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IMPROVED THEORY OF BENDING OF A SIMPLY SUPPORTED PLATE

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1. Definition of the third boundary-value problem in bending of a plate

In the theory of boundary value problems in bending of moderate thick plates, the first two boundary-value problems have already been dealt with several times and in principle solved generally [1—3]. If the following designations are introduced:

$$K_1 = -\frac{16\mu}{1-\nu}, \qquad \alpha_1 = -\frac{4h^2}{1-\nu} \tag{1—1}$$

$2h$ = plate thickness then, the bending w of the plate's middle plane can be expressed like:

$$K_1 w = K_1 w_0 + \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} + \alpha_1 [\varphi'(z) + \overline{\varphi'(z)}] \tag{1—2}$$

Here w_0 is the bending resulting from the particular solution of biharmonic equation for w.

The first boundary-value problem can now in case of clamped boundary be formulated in the form

$$\frac{\partial w}{\partial \bar{z}} = 0, \qquad z \in C \tag{1—3}$$

or:

$$-K_1 \frac{\partial w_0}{\partial \bar{z}} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha_1 \overline{\varphi''(z)}, \qquad z \in C \tag{1—4}$$

$\psi(z) = \chi'(z)$

if C is the boundary curve of domain D where the boundary value problem is being solved, and $\varphi(z)$ and $\psi(z)$ are unknown analytical functions in D.

If in continuation the following designations are introduced:

$$(1-5)\qquad \left.\begin{aligned} K_2 &= -\frac{6}{(1-\nu)\,h^2} \\ \alpha_2 &= -\frac{2\,(2-\nu)\,h^3}{5\,(1-\nu)} \end{aligned}\right\}$$

$$(1-6)\qquad \left.\begin{aligned} &G = \text{bending couple function} \\ &H = \text{generalized shear force function} \\ &N = \int^{z}\left[G + i\int^{s} H ds\right] dz \end{aligned}\right\}$$

then it is possible to formulate the second boundary-value problem for one unloaded boundary curve as:

$$(1-7)\qquad N(w) = 0, \qquad z \in C$$

or

$$(1-8)\qquad \begin{aligned} &-K_2 N(w_0) = -\varkappa\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha_2\overline{\varphi''(z)} \\ &z \in C \end{aligned}$$

In the third boundary-value problem the plate on the boundary curve is simply supported. This means that the boundary condition at such a bending is the following

$$(1-9)\qquad \begin{aligned} w &= 0 \\ G(w) &= 0, \qquad z \in C \end{aligned}$$

With designations $\frac{df}{ds} = \dot{f}$ and

$$(1-10)\qquad F(z) + \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha_1\overline{\varphi''(z)}$$

it is possible to write the boundary condition (1—9) in the following form

$$(1-11\,a)\qquad \begin{aligned} -[2K_2 G(w_0) - K_1\ddot{w}_0] &= 2\,Re\,[(\alpha_2-\alpha_1)\,\dot{z}^2\varphi'''(z) - (\varkappa+1)\,\varphi'(z)] - \\ &\quad -[\ddot{\bar{z}} F(z) + \ddot{z}\overline{F(z)}] \end{aligned}$$

$$(1-11\,b)\qquad \begin{aligned} &-K_1\dot{w}_0 = \dot{\bar{z}} F(z) + \dot{z}\overline{F(z)} \\ &z \in C \end{aligned}$$

If the above equations are multiplied in turn by 1 and i and summed up, we get

$$(1-12)\qquad \begin{aligned} [2K_2 G(w_0) - K_1\ddot{w}_0] + iK_1\dot{w}_0 &= 2\,Re\,[-\alpha_0\dot{z}^2\varphi'''(z) + (\varkappa+1)\,\varphi'(z)] + \\ &\quad + (-i\dot{\bar{z}} + \ddot{\bar{z}})\,F(z) + (-i\dot{z} + \ddot{z})\,\overline{F(z)} \\ &z \in C \end{aligned}$$

This is an improved formulation of the boundary valuc problem in Sherman's form [4] similar to that in the theory of thin plates. It is applicable to all the forms of the domain where there the boundary-value problem is solved.

But if we multiply the equations (1—11) in turn by $\dot{z}$ and $\ddot{z}$ and sum them up, taking into account

$$\frac{1}{\rho} = -i\,\dot{\bar{z}}\,\ddot{z} \tag{1—12'}$$

it follows:

$$\begin{aligned}\mathcal{F}(z) &= -\left\{[2K_2 G(w_0) - K_1 \ddot{w}_0] + i\frac{K_1}{\rho}\dot{w}_0\right\} = \\ &= 2\,Re\,[\alpha_0 \dot{z}^2 \varphi'''(z) - (\varkappa+1)\varphi'(z)] - 2\ddot{\bar{z}}\,[\varphi(z) + z\overline{\varphi'(z)} + \\ &+ \overline{\psi(z)} + \alpha_1\overline{\varphi''(z)}], \qquad z \in C\end{aligned} \tag{1—13}$$

where

$$\alpha_0 = \alpha_2 - \alpha_1 = \frac{2(8+\nu)}{5(1-\nu)} h^2 > 0 \tag{1—14}$$

The formulation (1—13) is simpler than Sherman's, but it is not generally applicable. It holds provided there is no straight section on the boundary curve C of domain D.

2. Simply Supported Circular Plate

A circular plate with $R=1$ will be discussed, thus $z = e^{is}$, $0 \leqslant s \leqslant 2\pi$, $\dot{z} = ie^{is} = iz$, $\ddot{z} = -e^{is} = -z$. From the equation (1—13) follows

$$\begin{aligned}\mathcal{F}(z) &= \alpha_0[\dot{z}^2\varphi'''(z) - \dot{\bar{z}}^2\overline{\varphi'''(z)}] - (\varkappa+1)[\varphi'(z) + \overline{\varphi'(z)}] - \\ &- 2\ddot{\bar{z}}\,[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha_1\overline{\varphi''(z)}] = \\ &= \{-\alpha_0[z^2\varphi'''(z) + \bar{z}^2\overline{\varphi'''(z)}] - (\varkappa+1)[\varphi'(z) + \overline{\varphi'(z)}] + \\ &+ 2\bar{z}\,[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha_1\overline{\varphi''(z)}]\}^+, \qquad z \in C\end{aligned} \tag{2—1}$$

Thus equation can be written in the following more suitable form:

$$\begin{aligned}&\left[\alpha_0 z^2\varphi'''(z) + (\varkappa+1)\varphi'(z) - \frac{2}{z}\varphi(z)\right]^+ - \\ &-\left[-\alpha_0\frac{1}{z^2}\bar{\varphi}'''\left(\frac{1}{z}\right) - (\varkappa+1)\bar{\varphi}'\left(\frac{1}{z}\right) + 2\bar{\varphi}'\left(\frac{1}{z}\right) + \right. \\ &\left. + \frac{2}{z}\bar{\psi}\left(\frac{1}{z}\right) + \frac{2\alpha_1}{z}\bar{\varphi}''\left(\frac{1}{z}\right)\right]^- = -\mathcal{F}(z), \qquad z \in C\end{aligned} \tag{2—2}$$

Let us define

$$\Phi(z) = \alpha_0 z^2 \varphi'''(z) + (\varkappa + 1)\varphi'(z) - \frac{2}{z}\varphi(z), \qquad z \in D$$

(2—3)
$$\Phi(z) = -\frac{\alpha_0}{z^2}\bar{\varphi}'''\left(\frac{1}{z}\right) - (\varkappa - 1)\bar{\varphi}'\left(\frac{1}{z}\right) + \frac{2}{z}\bar{\psi}\left(\frac{1}{z}\right) +$$
$$+ \frac{2\alpha_1}{z}\bar{\varphi}''\left(\frac{1}{z}\right), \qquad z \notin D \cup C$$

from (2—2) then follows

(2—4)
$$\Phi^+(z) - \Phi^-(z) = -\mathcal{F}(z), \qquad z \in C$$

Because

$$\varphi(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$\varphi'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

(2—5)
$$\varphi'\left(\frac{1}{z}\right) = \bar{a}_1 + \frac{2\bar{a}_2}{z} + \frac{3\bar{a}_3}{z^2} + \cdots$$

$$\psi(z) = b_0 + b_1 z + b_2 z^2 + \cdots$$

$$\bar{\psi}\left(\frac{1}{z}\right) = \bar{b}_0 + \frac{\bar{b}_1}{z} + \frac{\bar{b}_2}{z^2} + \cdots$$

then with the assumption $(\varphi(0) = a_0 = 0)$ we get

$$\lim_{z \to 0} \Phi(z) = (\varkappa - 1)\, a_1$$

(2—6)

$$\lim_{|z| \to \infty} \Phi(z) = -(\varkappa - 1)\, \bar{a}_1$$

From the above two equations and from (2—4) follows

(2—7)
$$\Phi(z) = -\frac{1}{2\pi i}\oint_{C+} \frac{\mathcal{F}(\zeta)\, d\zeta}{\zeta - z} - (\varkappa - 1)\,\bar{a}_1$$

and

(2—8)
$$(\varkappa - 1)(a_1 + \bar{a}_1) = -\frac{1}{2\pi i}\oint_{C+} \frac{\mathcal{F}(\zeta)}{\zeta}\, d\zeta$$

and thereof also

$$Im\left[-\frac{1}{2\pi i}\oint_{C+}\frac{\mathcal{F}(\zeta)}{\zeta}d\zeta\right]=0$$

From the equation (1—13) it follows that the right side of the equation (2—8) is real. Consequently $a_1 = Re[a_1]$ can be taken, and we get

(2—9) $$a_1 = -\frac{1}{4\pi i(\varkappa-1)}\oint_{C+}\frac{\mathcal{F}(\zeta)}{\zeta}d\zeta$$

and

(2—10) $$\Phi(z) = -\frac{1}{4\pi i}\oint_{C+}\frac{\zeta+z}{\zeta(\zeta-z)}\mathcal{F}(\zeta)\,d\zeta$$

Now from the first equation (2 – 3) the function $\varphi(z)$ has to be defined. The equation is written in the form:

(2—11) $$\alpha_0 z^3\varphi'''(z)+(\varkappa+1)z\varphi'(z)-2\varphi(z)=z\,\Phi(z) \qquad z\in D$$

The general solution contains three arbitrary constants that have to be defined so that $\varphi(z)$ in D is a holomorphic function everywhere. It proves that according to this requirement $\varphi(z)$ determined from (2—10) is unique. The solution can be given in a closed form, however for practical use it is more suitable to express it by the infinite series. If we write

(2—12) $$\Phi(z)=c_0+c_1 z+c_2 z^2+\cdot\cdot$$

considering (2 – 5) we get

(2—13) $$a_k=\frac{c_{k-1}}{[\alpha_0(k-1)(k-2)+(\varkappa+1)]\,k-2}, \qquad k=1,2,3,\ldots$$

thus

(2—13′) $$\varphi(z)=\sum_{k=1}^{k=\infty}\frac{c_{k-1}z^k}{[\alpha_0(k-1)(k-2)+(\varkappa+1)]\,k-2}$$

From the second equation (2 – 3) we now find the remaining function $\psi(z)$:

(2—14) $$2\psi(z)=\alpha_0 z\varphi'''(z)-2\alpha_1\varphi''(z)+\frac{1}{z}\left[(\varkappa-1)\varphi'(z)+\overline{\Phi}\left(\frac{1}{z}\right)\right]$$

From the equations (2 – 7) – (2 – 10) it follows that the expression in square brackets is at $z=0$ holomorphic and equal to 0, consequently $\psi(z)$ from equation (2 – 14) is everywhere in D a holomorphic function.

3. Simply Supported Infinite Plate with a Circular Hole

The procedure in this case is similar to that of the finite circular plate, except that here it is taken

$$\left.\begin{aligned}\varphi(z)&=\Gamma_1 z+\varphi_0(z), \quad \overline{\Gamma}_1=\Gamma_1\\ \psi(z)&=\Gamma_2 z+\psi_0(z)\end{aligned}\right\} \tag{3—1}$$

$$\left.\begin{aligned}\varphi_0(z)&=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots\\ \psi_0(z)&=b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots\end{aligned}\right\} \tag{3—2}$$

where Γ_1 is an arbitrarly given real constant and Γ_2 is arbitrarly given complex constant.

Further we take that

$$z=e^{-is}, \quad 0\leqslant s\leqslant 2\pi, \quad \dot{z}=-iz, \quad \ddot{z}=-z$$

and considering this in equation (1—13) we again get equation (2—1). For $\varphi(z)$ and $\psi(z)$ let the forms from (3—1) and (3—2) be taken into account. Thus we get quite analogue equations to (2—2), (2—3) and (2—4), provided that in these $\varphi(z)$ and $\psi(z)$ are substituted by $\varphi_0(z)$ and $\psi_0(z)$. We assume $b_0=0$ and instead of $\mathcal{F}(z)$ we write $\mathcal{F}(z)+2(\varkappa-1)$. $\Gamma_1-\frac{2}{z^2}\cdot\overline{\Gamma}_2=\mathcal{F}_0(z)$.

Finally is

$$\Phi(z)=\frac{1}{2\pi i}\oint_{C+}\frac{\mathcal{F}(\zeta)\,d\zeta}{\zeta-z}+\frac{2\overline{\Gamma}_2}{z^2}, \qquad z\in D \tag{3—3}$$

$$\Phi(z)=\frac{1}{2\pi i}\oint_{C+}\frac{\mathcal{F}(\zeta)\,d\zeta}{\zeta-z}+2(\varkappa-1)\Gamma_1, \qquad z\notin D\cup C \tag{3—3'}$$

If we express

$$\frac{1}{2\pi i}\oint_{C+}\frac{\mathcal{F}(\zeta)\,d\zeta}{\zeta-z}=\sum_{k=1}^{k=\infty}\frac{c_k}{z^k}$$

equations (2—3) and (2—11) yield

$$\varphi_0(z)=-\sum_{k=0}^{k=\infty}\frac{c_{k+1}+2\overline{\Gamma}_2\,\delta(k-1)}{\{[\alpha_0(k+1)(k+2)+(\varkappa+1)]\,k+2\}\,z^k} \tag{3—4}$$

where $\delta(0)$ is Kronecker's symbol. From equation (2—3) we get further the function $\psi_0(z)$

$$2z\psi_0(z)=\alpha_0 z^2\varphi'''(z)+(\varkappa-1)\varphi'(z)-2\alpha_1 z\varphi''(z)+\overline{\Phi}\left(\frac{1}{z}\right) \tag{3—5}$$

This is in D obviously holomorphic since it is

$$\overline{\Phi}\left(\frac{1}{z}\right)=-\frac{1}{2\pi i}\oint_{C+}\frac{z\overline{\mathcal{F}(\zeta)\,d\zeta}}{z\bar{\zeta}-1}+2(\varkappa-1)\Gamma_1, \qquad z\in D$$

where $\psi_0(\infty)=0$ is still valid. This is in accordance with the assumption $b_0=0$.

4. Simply supported semi-plane

In the case of a semi-plane $\ddot{z}$ is equal to 0 on the boundary curve, and thus the discussed problem cannot be solved in the form (1—13). From this equation follows only that

$$\begin{aligned}\mathcal{F}(z)&=-[2K_2G(w_0)-K_1\ddot{w}_0]=\\&=2\,Re\,[\alpha_0\dot{z}^2\varphi'''(z)-(\varkappa+1)\varphi'(z)]^-\end{aligned} \tag{4—1}$$

if D is the lower semi-plane. This is written in the form

$$\begin{aligned}&[-\alpha_0\overline{\varphi}'''(z)+(\varkappa+1)\overline{\varphi}'(z)]^+-\\&-[\alpha_0\varphi'''(z)-(\varkappa+1)\varphi'(z)]^-=-\mathcal{F}(z), \qquad z\in C\end{aligned} \tag{4—2}$$

Functions $\varphi(z)$ and $\psi(z)$ are being sought in the form

$$\begin{aligned}&\varphi(z)=\Gamma_1 z+\varphi_0(z), \qquad \overline{\Gamma}_1=\Gamma_1\\&\psi(z)=\Gamma_2 z+\psi_0(z)\end{aligned} \tag{4—3}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are holomorphic in the lower semi-plane and for lim $|z|=0$ it is:

$$\left.\begin{aligned}&\varphi_0(z)=a_0+o(1)\\&\psi_0(z)=b_0+o(1)\\&\varphi_0'(z)=o(|z|^{-1})\\&\varphi_0''(z)=o(1)\\&\varphi_0'''(z)=o(1)\end{aligned}\right\} \tag{4—4}$$

If we define

$$\text{(4—5)}\qquad \begin{aligned} &\Phi(z) = -\alpha_0 \overline{\varphi'''}(z) + (\varkappa+1)\overline{\varphi'}(z), && z \notin D \cup C \\ &\Phi(z) = \alpha_0 \varphi'''(z) - (\varkappa+1)\varphi'(z), && z \in D \end{aligned}$$

then the above equations yield

$$\text{(4—6)}\qquad \Phi^+(z) - \Phi^-(z) = -\mathcal{F}(z), \qquad z \in C$$

$$\text{(4—6')}\qquad \Gamma_1 = 0$$

and

$$\text{(4—7)}\qquad \Phi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathcal{F}(\xi)\, d\xi}{\xi - z}$$

From the second equation (4—5) we get

$$\text{(4—8)}\qquad \alpha_0 \varphi'''(z) - (\varkappa+1)\varphi'(z) = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} (\xi - z)^{-1} \cdot \mathcal{F}(\xi)\, d\xi \qquad z \in D$$

From this equation the function $\varphi(z)$ has to be defined with the properties (4—3) and (4—4). First of all it is:

$$\text{(4—9)}\qquad \alpha_0 \varphi''(z) - (\varkappa+1)\varphi(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \ln(\xi - z)\, \mathcal{F}(\xi)\, d\xi - (\varkappa+1)\, a_0$$

The general solution of this equation contains two constants, but they have to be defined in accordance to the required properties of (4—4). This definition is unique. After a short calculation we get

$$\text{(4—10)}\qquad \varphi(z) = -\frac{1}{4\pi i \sqrt{(\varkappa+1)\alpha_0}} \cdot \left\{ \int_{-\infty}^{z} e^{-\sqrt{\frac{\varkappa+1}{\alpha_0}}(z-\eta)} \left[\int_{-\infty}^{+\infty} \ln(\xi - \eta)\, \mathcal{F}(\xi)\, d\xi \right] d\eta + \int_{z}^{\infty} e^{-\sqrt{\frac{\varkappa+1}{\alpha_0}}(\eta - z)} \cdot \left[\int_{-\infty}^{+\infty} \ln(\xi - \eta)\, \mathcal{F}(\xi)\, d\xi \right] d\eta \right\} + a_0$$

The constant a_0 remains arbitrary.

The second function $\psi(z)$ has to be sought either from the general formulation of the boundary value problem (1—12) or simply from (1—11 b). For semi-plane the latter equation can be written:

$$-K_1 \dot{w}_0 = [F(z) + \overline{F(z)}]^-, \qquad z \in C \tag{4—11}$$

Introducing still

$$\begin{aligned} \Psi(z) &= \overline{\varphi}(z) + z\overline{\varphi}'(z) + \overline{\psi}(z) + \alpha_1 \overline{\varphi}''(z), \qquad z \notin D \cup C \\ \Psi(z) &= -\varphi(z) - z\varphi'(z) - \psi(z) - \alpha_1 \varphi''(z), \qquad z \in D \end{aligned} \tag{4—12}$$

we get from (4—11)

$$\Psi^+(z) - \Psi^-(z) = -K_1 \dot{w}_0, \qquad z \in C \tag{4—13}$$

and considering (4—3) and (4—4) at $\lim |z| = \infty$ we further obtain:

$$\begin{aligned} \Psi(z) &= -\Gamma_2 \cdot z - (a_0 + b_0) + o(1), \qquad z \in D \\ \Psi(z) &= \overline{\Gamma}_2 \cdot z + (\bar{a}_0 + \bar{b}_0) + o(1), \qquad z \notin D \cup C \end{aligned} \tag{4—14}$$

Out of these equations for $\Psi(z)$ follows that (4—14) is possible only at

$$Re[\Gamma_2] = 0 \tag{4—15}$$

thus

$$\Psi(z) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{K_1 \dot{w}_0 \, d\xi}{\xi - z} - i\, Im[\Gamma_2] z + K \tag{4—16}$$

Considering the equation

$$a_0 + \bar{a}_0 + b_0 + \bar{b}_0 = 0$$

it follows for the constant K

$$K = -(a_0 + b_0) = -i\, Im[a_0] - i\, Im[b_0] \tag{4—17}$$

Consequently it finally follows:

$$\psi(z) = -\varphi(z) - z\varphi'(z) - \alpha_1 \varphi''(z) - \Psi(z) \tag{4—18}$$

The constant $Im[b_0]$ remains arbitrary. It can be used e.g. for the fulfilment of the additional boundary value requirement:

$$\frac{\partial w}{\partial y}(0, 0) = 0.$$

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BIEGUNG DER GELENKIG GELAGERTEN PLATTE NACH DER VERBESSERTEN THEORIE

B. Krušić

Zusammenfassung

In dem Beitrag ist die Formulierung der Randweitbedingung der gelenkig gelagerten Platte in der komplexen Form in zwei Varianten gegeben Eine von denen ist einfacher, aber sie ist nicht allgemein verwendbar. Im weiteren sind die Lösungen der Randwertaufgabe für kreisförmige Platte, für die unendlich ausgedehnte Platte mit kreisförmiger Öffnung und für Halbebene dargestellt.

UPOGIB PROSTO PODPRTE PLOŠČE PO IZBOLJŠANI TEORIJI

B. Krušič

Rezime

V tem članku je dana formulacija robnega pogoje prosto podprte plošče v kompleksni obliki v dveh variantah. Ena od teh je preprostejša, vendar je ni moč splošno uporabiti. Dane so rešitve problema za krožno ploščno, za neskončno ploščo s krožno odprtino in za polravnino.

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