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ON CONSERVATION LAWS IN THERMOELASTICITY

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1. Introduction

Recent applications (e.g. [1], [2], [3]) of a twodimensional conservation law often referred to as the *J*-integral to the important field of fracture mechanics, have generated interest in the theoretical foundations of conservation laws in elasticity. The *J*-integral was obtained by Rice [4] in a paper devoted to the analysis of stress concentration near the tips of cracs and notches. Although the path-independent integral used by Rice can be traced to the earlier work of Eshelby [5] on she theory of dislocations, much of current interest in these integrals and their applications was stimulated by the results reported in [4].

In a recent paper by Knowles and Sternberg [6] it was shown that *J*-integral or the conservation law of Rice, as well as its three-dimensional analogue, may be generated systematically with the aid of a theorem due to Noether [7] on invariant variatinal principles in conjuction with the principle of stationary potencial energy. Roughly speaking, Noether's theorem states that if a given set of differential equations can be identified as the Euler-Lagrange equations corresponding to a variational principle which remains invariant under an *r*-parametar group of infinitesimal transformations, then there is an associated set of *r* conservation laws satisfied by all solutions of the original differential equations. This procedure, moreover, yields two additional conservation laws. Finally, it was shown that, within the context of linear isotropic, homogeneous elastostatics, the three conservation laws are complete in the sense that they are the only ones furnished by Noether's theorem.

In [6] and in later paper by Green [8], it was shown that analogous laws exist for finite deformations of homogeneous elastic materials, but the completness of these conservation laws, within the frame work of Noether's theorem, was not proved. In [8] a direct approach was used to demonstrate that the conservation laws follow from certain symmetries which are satisfied by the strain energy function.

The results of Knowles and Sternberg [6] are extended to linear elastodynamics by Fletcher [9].

More recently, Chen and Shield [10] have investigated, out of framework of Noether's theorem, completeness of the conservation laws for finite elastic deformations among the class of laws expressible as functionals linear in the stain energy W and its first derivatives with respect to to the deformation gradients. It appears that the completness of the conservation laws, using Nother's theorem, in nonlinear elasticity is considerably more complicated than the analogous question in the linearized theory.

Aifantis [11] considered conservation laws for linear isotropic stress fields in the presence of body forces derived from harmonic potentials, and for symmetric stress-diffusion fields surrounding line cracks.

The same problem was considered by Gurtin [13], [14], Smelser and Gurtin [15] for bi-material bodies. In [14] it was remarked that similar path--independent integrals can be derived for both dynamic and quasi-static linear viscoelasticity utilizing the convolution as the basic tool.

It is our purpose to extend the investigation of conservation law to the thermoelasticity. We show how they can be generated with the aid of Noether's theorems. These theorems are, in our opinion of considerable importance in field-theoretical applications since they establish the existence and precise nature of certain conservation laws which result from the given invariance requirement. Moreover, we shall show that the completness of conservation laws, using Noether's theorem, in nonlinear thermoelasticity is not more complicated than the analogous question in the linearised theory.

2. Preliminaries. Restricted version of Noether's theorem

Let $\xi=(\xi_\alpha)(\alpha=1, 2, \ldots, n)$ be a point in rectangular Cartesian coordinates of n-dimensional Euclidean space E_n, and let R be a bounded, closed, regular region in E_n. Let $\underset{a}{\mathbf{W}}(\underset{a}{W_r}(\xi))$ $(a=1, 2, \ldots, p;\ r=1, \ldots, m)$ and $\underset{b}{\varphi}(\xi)(b=1, 2, \ldots, q)$ be arbitrary p-vector fields of m components and q scalar fields, respectively defined and twice continuously differentiable on R.

Let us suppose that we are given some one-parametar family of transformations

$$\bar{\mathbf{X}}=\bar{\mathbf{X}}(\mathbf{X}, \eta), \tag{2.1}$$

in which η denotes the parametar of the family, and where by definition

$$\mathbf{X}=(\xi, \underset{a}{\mathbf{W}}(\xi), \underset{b}{\varphi}(\xi)). \tag{2.2}$$

For $\eta=0$ the transformations are required to reduce to the identity

$$\bar{\mathbf{X}}_{\eta=0}\equiv\bar{\mathbf{X}}_0=\mathbf{X} \tag{2.3}$$

Hence the infinitesimal transformations corresponding to (2.1) are given by

$$\bar{\mathbf{X}}=\mathbf{X}+\Phi\eta+\mathbf{0}(\eta^2), \tag{2.4}$$

so that

$$(2.5)\qquad \begin{gathered}\Phi=\Phi(\alpha,\ \underset{a}{\beta},\ \underset{b}{\gamma})\equiv\left(\frac{d\bar{\mathbf{X}}}{d\eta}\right)_{\eta=0}\equiv\left(\frac{d\bar{\mathbf{X}}}{d\eta}\right)_0\\ \alpha\equiv\left(\frac{d\bar{\xi}}{d\eta}\right)_0,\quad \underset{a}{\beta}\equiv\left(\frac{d\underset{a}{\mathbf{W}}}{d\eta}\right)_0,\quad \underset{b}{\gamma}\equiv\left(\frac{d\underset{b}{\varphi}}{d\eta}\right)_0.\end{gathered}$$

Further, suppose

$$(2.6)\qquad L=L(Z).$$

is a real scalar function defined and differentiable for all values of its arguments

$$(2.7)\qquad \begin{gathered}\mathbf{Z}\equiv(\xi,\ \underset{a}{\mathbf{W}},\ \nabla\underset{a}{\mathbf{W}},\ \underset{b}{\varphi},\ \nabla\underset{b}{\varphi}),\\ \nabla\underset{a}{\mathbf{W}}=\operatorname{grad}\underset{a}{\mathbf{W}}=(\underset{a}{W}_{i,\alpha}),\quad \nabla\underset{b}{\varphi}=\operatorname{grad}\underset{b}{\varphi}=(\underset{b}{\varphi},\alpha)\end{gathered}$$

Then we write

$$(2.8)\qquad \begin{gathered}L,\alpha=\frac{\partial}{\partial\xi^\alpha}L(\mathbf{Z}),\quad L,\underset{a}{w}_i=\frac{\partial}{\partial\underset{a}{W}_i}L(\mathbf{Z}),\quad L,w_{i,\alpha}=\frac{\partial}{\partial\underset{a}{W}_{i,\alpha}}(\mathbf{Z}),\\ L,\underset{b}{\varphi}=\frac{\partial}{\partial\underset{b}{\varphi}}L(\mathbf{Z}),\quad L,\underset{b}{\varphi},_\alpha=\frac{\partial}{\partial\underset{b}{\varphi},_\alpha}L(\mathbf{Z}).\end{gathered}$$

provided the foregoing differentiations are meaningful. By L,α we denote the partial derivative of L with respect to ξ_α to distinguish it from the total derivative $L\ \alpha$.

Now we define a functional $\mathscr{L}$ on class of a given fields X by the formula

$$(2.9)\qquad \mathscr{L}(\underset{a}{\mathbf{W}},\ \underset{b}{\varphi})=\int_R L(\mathbf{Z})\,d\xi,$$

where L is given by (2.6). The functional $\mathscr{L}$ in (2.9) is said to be invariant at

$$(2.10)\qquad \mathbf{Y}=(\underset{a}{\mathbf{W}},\ \underset{b}{\varphi})$$

under transformation (2.1) if

$$(2.11)\qquad \int_{\bar{R}} L(\bar{\mathbf{Z}})\,d\bar{\xi}=\int_R L(\mathbf{Z})\,d\xi,$$

for all sufficiently small values of (η). If, for a given Y,

$$(2.12)\qquad \left\{\frac{d}{d\eta}\int_R L(\bar{\mathbf{Z}})\,d\bar{\xi}\right\}_0=0,$$

then $\mathscr{L}$ is said to be infinitiesimally invariant at Y. Evidently, if $\mathscr{L}$ is invariant at Y then $\mathscr{L}$ is infinitesimally invariant at Y.

Now we can state a restricted version of *Noether's theorem* 1:

Let R be a domain in E_n, and suppose Y satisfy the Euler-Lagrange equations

$$L, \underset{a}{w_i}(\mathbf{Z}) - \frac{\partial}{\partial \xi_\alpha}[L, \underset{a}{w_{i,\alpha}}(\mathbf{Z})] = 0. \tag{2.13}$$

Then $\mathscr{L}$ in (2.9) is infinitesimally invariant at Y under transformatons (2.1) for every bounded, regular subregion R of $\hat{R}$ iff satisfies

$$\begin{aligned}&\frac{\partial}{\partial \xi_\alpha}(L\,\alpha_\alpha(X) + L, \underset{a}{w_{i,\alpha}}\, \underset{a}{r_i}(\mathbf{X}) + L, \underset{b}{\varphi_{,\alpha}}\, \underset{b}{q}(\mathbf{X})) + \\ &+ \underset{b}{q}\left(L, \underset{b}{\varphi} - \frac{\partial}{\partial \xi_\alpha} L, \underset{b}{\varphi_{,\alpha}}\right) = 0.\end{aligned} \tag{2.14}$$

where

$$\underset{a}{r_i} = \underset{a}{\beta_i} - \underset{a}{W_i},\ {}_\delta\alpha_\delta \qquad \underset{b}{q} = \underset{b}{\gamma} - \underset{b}{\varphi},\ {}_\delta\alpha_\delta. \tag{2.15}$$

In a case when Y also satisfies the equation

$$L, \underset{b}{\varphi}(\mathbf{Z}) - \frac{\partial}{\partial \xi_\alpha} L, \underset{b}{\varphi_{,\alpha}}(\mathbf{Z}) = 0, \tag{2.16}$$

then $\mathscr{L}$ in (2.9) is infinitesimally invariant at Y under transformations (2.1) iff Y satisfies

$$\frac{\partial}{\partial \xi_\alpha}(L\,\alpha_\alpha + L, w_{i,\alpha}\underset{a}{r} + , \underset{b}{\varphi_\alpha}\underset{b}{q}) = 0.^* \tag{2.17}$$

P r o o f of theorem 1: Let $\mathscr{L}$ in (2.9) be infinitesimally invariant at Y under transformations (2.1). Then (2.11) holds. Using the Jacobian determinant

$$J(\bar{\xi}, \xi) = \det\left(\frac{\partial \bar{\xi}_\alpha}{\partial \xi_\beta}\right) \tag{2.18}$$

we can write (2.12) as

$$\left\{\frac{d}{d\eta}\int_R L(\bar{\mathbf{Z}})\,Jd\xi\right\}_0 = 0. \tag{2.19}$$

Since the region of integration R is independent of the parameter η and arbitrary we obtain the following necessary condition

$$\frac{d}{d\eta}(L(\bar{\mathbf{Z}}))_0 J_0 + L(\bar{\mathbf{Z}})_0\left(\frac{dJ}{d\eta}\right)_0 = 0. \tag{2.20}$$

Here throughout this paper summation over repeated subscript is implied.

It is evident, from (2.4) and (2.5), that

$$J_0 = 1, \qquad \left(\frac{dJ}{d\eta}\right)_0 = \alpha_{\delta,\delta}, \qquad L(\overline{\mathbf{Z}})_0 = L(\mathbf{Z}) \tag{2.21}$$

When $\dfrac{dL(\overline{Z})}{d\eta}$ is differentiated with respect to η, one obtains

$$\begin{aligned}\frac{dL(\overline{\mathbf{Z}})}{d\eta} &= L,_{\bar{\alpha}} \frac{d\bar{\xi}_\alpha}{d\eta} + L,_{\underset{a}{\bar{w}_i}} \frac{d\underset{a}{\overline{W}_i}}{d\eta} + L,_{\underset{a}{\bar{w}_{i,\alpha}}} \frac{d\underset{a}{\overline{W}_{i,\alpha}}}{d\eta} + \\ &L,_{\underset{b}{\bar{\varphi}}} \frac{d\underset{b}{\bar{\varphi}}}{d\eta} + L,_{\underset{b}{\bar{\varphi}},\alpha} \frac{d\underset{b}{\bar{\varphi}},_\alpha}{d\eta}\end{aligned} \tag{2.22}$$

Differentiation of (2.4) and (2.5) yield the following relations

$$\begin{aligned}&\left(\frac{d\bar{\xi}_\alpha}{d\eta}\right)_0 = \alpha_\alpha, \qquad \left(\frac{d\underset{a}{\overline{W}_i}}{d\eta}\right)_0 = \underset{a}{\beta_i}, \qquad \left(\frac{d\underset{a}{\overline{W}_{i,\alpha}}}{d\eta}\right) = \underset{a}{\beta_{i,\alpha}} - \underset{a}{W_{i.\beta}}\,\alpha_{\beta,\alpha} \\ &\left(\frac{d\underset{b}{\bar{\varphi}}}{d\eta}\right)_0 = \underset{b}{\gamma}, \qquad \left(\frac{d\underset{b}{\varphi},_\alpha}{d\eta}\right) = \underset{b}{\gamma},_\alpha - \underset{b}{\varphi},_\beta\,\alpha_{\beta,\alpha}\end{aligned} \tag{2.23}$$

We also need the relation

$$\begin{aligned}(L\,\alpha_\alpha),_\alpha &= L,_\alpha \alpha_\alpha + L,_{\underset{a}{w_i}} \underset{a}{W_{i.\alpha}}\,\alpha_\alpha + L,_{\underset{a}{w_{i,\beta}}} \underset{a}{W_{i,\beta\alpha}}\,\alpha_\alpha + \\ &+ L,_{\underset{b}{\varphi}} \underset{b}{\varphi},_\alpha \alpha_\alpha + L,_{\underset{b}{\varphi}\beta} \underset{b}{\varphi},_{\beta\alpha}\,\alpha_\alpha + L\,\alpha_{\alpha,\alpha}.\end{aligned} \tag{2.24}$$

Using (2.21-24), after some manipulation, we obtain

$$\begin{aligned}&\frac{\partial}{\partial \xi_\alpha}\left(L\,\alpha_\alpha + L,_{\underset{a}{w_{i,\alpha}}} \underset{a}{r_i} + L,,_{\underset{b}{\varphi},\alpha} \underset{b}{q}\right) + \underset{b}{q}\left(L,_{\underset{b}{\varphi}} - \frac{\partial}{\partial \xi_\alpha} L,_{\underset{b}{\varphi},\alpha}\right) + \\ &+ \underset{a}{r_i} \frac{\partial}{\partial \xi_\alpha}\left(L,_{\underset{a}{w_i}} - \frac{\partial}{\partial \xi_\alpha} L,_{\underset{a}{w_{i,\alpha}}}\right) = 0,\end{aligned} \tag{2.25}$$

where $\underset{a}{r_i}$ and $\underset{b}{q}$ are given by (2.15). Then (2.25) and (2.13) yield (2.14). If Y also satisfies (2.16), we have (2.17).

The sufficiency follows immediately from (2.13), (2.14) and (2.25) or from (2.13), (2.16), (2.17) and (2.25). Q.E.D.

We note that the statement of the theorem given above is more general then that used in [6] and [9].

In general, we shall speak of (2.14) and (2.16) as the conservation laws. The usefulness of the Theorem 1 as a device for generating conservation laws in any particular branch of mathematical physics depends on the existence of regular mapping (2.1) with respect to which the stationary functional (2.9) is infinitesimally invariant.

In this form Noether's theorem may be applied to the theory of elastic dielectrics, thermoelastic diffusion and micropolar theory. We shall confine our attention to the theory of thermoelastic materials when a, $b=1$ and $\underset{1}{w_i}=x_i$ and $\underset{1}{\varphi}=\theta$.

3. Thermoelastic materials

We now recall certain results from the theory of finitely deformed homogeneous and isotropic thermoelastic solids. In this connection we assume the absence of body forces and presuppose the existence of an elastic potential.

We use coordinates referred to a fixed rectangular Cartesian coordinate system to describe particle locations. During a deformation of the unstrained body $\mathscr{B}$, a particle at the point X_K in $\mathscr{B}$ is displaced to the point x_k in the deformed body $\mathscr{B}^*$, with

$$x_k = x_k(X_K, t), \qquad (k, K = 1, 2, 3). \tag{3.1}$$

Further, we require that this mapping is one-to-one so that the Jacobian

$$J = \det(x_{k,K}) \neq 0 \tag{3.2}$$

at all points of $\mathscr{B}$. The deformation gradients $x_{k,K}$ satisfy the nine linear equations

$$x_{k,K} X_{K,l} = \delta_{kl}, \qquad X_{K,k} x_{k,L} = \delta_{KL}. \tag{3.3}$$

The local balance laws in the reference frame X_K are listed below for simple thermodynamic processes [15]:

the balance of linear momentum

$$T_{Kk,K} = \rho_0 \ddot{x}_k, \tag{3.4}$$

the balance of moment of momentum

$$T_{Kk} x_{l,K} = T_{Kl} x_{k,K}, \tag{3.5}$$

the balance of energy

$$\rho\dot{\varepsilon} = T_{Kk} \dot{x}_{k,K} + Q_{K,K} + \rho_0 h, \tag{3.6}$$

where the above given quantities are

T_{Kk} — the first Piola-Kirchhof stress tensor,
Q_K — the heat vector,
ρ_0 — the initial mass dens.ty,
$\varepsilon = \varepsilon(x_{k,K}; \eta)$ — the internal energy per unit mass,
$(\dot{\ })$ — the material derivative,
h — the heat supply per unit mass,
η — the entropy density

Finally we write the constitutive equations

(3.7) $$T_{Kk} = \rho_0 \frac{\partial \psi}{\partial x_{k,K}} = \rho_0 \frac{\partial \varepsilon}{\partial x_{k,K}}; \quad \theta = \frac{\partial \varepsilon}{\partial \eta}; \quad \eta = -\frac{\partial \psi}{\partial \theta},$$

where θ is the absolute temperature and

(3.8) $$\psi = \psi(x_{k,K}; \theta) = \varepsilon - \theta\eta,$$

the free energy.

From (3.7) and (3.6) we obtain

(3.9) $$\rho_0 \theta \dot{\eta} = Q_{K,K} + \rho_0 h.$$

In order to write the differential form of the balance law (3.4) in more compact form it is convienient to inroduce some additional notation. We set

(3.10) $$\xi_\alpha = \begin{cases} X_K & \alpha = K \\ t & \alpha = 4, \end{cases} \quad \frac{d}{dt}(\ \ ) = (\dot{\overline{\ \ }}) = (\ \ )_{,4},$$

and

(3.11) $$H = \rho_0 \left(\psi - \frac{1}{2} \dot{x}_k \dot{x}_k \right) = H(x_{k,\alpha}; \theta).$$

It may be verified that

(3.12) $$H,_{x_{k,K}} = \rho_0 \psi,_{x_{k,K}} = T_{Kk}, \qquad H,_{\dot{x}_k} = -\rho_0 \dot{x}_k, \quad H,_{x_k} = 0,$$

and the balance law (3.4) is equivalent to the Euler-Lagrange equations

(3.13) $$\frac{\partial}{\partial \xi_\alpha} H,_{x_{k,\alpha}} = 0.$$

Then the Noether's theorem 1 can be applied to our case. To confirm this claim we choose

$$L \equiv H, \qquad \underset{a}{w_k} = x_k, \qquad \underset{b}{\varphi} = \theta \qquad (a,\ b = 1)$$

(3.14)

$$\mathbf{X} = (\xi;\ x;\ \theta), \qquad \mathbf{Y} = (x;\ \theta), \qquad \mathbf{Z} = (x_{k,\,\alpha};\ \theta).$$

The corresponding conservation law (2.14) now reads

$$\frac{\partial}{\partial \xi_\alpha}(H\,\alpha_\alpha + H,\ x_{k,\,\alpha} r_k) + H,\ \theta\,(\gamma - \theta,\ _\beta \alpha_\beta) = 0, \tag{3.15}$$

where L is given by (3.14_1). We note that (2.15) and (2.23) hold and we will use them hereafter but without indeces a and b, i.e.

$$\text{(a)} \qquad r_i = \beta_i - x_{k,\,\beta}\alpha_\beta; \qquad q = \gamma - \theta,\ _\beta\alpha_\beta.$$

Before proceeding further we give the integral form of this conservation law using the divergence theorem, i.e.

$$\frac{d}{dt}\int_V (H\,\alpha_4 + H,\ \dot{x}_k r_k) + \int_S (H\,\alpha_K + H,\ x_{k,\,K} r_k)\,N_K\,dS +$$

(3.16)

$$\int_V H,_\theta(\gamma - \theta,\ _\beta\alpha_\beta)\,dV = 0,$$

where V is the region in the space occupied by the body in the reference state, S the boundary of V and N_K is the unit outward normal on S.

Our next task is to show that they do in fact exist mapping family (2.1) with respect to which $\mathscr{L}$ is infinitesimally invariant at $\mathbf{X} = (\xi,\ x,\ \theta)$.

4. Completness for arbitrary objective and isotropic

If we impose the condition that the form of the free energy must satisfy material objectivity but otherwise remains arbitrary, we must have

$$e_{klm}\frac{\partial \psi}{\partial x_{l,\,K}}x_{m,\,K} = 0, \tag{4.1}$$

where e_{klm} is the usual permutation symbol. Moreover, if the material is initially isotropic, we have

$$e_{KLM}\frac{\partial \psi}{\partial x_{k,\,L}}x_{k,\,M} = 0. \tag{4.2}$$

Then we can formulate the following *theorem* 2:

Suppose the thermoelastic material under consideration is isotropic and satisfies material objectivity. Let $\mathscr{L}$ be the Lagrangian

$$\mathscr{L}(\mathbf{Y})=\int_R H(\mathbf{Z})\, d\xi, \tag{4.3}$$

where R it a bounded regular region in E_4, H, Z, Y are given by (3.14) and ψ satisfies (4.1) and (4.2). Then $\mathscr{L}(\mathbf{Y})$ is infinitesimally invariant at Y under transformation (2.1) for every $x_k=x_k(\xi_\alpha)$ satisfying the equations of motion (3.13) iff

$$\begin{aligned} \alpha_K &= e_{KLM} X_L \Lambda_M + C_K \\ \alpha_4 &= C_4 \\ \beta_k &= e_{klm} x_l \Omega_m + a_k \\ \gamma &= 0 \end{aligned} \tag{4.4}$$

where a_k, C_k, C_4 Ω_k and Λ_k are arbitrary real constants.

P r o o f of theorem 2: To prove sufficiency we may directly verify the infinitesimal invariance of (3.4) under (2.4) and (4.4).

We now proceed to establish the nessecity of the Theorem 2. To this end we note that if there exist some one-parametar family of transformations (2.1) under which $\mathscr{L}$ in (3.4) is infinitesimally invariant then (3.15) holds for any $H(Z)$ given by (3.11) and any ψ in (3.8) satisfying (4.1) and (4.2). Using (3.13) in (3.15) we obtain

$$H,_{\alpha}\alpha_\alpha + H,_{x_{k,\alpha}} r_{k,\alpha} + H,_{\theta}(\gamma-\theta,_{\beta}\alpha_\beta) + H\alpha_{\alpha,\alpha}=0. \tag{4.5}$$

If we substitute (a), (3.10) and (3.11) into (4.5) we find that

$$\begin{aligned} &\psi,_{x_{k,K}}(\beta_{k,K}-x_{k,\beta}\alpha_{\beta,K})-\dot{x}_k(\beta_{k,4}-x_{k,\beta}\alpha_{\beta,4})+\psi,_{\theta}\gamma \\ &+\psi\alpha_{\beta,\beta}-\frac{1}{2}\dot{x}_k\dot{x}_k\alpha_{\beta,\beta}=0. \end{aligned} \tag{4.6}$$

Now the values of $\psi,_{x_{k,K}}$ are restricted by equations (4.1) and (4.2). Utilizing Lagrange's method of multipliers, we set

$$\begin{aligned} &\psi,_{x_{k,K}}(\beta_{k,K}-x_{k,\beta}\alpha_{\beta,k}-\tau_i e_{jkl}x_{l,k}-\Lambda_j e_{JKL}x_{K,L})+ \\ &+\psi,_{\theta\gamma}+\psi_{\alpha\beta,\beta}-\dot{x}_k(\beta_{k,4}-x_{k,\beta}\alpha_{\beta,4})-\frac{1}{2}\dot{x}_k\dot{x}_k\alpha_{\beta,\beta}=0. \end{aligned} \tag{4.7}$$

where Ω_k and Λ_K are the unknown Lagrange multipliers. Since the function $\psi(x_k;\alpha)$ in (4.7) is now arbitrary, the values of ψ and $\psi,_{x_{k,\alpha}}$ at a point may be chosen independently of the values of higher derivatives of ψ. Thus, (4.7) is satisfied for arbitrary ψ, $\psi,_0$ and ψ, $x_{k,K}$ if

$$\beta_{k,K} - x_{k,\beta}\alpha_{\beta,K} - \Omega_j e_{jkl} x_{l,K} - \Lambda_J e_{JKL} x_{k,L} = 0,$$

(4.8)
$$\gamma = 0,$$

$$\alpha_{\beta,\beta} = 0,$$

$$\dot{x}_k(\beta_{k,4} - x_{k},{}_\beta\alpha_{\beta,4}) = 0.$$

We write explicitly

(4.9)
$$\alpha_{\beta,\gamma} = \alpha_{\beta,\gamma} + \alpha_{\beta,k} x_{k,\gamma} + \alpha_{\beta,\theta}\theta,_\gamma,$$

$$\beta_{\beta,\gamma} = \beta_{\beta,\gamma} + \beta_{\beta,k} x_{k,\gamma} + \beta_{\beta,\theta}\theta,_\gamma.$$

At any point of R the values of (x_k, $x_{k,\alpha}$, θ) may be chosen arbitrary and independently of the values of higher derivatives of ξ_α. Then from (4.9_1) and (4.8_3), we conclude that

(4.10)
$$\alpha_{\beta,\beta} = 0; \qquad \alpha_{\beta,k} = 0 \qquad \alpha_{\beta,\theta} = 0,$$

and ($4.8_{1,4}$) may be written as

(4.11)
$$\beta_{k/K} + \beta_{k,l} x_{l,K} + \beta_{k,\theta}\theta,_K - x_{k,L}\alpha_{L,K} - \dot{x}_k\alpha_{4/K} =$$

$$\Omega_j e_{jkl} x_{l,k} - \Lambda_J e_{JKL} x_{k,L} = 0,$$

$$\dot{x}_k(\beta_{k/4} + \beta_{k,l}\dot{x}_l + \beta_{k,\theta}\dot{\theta} = x_{k,K}\alpha_{K,4} - \dot{x}_k\alpha_{4,4}) = 0$$

since α_β and β_k are independent of $x_{k,\alpha}$ and $\theta,_\alpha$ it follows that the linear and quadratic parts in $x_{k,\alpha}$ $\theta,_\alpha$ of (4.11) must vanish separately. This leads to

(4.12)
$$\beta_{k/\alpha} = 0; \qquad \beta_{k,\theta} = 0;$$

$$\alpha_{K/4} = 0; \qquad \alpha_{4/K} = 0;$$

and

(4.13)
$$(\beta_{k,l} - \Omega_j e_{jkl}) x_{l,K} - (\alpha_{L,K} - \Lambda_J e_{JLK}) x_{k,L} = 0,$$

$$\dot{x}_i\dot{x}_k(\beta_{k,l} - \delta_{kl}\alpha_{4,4}) = 0.$$

From (4.10) and (4.12) we deduced that α_k is a function of X_K and β_k is a function of x_k.

Using (3.3) ın (4.13) we obtain

(4.14)
$$\beta_{k,l} - \Omega_j e_{jkl} - (\alpha_{L,K} - \Lambda_J e_{JKL}) x_{k,L} X_{K,l} = 0,$$

$$\alpha_{L,K} - \Lambda_J e_{JLK} - (\beta_{k,l} - \Omega_j e_{jkl}) x_{l,k} X_{L,k} = 0,$$

form which follows that

$$\beta_{(k,l)} - (\alpha_{L,K} - \Lambda_J e_{JLK}) X_{K,}({}_l x_k)_{,L} = 0, \tag{4.15}$$
$$\alpha_{(L,K)} - (\beta_{k,l} - \Omega_j e_{jkl}) x_{l,}({}_K X_L)_{,l} = 0.$$

Thus

$$\beta_{(k,l)} = 0 \qquad \alpha_{(L,K)} = 0, \tag{4.16}$$

or

$$\beta_k = e_{klm} x_l d_m + a_k \tag{4.17}$$
$$\alpha_K = e_{KLM} X_L B_M + C_K$$

where a_k, B_k, C_K and d_k are arbitrary real constants.

From (4.16) and (4.13$_2$) we obtain

$$\alpha_{4/4} = 0, \tag{4.18}$$

and from this and (4.10) and (4.12) we conclude that

$$\alpha_4 = B_4, \tag{4.19}$$

where B_4 is arbitrary real constant. Q.E.D.

Then the infinitesimal part of the transformations (2.1) are

$$\bar{x}_k = x_k + (e_{kim} x_l d_m + a_k)\eta \tag{4.20}$$
$$\bar{X}_k = X_k + (e_{KLM} X_L D_M + C_K)\eta$$
$$\bar{t} = t \quad + B_4 \eta$$

However, this is all that is required for the corresponding conservation laws.

Lie [12] introduced the very convenient symbol *Uf*, being called the generator of group, for the coefficients of η (5.1), i.e.

$$Uf = \beta_k \frac{\partial f}{\partial x_k}. \tag{4.21}$$

Since *Uf* can be written when the infinitesimal transformation of a group is known, and conversely, (5.2) is known when *Uf* is given, *Uf* is said to represent β_k. Then the finite transformation is known. In particular the finite transformation of the groups, whose infinitesimal transformations are given by (5.1), are

$$\bar{x}_k = q_{kl}(\eta)\, x_l + p_k(\eta) \tag{4.22}$$
$$\bar{X}_K = Q_{KL}(\eta)\, X_L + P_K(\eta)$$
$$\bar{t} = t + P(\eta)$$

(b) $\quad q_{kl}(0) = \delta_{kl}; \quad Q_{KL}(0) = 0; \quad q_{km} q_{lm} = \delta_{kl}; \quad Q_{KM} Q_{LM} = \delta_{KM}$

which represent the full group of orthogonal transformations and a time shift.

5. The conservation laws

We now proceed to write the integral form of the conservation law (3.16) wkich correspond to the particular transformations (4.4). By taking all of the arbitrary constants in (4.4) zero exept one in turn, we obtain the corresponding conservation laws. There are the following five transformations under which the functional is infinitesimally invariant:

I. $$a_k = 0\ (r_k = a_k;\quad q = 0)$$

Then we introduce the family of transformations

(5.1) $$\bar{x}_k = x_k + a_k \eta \qquad \bar{t} = t, \qquad \bar{X}_k = X_k,$$

which represents rigid body translation. The corresponding conservation law (3.16) now reads

(5.2) $$\frac{d}{dt}\int_V \rho_0 \dot{x}_k \, dV - \int_S T_{Kk} N_K \, dS = 0,$$

where we have used (3.12).

II. $$d_m \neq 0\ (a_k = C_k;\quad B_\alpha = 0 \quad \gamma = 0)$$

$$r_k = e_{klm} x_l d_m, \quad q = 0$$

So induced family of transformations

(5.3) $$\bar{x}_k = Q_{kl}(\eta)\, x_l \qquad \bar{t} = t \qquad \bar{X}_K = X_K$$

represents rigid body rotation, and the corresponding conservation law reads

(5.4) $$\frac{d}{dt}\int_V \varepsilon_{jkl} x_k (\rho_0 \dot{x}_l)\, dV = \int_S e_{jki} x_k T_{Kl} N_K \, dS = 0$$

III. $$B_4 \neq 0\ (a_k = d_k = C_k = B_k = 0;\quad \gamma = 0)$$

$$(r_k = -\dot{x}_k B_4;\quad q = -\dot{\theta} B_4)$$

The family of transformations

$$\bar{x}_k = x_k$$
$$(5.5) \qquad \bar{X}_K = X_K$$
$$\bar{t} = t + B_4 \eta$$

represents a shift of time, and the corresponding conservation law may be written as

$$(5.6) \qquad \frac{d}{dt}\int_V (H - H,_{\dot{x}_k} \dot{x}_k)\, dV - \int_S H,_{x_{k,K}} \dot{x}_k N_K\, dS = \int_V H,_\theta \dot{\theta}\, dV = 0.$$

Upon using (3.7-12) this reads

$$(5.7) \qquad \frac{d}{dt}\int_V \rho_0 \left(\varepsilon + \frac{1}{2}\dot{x}_k \dot{x}_k\right) dV - \int_S T_{Kk} \dot{x}_k N_K\, dS - \int_S Q_K N_K\, dS - \int_V \rho_0 h dV = 0,$$

The three conservation laws, (5.2), (5.4) and (5.7), correspond respectively to the conservations of linear momentum, angular momentum and energy. They are the consequence of our requirement that the form of the free energy function $\psi(x_{k,K}; \theta)$ must satisfy material objectivity but otherwise remains arbitrary and are easily verified in the usual way from the basic equations (3.4-7).

Thus we have established the basic theorem of the equivalence between conservation and invariance [16]:

Linear momentum, angular momentum and energy are conserved in a thermoelastic medium iff the action density $\mathscr{L}$ is invariant under the group of Euclidean displacements.

IV. Now we consider the case when

$$\alpha_K = C_K \; (\beta_k = 0, \;\; \alpha_4 = 0, \;\; \gamma = 0$$
$$r_k = -x_{k,K} C_K, \;\; q = -\theta,_K C_K$$

The corresponding family of transformations and the conservation law are respectively

$$\bar{x}_k = x_k$$
$$(5.8) \qquad \bar{t} = t$$
$$\bar{X}_K = X_K + C_K \eta$$

$$(5.9) \qquad \frac{d}{dt}\int_V \rho_0 \dot{x}_k x_{k,K}\, dV + \int_S \left[\rho_0 \left(\psi - \frac{1}{2}\dot{x}_k \dot{x}_k\right)\delta_{KL} - T_{Lk} x_{k,K}\right] N_L\, dS -$$
$$- \int_V \rho_0 \psi,_\theta \theta,_K dV = 0.$$

V. The last case follows from $B_K = 0$. The corresponding family of transformation is

$$\bar{x}_k = x_k$$

(5.10) $$\bar{t} = t$$

$$\bar{X}_K = Q_{KL} X_L.$$

Then the conservation law reads

(5.11) $$e_{KLM}\left\{\frac{d}{dt}\int_V \rho_0 \dot{x}_k x_{k,K} X_L dV - \int_{\dot{S}} \left[\rho_0\left(\psi - \frac{1}{2}\dot{x}_k \dot{x}_k\right) - T_{Kk}\right] X_L N_K dS - \right.$$

$$\left. - \int_V \rho_0 \psi,_{\theta}\theta,_K X_L dV\right\} = 0.$$

Now we can conclude:

The conservation laws (5.9) and (5.11) hold when the material is initially isotropic, but does not necessarily satisfy material objectivity.

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ОБ ЗАКОНАХ СОХРАНЕНИЯ В ТЕРМОУПРУГОСТИ

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Резюме

В работе расматривается приложения Теоремы Нетеровой в нелинейной термоупругости Этим способом полученная система законов сохранения единственна в смысле что ими соответствующие группы преобразования координати и температуры однозначно определённы. В специальном случае получаюшая, в отсуствие температуры, законы сохранения не линейний теории упругости.

ZAKONI ODRŽANJA U TERMOELASTIČNOSTI

Jova Jarić

Rezime

U nedavno objavljenom radu Knowles i Sternberg [6] pokazuju da J-integral ili zakon konzervacije Rice, kao i njegov trodimenzioni analogon, sledi iz Neterove teoreme [7].

U istom radu je pokazano da postoje još dva zakona konzervacija. Pokazuje se da su ovako dobijeni zakoni konzervacije jedini kada je u pitanju linearna, izotropna homogena elastostatika.

Kasnije je pokazano da analogni zakoni postoje i za konačne deformacije homogenih elastičnih materijala ali njihova jedinstvenost, u okviru Neterove teoreme, nije dokazana. U radu [8] je pokazano da zakoni konzervacije slede iz simetričnih svojstava koje zadovoljava funkcija energije deformacije.

Namera ovog rada je da proširi ispitivanje zakona konzervacija na domen termoelastičnosti i pokaže kako oni slede primenom Neterovih teorema. Ove teoreme su, po našem mišljenju od velikog značaja jer utvrđuju egzistenciju i prirodu nekih zakona konzervacija koji slede na osnovu zahteva invarijantnosti. Dalje, dokazujemo da kompletnost zakona konzervacija, koristeći Neterovu teoremu, nelinearnoj termoelastičnosti nije ništa složenija od analognog problema u linearnoj teoriji.

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