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FLOW OF A SECOND ORDER FLUID IN A DOUBLE ARRAY OF VORTICES AND BEHIND A TWO-DIMENSIONAL GRID

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1. Introduction

Taylor [1] solved the problem of flow in a double array of vortices in a viscous liquid and found that such vortices decay exponentially with time. This decay or vortices is mainly due to the viscous action of the liquid. The non-linear terms of the vorticity equation of such flows cancel out among themselves and this fact led Taylor to obtain an exact solution of the problem.

Kovásznay [2] studied a similar type of problem on the flow behind a two-dimensional grid. Assuming periodicity of the flow in one direction, he obtained an exact solution of the Navier-Stokes equations. This reveals he existence of pair of bound eddies behind the single element of the grid with stream lines becoming parallel at large distance down stream. Gupta [3] solved a similar problem for the Watters [4] B' liquid.

In this paper our aim is to study these two problems replacing viscous liquid by Second order fluid. The constitutive equation of an incompressible second order fluid had been suggested by Noll and Coleman [5] as

(1) $$p_{ij} = -p\delta_{ij} + \mu_1 A_{(1)}ij - \mu_1 A_{(2)}ij + \mu_3 A_{(1)ik} A_{(1)kj}$$

wh·re

(2) $$\left.\begin{aligned} A_{(1)ij} &= v_{i,j} + v_{j,i} \\ A_{(2)ij} &= a_{i,j} + a_{j,i} + 2v_{m,i}v_{m,j} \end{aligned}\right\}$$

p_{ij} is the stress tensor, v_i and a_i are the velocity and the acceleration vectors respectively, μ_1, μ_2 and μ_3 are the material constants and p is an indeterminate hydrostatic pressure.

The equation of continuity and motion are

(3) $$A_{(1)ii} = 0$$

(4) $$\rho\left[\frac{\partial v_i}{\partial t} + v_j v_{i,j}\right] = p_{ij,j}$$

From Eqs. (1), (3) and (4), we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu_1\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$-\nu_2\left[\frac{\partial^3 u}{\partial t\,\partial x^2}+\frac{\partial^3 u}{\partial t\,\partial y^2}\right.$$

$$+10\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}$$

$$+u\frac{\partial^3 u}{\partial x^3}+3\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}$$

$$+v\frac{\partial^3 u}{\partial x^2\,\partial y}+3\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y}+u\frac{\partial^3 u}{\partial x\,\partial y^2}$$

$$+v\frac{\partial^3 u}{\partial y^3}+3\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\,\partial y}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}$$

$$\left.+\frac{\partial u}{\partial x}\cdot\frac{\partial^2 y}{\partial y^2}\right]+\nu_3\frac{\partial}{\partial x}\left[4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right], \tag{5}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1\,\partial p}{\rho\,\partial y}+\nu_1\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\nu_2\left[\frac{\partial^3 v}{\partial t\,\partial x^2}\right.$$

$$+\frac{\partial^3 v}{\partial t\,\partial y^2}+3\frac{\partial y}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}+3\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}$$

$$+u\frac{\partial^3 v}{\partial x^3}+3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\,\partial y}+v\frac{\partial^3 v}{\partial x^2\,\partial y}$$

$$+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\,\partial y}+u\frac{\partial^3 v}{\partial x\,\partial y^2}+10\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 v}{\partial y^3}$$

$$\left.+4\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right]+\nu_3\frac{\partial}{\partial y}\left[4\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]. \tag{6}$$

2. Flow in a double array of vortices

Let us consider a double array of vortices in the flow of second order fluid. We can define a stream function ψ satisfying th continuity equation (3), given by

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x}. \tag{7}$$

Eliminating p between (5) and (6) and putting the values of u and v from (7), we get

$$\frac{\partial^3\psi}{\partial t\,\partial y^2}+\frac{\partial^3\psi}{\partial t\,\partial x^2}+\frac{\partial\psi}{\partial y}\,\frac{\partial^3\psi}{\partial x\,\partial y^2}-\frac{\partial\psi}{\partial x}\,\frac{\partial^3\psi}{\partial x^2\,\partial y}+\frac{\partial\psi}{\partial y}\,\frac{\partial^3\psi}{\partial x^3}-\frac{\partial\psi}{\partial x}\,\frac{\partial^3\psi}{\partial y^3}$$

$$=\nu_1\left[\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}+2\frac{\partial^4\psi}{\partial x^2\,\partial y^2}\right]-\nu_2\left[2\frac{\partial\psi}{\partial y}\,\frac{\partial^5\psi}{\partial x^3\,\partial y^2}-2\frac{\partial\psi}{\partial x}\,\frac{\partial^5\psi}{\partial x^2\,\partial y^2}\right.$$

$$-2\frac{\partial^2\psi}{\partial y^2}\,\frac{\partial^4\psi}{\partial x^3\,\partial y}+2\frac{\partial^2\psi}{\partial x^2}\,\frac{\partial^4\psi}{\partial x^3\,\partial y}+2\frac{\partial^2\psi}{\partial x^2}\,\frac{\partial^4\psi}{\partial x\,\partial y^3}$$

$$-2\frac{\partial^2\psi}{\partial y^2}\,\frac{\partial^4\psi}{\partial x\,\partial y^3}+\frac{\partial^5\psi}{\partial t\,\partial y^4}+\frac{\partial^5\psi}{\partial t\,\partial x^4}+\frac{\partial\psi}{\partial y}\,\frac{\partial^5\psi}{\partial x\,\partial y^4}$$

$$-\frac{\partial\psi}{\partial x}\,\frac{\partial^5\psi}{\partial x^4\,\partial y}+\frac{\partial\psi}{\partial y}\,\frac{\partial^5\psi}{\partial x^5}-\frac{\partial\psi}{\partial x}\,\frac{\partial^5\psi}{\partial x^5}$$

$$\left.+2\frac{\partial^2\psi}{\partial x\,\partial y}\,\frac{\partial^4\psi}{\partial y^4}-2\frac{\partial^2\psi}{\partial x\,\partial y}\,\frac{\partial^4\psi}{\partial y^4}\right]. \tag{8}$$

Following Taylor, we take

$$\psi=A\cos\frac{\pi x}{d}\cdot\cos\frac{\pi y}{d}\cdot e^{-\lambda t}, \tag{9}$$

where "d" is the space between two vortices, A is a constant and λ is the vorticity parameter. Substituting (9) into (8) and simplifying, we get

$$\lambda-\lambda\nu_2\frac{\pi^2}{d^2}=\frac{2\nu_1\pi^2}{d^2}$$

$$\Rightarrow\ \lambda=\frac{2\nu_1\pi^2/d^2}{1-\nu_2\dfrac{\pi^2}{d^2}}. \tag{10}$$

3. Flow behind a two-dimensinal grid

We now study the laminar flow of a second order liquid behind a two-dimensional grid. We take x — axis normal to the grid. Eliminating p between (5) and (6) and using the fact that

$$\eta=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x},$$

we get

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial y}=\nu_1\nabla^2\eta-\nu_2\Bigg[\frac{\partial}{\partial t}\nabla^2\eta+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla^2\eta$$

$$+4\left\{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right)\right.$$

$$\left.-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}\right)\right\}$$

$$+2\left\{\frac{\partial^2}{\partial y^2}\left(\frac{\partial y}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right)\right.$$

$$\left.-\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right)\right\}\Bigg]. \tag{11}$$

Let be U the average velocity along the axis of x. Denoting the local velocity components along the x and y directions by $U+u'(x', y')$ and $v'(x', y')$ we get from (11)

$$\frac{\partial\omega}{\partial t}+(1+u)\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{R}\nabla^2\omega-\Lambda\Bigg[\frac{\partial}{\partial t}(\Delta^2\omega)+(1+u)\frac{\partial}{\partial x}(\nabla^2\omega)$$

$$+v\frac{\partial}{\partial y}(\nabla^2\omega)+2\left\{\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^3 v}{\partial x^3}\right.$$

$$\left.-\frac{\partial u}{\partial y}\frac{\partial}{\partial x}(\nabla^2 u)-\frac{\partial v}{\partial x}\frac{\partial^2\omega}{\partial x\,\partial y}\right\}\Bigg], \tag{12}$$

where

$$x'=hx;\quad y'=hy;\quad Ut'=ht;\quad u'=Uu$$

$$v'=Uv;\quad h\eta=U\omega;\quad R=\frac{hU}{\nu_1};\quad \Lambda=\frac{\nu_1}{h^2}.$$

We define a stream function ψ as in (7), whence we get

$$\omega=\nabla^2\psi. \tag{13}$$

We assume a periodic solution in the y-direction, in the form

$$\psi(x, y)=f(x)\sin 2\pi y. \tag{14}$$

This form of solution was assumed by Kovásznay. From (13) and (14), we get

$$w=(f''-4\pi^2 f)\sin 2\pi y, \tag{14a}$$

where a dash denotes differentiation with respect to x.

In case of steady motion from (12), we get

$$\nabla^2 \omega - \Lambda R \frac{\partial}{\partial x}(\nabla^2 \omega) - R\frac{\partial \omega}{\partial x} = R\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) + \Lambda R\left[u\frac{\partial}{\partial x}(\nabla^2 \omega)\right.$$

$$+ v\frac{\partial}{\partial y}(\nabla^2 \omega) + 2\left\{\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^3 v}{\partial x^3} - \frac{\partial u}{\partial y}\frac{\partial}{\partial x}(\nabla^2 u)\right.$$

$$\left.\left. - \frac{\partial v}{\partial x}\frac{\partial^2 \omega}{\partial x \partial y}\right\}\right]. \tag{15}$$

In order that the term $u\frac{\partial \omega}{\partial y} + v\frac{\partial \omega}{\partial y}$ in (15) should vanish, we must have

$$ff''' - f'f'' = 0,$$

which gives

$$f = Ae^{mx}, \tag{16}$$

where A and m are constants. So now we get

$$\begin{aligned} \psi &= Ae^{mx}\sin 2\pi y, \\ u &= 2\pi Ae^{mx}\cos 2\pi y, \\ v &= -Ame^{mx}\sin 2\pi y. \end{aligned} \tag{17}$$

With (16) and (17), the right hand side of (15) vanishes identically, and we get

$$\nabla^2 \omega = \Lambda R \frac{\partial}{\partial x}(\nabla^2 \omega) - R\frac{\partial \omega}{\partial x} = 0. \tag{18}$$

To solve (18), we set

$$w = g(x)\sin 2\pi y, \tag{19}$$

when we get from (18)

$$\Lambda R g''' - g'' + (R - 4\pi^2 \Lambda R) g' + 4\pi^2 g = 0. \tag{20}$$

To solve (20), we take a trial solution

$$g = Be^{\lambda x}$$

where λ is a root of the cubic equation

$$\Lambda R \lambda^3 - \lambda^2 + R(1 - 4\pi^2 \Delta)\lambda + 4\pi^2 = 0. \tag{21}$$

In this equation $\Lambda > 0$ and $R > 0$. So the coefficients are real. Hence, cubic (21) has at least one real root. If $\Lambda < \frac{1}{4\pi^2}$, then there are two changes of sign in (21). Hence, maximum number of positive roots is two. Changing λ

to $-\lambda$ we get only one change of sign. So at least there is only one negative root. Thus, for $\Lambda < \frac{1}{4\pi^2}$ and all values of R, e_q, (21) has a negative real root, say λ_1, and two positive real roots λ_1, λ_2. We are interested only with the root λ_1 and the roots λ_2, λ_3 are rejected as they will make ω infinite at large distance down stream which is physically not possible. In order that the solution

(22) $$\omega = Be^{\lambda_1 x} \sin e\, 2\pi y$$

is consistent with (14a) and(16), we must have

$$\omega = A(m^2 - 4\pi^2)\, e^{mx} \sin 2\pi y = Be^{\lambda_1 x} \sin e\, 2\pi y$$

and this is so if

(23) $$\lambda_1 = m \quad \text{and} \quad B = A(m^2 - 4\pi^2)$$

If ψ_0 is the stream function for the average flow, then

(24) $$\psi_0 = y,$$

and the stream function for the flow behind the grid from (17), (23) and (24)

(25) $$\psi + \psi_0 = y + Ae^{\lambda_1 x} \sin 2\pi y$$

If we choose the stagnation point at (0, 0), then

$$\frac{\partial \psi}{\partial y} = 1 + 2\pi A e^{\lambda_1 x} \cos 2\pi y = 0,$$

and hence

(26) $$A = -\frac{1}{2\pi}.$$

From (22), (25) and (26)

(27)
$$1 + u = 1 - e^{\lambda_1 x} \cos 1\pi y,$$
$$v = \frac{\lambda_1}{2\pi} e^{\lambda_1 x} \sin 2\pi y,$$
$$\omega = -\frac{1}{2\pi} (\lambda_1{}^2 - 4\pi^2)\, e^{\lambda_1 x} \sin 2\pi y.$$

4. Conclusions

Equation (10) shows that v_3 does not play any role to modify the vorticity decay or growth. Putting $v_2 = 0$, that is, for a viscous liquid, the growth or decay rate of vortices is

$$\lambda = \frac{2 v_1 \pi^2}{d^2}.$$

Hence, for an ordinary Newtonian liquid the vortices decay as time increases. But for a visco-elastic liquid, if $d^2 < \pi^2 \nu_2$ then λ is negative and the vorticity increases with time. But if $d^2 > \pi^2 \nu_2$, the vorticity decreases as time increases From (25) and (27), it is seen that at large distance down stream,

$$u = 1, \quad v = 0, \quad \psi + \psi_0 = y$$

and this proves that at large distance down stream the stream lines become parallel as in the viscous case. Equation (21) has been solved by Newton's method to obtain th. negative root for different values of R and Λ and these values are presented in the table — 1. An examination of this table shows that for fixed values of R, λ_1 increases with Λ, the elastic parameter. This implies that the scale of the eddies behind the grid decreases with the increase in the value of the elastic parameter. For a fixed value of Λ the value of λ_1 increases with the Reynolds number R.

Table — 1

Value of λ_1

R \ Λ	0.000	0.010	0.020
20	−1.81010	−2.49245	−3.31359
40	−1.13702	−1.48545	−1.84325
60	−0.45639	−1.03892	−1.61145

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DIE STRÖMUNG DER FLÜSSIGKEIT ZWEITER ORDNUNG IN EINER DOPPELTEN DER WIRBELN UND HINTER DEN ZWEI-DIMENSIONALEN RAHMEN

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Zusammenfassung

In dieser Arbeit die Verfasser behandeln ein interessantes Problem zweireihigen Wirbeln in einer nichtnewtonischen Flüssigkeit zweiter Ordnung. Im ersten Falle wurde gefunden dass in dieser zähelastischen Flüssigkeit das Wirbeln zeitlich wächst, insofern die Entfernung zwischen diesen zweireihigen Wirbeln unter dem kritischen Wert liegt. Dagegen, im Falle dass diese Entfernung den kritischen Wert übersteigt, das Wirbeln nimmt mit der Zeit ab. Dieser kritische Wert der Entfernung hängt von der Elastizität der Flüssigkeit ab.

DVOREDNI VRTLOG U (NENJUTNOVSKOM) FLUIDU DRUGOG REDA KAO I PROBLEM STRUJANJA ISTOG FLUIDA OKO DVO-DIMENZIONOG RAMA

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Rezime

Autori rešavaju problem dvorednih vrtloga u (nenjutnovskom fluidu drugoga reda, kao i problem strujanja istog fluida oko dvo-dimenzionog rama. U prvom slučaju je nađeno da u visko-elastičnom fluidu vrtložnost narasta u vremenu, ukoliko je rastojanje između ovih dvorednih vrtloga manje od kritične vrednosti. Naprotiv, ako je ovo rastojanje veće od kritične vrednosti, vrtložnost opada u vremenu. Ova kritična vrednost rastojanja zavisi od elastičnosti tečnosti.

Rešavajući drugi problem, autori dokazuju da na velikom rastojanju nizvodno, strujnice postaju nezavisne od efekta drugog reda i postaju, ustvari, paralelne strujnicama za slučaj strujanja viskoznog fluida.

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