{1}{2}} + (\gamma^2 - \xi^2)^{\frac{1}{2}}\}^{-1}$$

$$\beta(\xi) = (1 + \mu) \xi^{n+2} (\gamma^2 - \xi^2)^{\frac{1}{2}} \{\mu^2(\xi^2 - 1) + (\gamma^2 - \xi^2)\}^{-1}$$

$$\mu = \mu_1/\mu_2.$$

Next let us assume the solution of (3.9) in the form

(4.4) $$\theta(y)=\theta_0(y)+k_1^2\log k_1\,\theta_1(y)=k_1^2\theta_2(y)+(k_1^2\log k_1)^2\theta_3(y)$$
$$+\,k_1^4\log k_1\,\theta_4(y)+k_1^4\theta_5(y)+\theta(k_1^6)$$

Substituting this expansion as well as the expansion (4.1) of $\mathfrak{G}$ (t, y) in the integral equation (3.9) and equating the coefficients of equal powers of k_1, we get

$$\theta_0(y)=\frac{2d}{\lambda\,dy}\int_0^y\frac{u_0(x)\,dx}{(y^2-x^2)^{\frac{1}{2}}},$$

$$\theta_1(y)=\theta_3(y)=0,$$

$$\theta_2(y)=-\int_0^1 t\,\theta_0(t)\,\mathfrak{G}_1(t,\ y)\,dt$$

$$\theta_4(y)=-\int_0^1 t\,\theta_0(t)\,\mathfrak{G}_3(t,\ y)\,dt$$

$$\theta_5(t)=-\int_0^1 t\,[\theta_2(t)\,\mathfrak{G}(t,\ y)+\theta_0(t)\,\mathfrak{G}_2(t,\ y)]\,dt$$

In a particular case if we choose $u_0(x)=\dfrac{2\varepsilon}{\lambda}(1-x)$, ε being a known constant, then it can easily be seen that

$$\theta_0(y)=-\varepsilon,$$

$$\theta_2(y)=-a_1\,b_i\,M_0\,\varepsilon\,y\,(1-2y/3),$$

$$\theta_4(y)=\frac{2\,\varepsilon\,a_1^2\,M_2\,y}{3\lambda},\ \text{etc}$$

For most practical cases, the dimensionless frequency k_1 is taken less then 0.6. For these values, (4.4) gives approximate solutions to a sufficient degree of accuracy.

5. Stress intensity factor

The stress component has singularity *a−t* the edge of the strip. The singular part of the stress component can be obtained by using equations (2.7), (3.4) and (3.8).

Hence for $|x|<|$, we have

$$\sigma_{yz}(x, 0+)=\sigma_{yz}(x, 0-)=-\mu_1\int_0^1 \Psi(\xi)\cos\xi x\,d\xi+0(1)$$

$$=\mu_1\,\theta(1)\int_0^\infty \tau_0(\xi)\cos\xi x\,d\xi+0(1)$$

(5.1)
$$=\mu_1\frac{\theta(1)}{(1-x^2)^{\frac{1}{2}}},\quad |x|<1$$

Defining stress intenstty factor by the relation

(5.2)
$$N_z=\lim_{x\to 1-}(1-x)^{\frac{1}{2}}|\sigma_{yz}(x, 0)|_{x<1}$$

We obtain, on using (5. 1)

(5.3)
$$N_z=\mu_1\frac{|\theta(1)|}{\sqrt{2}}$$

Solutions derived in section 4 may be used to calculate the value of stress intensity factor for different combinations of the two elastic media and for a range of circular wave frequency. We have made calculations for the following values of the material constants which correspond to steel and Aluminium respectively:

$$\rho_1=7.6\text{ gm/cm}^3,\quad \mu_1=8.32\times 10^{11}\text{ dyne/cm}^2$$

$$\rho_2=2.7\text{ gm/cm}^3,\quad \mu_2=2.63\times 10^{11}\text{ dyne/cm}^2$$

The results are given in the following table:

k_1	0	0.1	0.2	0.3	0.4	0.5	0.6
$\varepsilon^{-1}\lvert\theta(1)\rvert$	1.000	0.9996	0.9981	0.9941	0.9872	0.9815	0.9683

It may be observed that the stress intensity factor decreases with the increase of the dimensionless frequency. This is in contrast to the results derived by Srivastava, Palaiya and Karolia [3] in the case of Griffith crack.

REFERENCES

[1] Srivastava, K.N., Gupta, O.P and Palaiya, R.M., *Interaction of elastic waves in two bonded dissimilar elastic half-planes having Griffith crack at interface — I**, Int. J. Fracture, 14 (2), 145 (1978).

[2] Srivastava, K.N., Palaiya, R.M. and Gupta, O.P., *Interaction of elastic waves with a penny-shaped crack at the interface of two bonded elastic half-spaces*, Int. J. Fracture, 15 (1979).

[3] Srivastava, K.M., Palaiya, R.M. and Karolia, D.S. *Interaction of antiplane shear waves by Griffith crack at the interface of two bonded dissimilar elastic half-planes* Int. J. Fracture (to appear).

[4] Srivastava, K.N., Palaiya, R.M. and Karolia, D.S. *Interaction of torsional waves by penny-shaped crack at the interface of two bonded dissimilar elastic half-spaces,* Zeit, Ang. Math. Mech. (to appear).

[5] Sneddon, I.N., *Mixed Boundary value problems in Potential Theory,* North Holland Publishing company, (1966).

[6] Mal, A.K., *Interaction of elastic waves with a Griffith crack,* Int. J. Engng. Sci. 8, 763 (1970).

[7] Abramowitz, M. and Stegun I.A., *Handbook of Mathematical functions,* Dover N.Y. (1965).

LA DIFFRACTION DES ONDES ANTI-PLANES PAR BANDE RIGIDE À LINTERFACE DE DEUX DEMI-ESPACES DISSIMILAIRE ÉLASTIQUE

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Résumé

Le problème de la diffraction des ondes orthegonales incidentes par une bande rigide située a l'integrrace de deux demi-espaces élastiques est résolu. Le probleme est reduit à la solution de l'équation intégrale dual. La solution approximative de l'équation intégrale dual est obtenu obtenu en réduisant la même a l'équation inteégrale de Fredholm de deuxième tupe. On use cette solution pour calculer le coéfficient d'intensité de tension a la lisière de la bande.

DIFRAKCIJA ANTIRAVNIH SMIČUĆIH TALASA NA ČVRSTOJ TRACI KOJA LEŽI NA POVRŠI KOJA RAZDVAJA DVA RAZLIČITA ELASTIČNA POLUPROSTORA

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Izvod

U radu se rešava problem difrakcije smičućih talasa koji padaju normalno na čvrstu traku koja se nalazi na međupovrši između dva različita elastična poluprostora. Problem je sveden na rešavanje dualne integralne jednačine. Dobijeno je približno rešenje dualne integralne jednačine i to svođenjem na Fredholmovu integralnu jednačinu druge vrste. Rešenje je iskorišćeno za izračunavanje faktora intenziteta napona na ivici trake.

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