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NOETHER'S THEOREM AND THE ISOPERIMETRIC PROBLEM IN CONTINUUM MECHANICS

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Introudction. In general, all problems in which one integral is to be made maximum or minimum subject to the subsidiary condition that a second integral has a given value, is defined as isoperimetric problems. This name stems from the famous problem, namely the problem that of finding the closed curve of given perimeter for which the area is maximum.

If we have the Lagrangian formulation for continuum mechanics (1), i.e. if development of the equations of motion is in general based upon variational methods and if we have one constraint of integral type on this motions then we call this problem "isoperimetric" for continuous systems.

The purpose of this paper is to present rigorous derivations of the conservation laws for this isoperimetric problem based on the Noether's theorem (2) and Lie's [3] groups theory.

Preliminaries. To alleviate the exposition, we make here a number of simplifying assumptions and before describing the class of conservations laws, to ivestigate we must introduce a few notions.

The independent variables describing the nature of the mechanical system under discussion to be designated as x^i where unless otherwise stated, indices $i, j, \dots, n$ range over 1, 2, .. , n, while the dependent variables will be designated as u^a and where indices $a, b, \dots,$ range over 1, 2, ... , m. The partial derivatives of the dependent variables with respect to the independent variables will be indicated by the index notation as by the following examples $\frac{\partial}{\partial x^i} \equiv \partial_i$, and partial derivation of some function with respect on dependent variables as $\frac{\partial}{\partial u^a} = \partial_a$. Summation convention will be used.

Isoperimetric problem. Suppose that we have a motion of a continuous system with integral of action in the form

$$L = \int_D l(x^i, u^a, u^a_{i,})\, d\omega, \tag{1}$$

where l is Lagrangian density, D -"region" of integration and $d\omega$ volume element. Next suppose that this motion is constraint by given integrals in the form

(2) $$\mathcal{K}=\int_D k(x^i, u^a, u^a_{,i})\, d\omega,$$

and where k is some Lagrangian density. Then a statement of isoperimetric problem for continuous system is to find the path in configuration spaces such that integral L in (1) is an extremum and such that the class of all admissible curves in configuration space assumes specified values on the boundary of D denoted by ∂D and simultaneously render the functional $\mathcal{K}$ equal to some specified value $\mathcal{K}_0$.

To solve this problem we can now use Lagrange's method multipliers for extremizing functions of several variables where these variables are subject to various constraints. Let a new functional be defined by

(3) $$\mathcal{F}=\int_D (l+\lambda k)\, d\omega,$$

or in the form

(4) $$\mathcal{F}=\int_D \mathcal{L}(x^i, u^a, u^a_{i,})\, d\omega,$$

where

(5) $$\mathcal{L}=l+\lambda k,$$

and λ is Lagrange's multiplier and it is constant.

Lie groups and its extension. Suppose that we have r-parameter's Lie's group of transformation

(6) $$\bar{x}=f^i(x^i, u^a, a),$$

(7) $$\bar{u}^a=\varphi^a(x^i, u^a, a),$$

where $a=(a^1, a^2, \ldots\ldots, a^r)$ are parameters of groups and functions f^i and φ^a are class C^∞ in each of their arguments. Then as it is known (3) infinitesimal operators of the group G have the form

(8) $$X_\alpha=\xi^i_\alpha\,\partial_i+\eta^a_\alpha\,\partial_a,$$

where

(9) $$\xi^i_\alpha=\left.\frac{\partial f^i}{\partial a^\alpha}\right|_{a=0};\quad \eta^a_\alpha=\left.\frac{\partial \varphi^a}{\partial a^\alpha}\right|_{a=0},\quad \alpha=1, 2, \ldots r.$$

Thiese operators satisfy the folowing relations

(10) $$[X_\alpha, X_\beta]=C^\gamma_{\alpha\beta}\, X_\gamma,$$

where $C^{\gamma}_{\alpha\beta}$ are structure constants (3). Operator of the extended group denoted by $\tilde{X}$ is given as

$$\tilde{X}_\alpha = X_\alpha + \zeta^a_{i\alpha}\frac{\partial}{\partial u^a_{,i}} + \zeta_\alpha \frac{\partial}{\partial (d\omega)}, \quad * \tag{11}$$

where

$$\zeta^a_{i\alpha} = D_i \eta^a_\alpha + u^a_{,j} D_i \xi^i_\alpha, \tag{12}$$

$$D_i = \partial_i + u^a_{,i}\frac{\partial}{\partial u^a}, \tag{13}$$

$$\zeta_\alpha = \frac{\partial (d\bar{w})}{\partial a^\alpha}\bigg|_{\alpha=0} = \frac{\partial J\left(\frac{\bar{x}}{x}\right)}{\partial a^\alpha}\bigg|_{\alpha=1} = D_i \xi^i_\alpha. \tag{14}$$

Noether's theorem for isoperimetric problem. Suppose that we know the functions $u^\alpha = u^a(x)$ of their arguments and if we substitute it in (6) and solve it with respect to and use (7) we obtain

$$\bar{u} = \varphi^a(x^i, a^\alpha), \tag{15}$$

If for all function u^a and for all transformations (6) and (7) the following relations

$$\int_D \mathcal{L}(x^i, u^a, u^a_{,i})\, d\omega = \int_D \mathcal{L}(\bar{x}^i, \bar{u}^a, \bar{u}^a_{,i}) J\left(\frac{\bar{x}}{x}\right) d\omega, \tag{16}$$

are valid then we say that the functional is invariant in respect to transformations (6) and (7), where D is the region of integration and $J\left(\frac{\bar{x}}{x}\right)$ is Jacobian of transformation. The relation (16) shows that the functional (4) is invariant with respect to the extended group $\tilde{G}$, namely groups extended for $d\omega$ and $u^a_{,i}$. On the one hand it is well known (4), that a functional F be invariant in respect to the group G is, that following relations are satisfied

$$\bar{X}_\alpha \mathcal{L} = \left(\xi^i_\alpha \partial_i \mathcal{L} + \eta^a_\alpha \partial_a \mathcal{L} + \zeta^a_{\alpha i}\frac{\partial \mathcal{L}}{\partial u^a_{,i}}\right) + D_i \xi^i_\alpha = 0, \tag{17}$$

or after some calculations and rearrangements these expressions we can write in the following form

$$\mu^a_\alpha \frac{\delta \mathcal{L}}{\delta u^a} + D_i (A^i_\alpha) = 0, \tag{18}$$

$* \ \frac{\partial \cdot}{\partial x^i} = \partial_i \cdot = \cdot_{,i}$

where we have used the relation (11) and where

(19) $$\mu^a_\alpha = \eta^a_\alpha - u^a_{,i}\,\xi^i_\alpha,$$

(20) $$\frac{\delta \mathcal{L}}{\delta u^a} = \frac{\partial \mathcal{L}}{\partial u^a} - D_i \frac{\partial \mathcal{L}}{\partial u^a_{,i}},$$

(21) $$A^i_\alpha = (\eta^a_\alpha - u^a_j\,\xi^j_\alpha)\frac{\partial \mathcal{L}}{\partial u^a_{,i}} + \mathcal{L}\,\xi^i_\alpha.$$

After substituting (21) in (18) the relation (18) can be written in the form

(22) $$\mu^a_\alpha \frac{\delta \mathcal{L}}{\delta u^a} + D_i\left(\mu^a_\alpha \frac{\partial \mathcal{L}}{\partial u^a_{,i}} + \mathcal{L}\,\xi^i_\alpha\right),$$

or in the form

(23) $$(\eta^a_\alpha - u^a_{,j}\,\xi^j_\alpha)\left(\frac{\delta l}{\delta u^a} + \lambda\frac{\delta k}{\delta u^a}\right) + D_i\left[(\eta^a_\alpha - u^a_{,j}\,\xi^j_\alpha)\left(\frac{\partial l}{\partial u^a_{,i}} + \right.\right.$$
$$\left.\left. + \lambda\frac{\partial k}{\partial u^a_{,i}}\right) + (l + \lambda k)\,\xi^i_a\right] = 0.$$

As it is well known (2) that Nother's theorem is concerned with consequence of invariance functional $\mathcal{F}$ at (3) under special circumstance in which u^a satisfies the Euler-Lagrange equations

(24) $$\frac{\delta l}{\delta u^a} + \lambda\frac{\delta k}{\delta u^a} = 0.$$

After it, we have that following relations are valid

(25) $$D_i(\underset{l}{A}{}^i_\alpha + \lambda\,\underset{k}{A}{}^i_\alpha) = 0,$$

where

(26) $$\underset{l}{A}{}^i_\alpha = (\eta^a_\alpha - u^a_{,j}\,\xi^j_\alpha)\,\frac{\partial l}{\partial u^a_{,i}} + l\,\xi^i_\alpha,$$

and

(27) $$\underset{k}{A}{}^i_\alpha = (\eta^a_\alpha - u^a_{,j}\,\xi^j_\alpha)\,\frac{\partial k}{\partial u^a_{,i}} + k\,\xi^i_\alpha.$$

Now we are in position to formulate the following (Noether's) theorem: If the functional $\mathcal{F}$ is invariant under extended group $\bar{G}$ of transformations then the following r identity is true

(28) $$D_i(\underset{l}{A}{}^i_\alpha + \lambda\,\underset{k}{A}{}^i_\alpha) = 0.$$

for $\alpha = 1, 2, \ldots, r$, where $\underset{l}{A}{}^{i}_{\alpha}$ and $\underset{k}{A}{}^{i}_{\alpha}$ are given explicitly in (26) and (27) and where D_i is given by (13). The formula (28) my be interpreted in a well known manner as expressing conservation laws in differential form.

Specification on the case of continuous mechanics. If we wish to use the previous theory on continuum mechanics we must accommodate it for this case. Therefore, let us have a vector field u^a and it is a displacement field where $a = 1, 2, 3$ and $x^0 = t$-time, $x^i (i, j, \ldots = 1, 2, 3)$ be rectangular cartesian coordinates.

As has been shown (5,6), that for the case of a linear elastodynamic for a Lagrangian density we can give

$$l = \frac{1}{2} l_{ijkl} u_{i,j} u_{k,l} - \frac{1}{2} \dot{u}_i \dot{u}_i . \tag{29}$$

In the same paper was shown that the group of invariants of Lagrangian density has the form

$$\bar{t} = (\alpha_1 t + \alpha_0), \ \bar{x}^i = \alpha_1 x^i + \varepsilon^i_{jk} \alpha^j_2 x^k + \alpha^i_3, \ \bar{u}^a = \alpha_1 u^a + \varepsilon^a_{ij} \alpha^i_4 u^j + \varepsilon^a_{ij} \alpha^j_2 x^i + \alpha^a_5 . \tag{30}$$

If we take this group as permissible and make the corresponding substitution, then as laws of conservation we obtain the following relations

$$\frac{\partial}{\partial t}\left(S^a_\alpha \frac{\partial l}{\partial u^a_{,t}} + l \xi^t_\alpha\right) + \frac{\partial}{\partial x^i}\left(S^a_\alpha \frac{\partial l}{\partial u^a_{,i}} + l \xi^i_\alpha\right) + \\ + \lambda\left[\frac{\partial}{\partial t}\left(S^a_\alpha \frac{\partial k}{\partial u^a_{,t}} + k \xi^t_\alpha\right) + \frac{\partial}{\partial x^i}\left(S^a_\alpha \frac{\partial k}{\partial u^a_{i,}} + l \xi^i_\alpha\right)\right] = 0, \tag{31}$$

where

$$S^a_\alpha = \eta^a_\alpha - u^a_{,t} \xi^t_\alpha - u^a_j \xi^j_\alpha . \tag{32}$$

The expression (31) represents the differential form of the conservation laws.

For application of above conservation laws it is necessary to know the constrain respectively the density of constrain k. It is obvious that k cannot be arbitrary and it is must be such a one which permits the group of invariant as l.

As a simple example let as have the constrain be given as a functional of the form

$$\mathcal{K} = \int_{t_1}^{t_2} \int_{v} k(x^i, u^a, u^a_{,i})\, dv\, dt \tag{33}$$

where $k = \delta_{ij} \delta_{kl} u_{i,j} u_{k,l}$ and $x^i = x_i$.

$\alpha_{1, 2, 3, 4, 5}$ are constants and $\varepsilon^i_{jk} = \varepsilon_{ijk}$

If we substitute the expression (33) respectively the density k in the form (31) then for the differential form of the laws of coservations we obtain the following relations

(34)
$$\partial_i\left(\frac{1}{2}l_{ijkl}u_{i,j}u_{k,l}+\frac{1}{2}\dot{u}_i\dot{u}_j\right)+\partial_f(-\dot{u}_j\sigma_{ij})+$$
$$+\lambda[\partial_t k+\partial_i(-\dot{u}_i u_{j,j})]=0,$$

(35)
$$\partial_t(\rho\dot{u}_i)+\partial_j(-\sigma_{ij})+\lambda\,\partial_i(u_{j,j})=0,$$

(36)
$$\partial_t(\rho\varepsilon_{ijk}x_j\dot{u}_k)+\partial_k(\varepsilon_{ijm}x_j\sigma_{mk})+$$
$$+\lambda\,\partial_k(\varepsilon_{ijk}x_j u_{l,l})=0,$$

(37)
$$\partial_t(\rho\dot{u}_j u_{j,i})+\partial_k(-u_{j,i}\sigma_{jk}+\delta_{in}l)+$$
$$+\lambda\,\partial_j(-u_{j,i}u_{n,k}=0,$$

(38)
$$\partial_t[\rho\dot{u}_j(u_j+x_m u_{j,m}+t\dot{u}_j)+tl]+$$
$$+\partial_k[-\sigma_{jk}(u_j+x_m u_{j,m}+t\dot{u}_j)+lx_k]+$$
$$+\lambda\{\partial_t(tk)+\partial_i[(u_i-\dot{u}_i-u_{i,j}x_j)u_{l,l}+x_i k\}=0,$$

(39)
$$\partial_t(\rho\varepsilon_{ijk}u_k\dot{u}_j+\rho\varepsilon_{ijk}x_j\dot{u}_m u_{m,k})+$$
$$+\partial_k(\varepsilon_{imj}u_m\sigma_{jk}-\varepsilon_{imj}x_j u_{l,m}\sigma_{lk}+$$
$$+\varepsilon_{imk}x_m l)+\lambda\,\partial_l[(\varepsilon_{ijl}u_j-$$
$$-\varepsilon_{ijk}u_{l,j}x_k)u_{c,c}+\varepsilon_{ijk}u_{l,j}x_k\cdot l]=0.$$

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* $\dot{u}-\partial_t u$

ТЕОРЕМА НЁТЕР И ИЗОПЕРИМЕТРИЧЕСКАЯ ЗАДАЧА В МЕХАНИКЕ СПЛОШНОЙ СРЕДЫ

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Резюме

В этой работе исследуется образование законов сохранения изопериметрического проблема в механике сплошной среды.

Ползуясь группой Ли инвариантности интеграла действвйя и интеграла каторий називается принуждение и теоремом Нётер формируяется законы сохранения.

Приложение статьи иллюстрируются одным примером.

NETEREVA TEOREMA I IZOPERIMETRIJSKI PROBLEM U MEHANICI KONTINUUMA

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Rezime

U radu je pokazano da je uslov invarijantnosti funkcionala (16) u odnosu na neku dopustivu Liovu grupu G dat izrazom (23). Polazeći od pretpostavke da je sistem konzervativan i da važi uslov (24) koji predstavlja Ojler-Langraževe jednačine problema kao uslov invarijantnosti dobijaju se izrazi opisani relacijama (25).

Kao ilustracija daje se primer teorije elastičnosti za koju je Langraževa gustina data izrazom (29) a integral prinude izrazom (33). Grupe invarijantnosti funkcionala date su konačnim transformacijama (30). Za tako definisani problem formirani su zakoni održanja (34 — 39).

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