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# EIGEN VALUES AND PRINCIPAL DIRECTIONS OF THE ENERGY MOMENTUM TENSOR OF A PERFECT CHARGED FLUID

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In this paper we investigate the eigen values and principal directions of the energy momentum tensor of a perfect charged fluid. The model, we are concerned with, is a scheme called perfect charged fluid with finite conductivity. This energy momentum tensor is composed by the energy momentum tensor of a perfect fluid and the one of the electromagnetic field; more precisely it is a sum of these two parts.

The energy momentum tensor of a perfect fluid is in its well known form

$$\mathcal{T}_{\alpha\beta}=(\rho+p)\,u_\alpha u_\beta-p g_{\alpha\beta}$$

where $\rho$ is the proper mass density, $p$ is the pressure, $u_\alpha$ is the unitary four--velocity and $g_{\alpha\beta}$ is the metric tensor of space-time of Minkowski, i. e. $g_{\alpha\beta}=\mathrm{diag}(-1,\ -1,\ -1,\ 1)$; the velocity of light $c=1$.

Because of induction, the electromagnetic field is defined by two tensor fields namely by the electric field-magnetic induction tensor, $H_{\alpha\beta}$, and the electric induction-magnetic field tensor, $G_{\alpha\beta}$. The electomagnetic part of the energy momentum tensor is of the form

$$\tau_{\alpha\beta}=\frac{1}{4}\,g_{\alpha\beta}\,H_{\xi\eta}\,G^{\xi\eta}-H_{\alpha\xi}\,G_\beta{}^\xi.$$

The two tensor treatment of the electomagnetic field was introduced by Minkowski, and further investigated by Gordon, Weyl, Lichnerowicz and others. Because of the presence of induction the expression for $\tau_{\alpha\beta}$ is not unique and other forms were also proposed.

Tensor $\tau_{\alpha\beta}$ is not symmetric. Abraham tried to symmetrise it and proposed one symmetric form. However, Tamm believed that the expression for $\tau_{\alpha\beta}$ is correct. He remarked that electromagnetic field with induction forms a system which is not closed and that there was no reason to prerequisite the symmetry of the energy momentum tensor. Moreover, though the microscopic tensor is symmetric, the macroscopic one has not to be such; the last one is not the average value of the former. Tamm showed that in some special cases more adequate results could be obtained using the expression proposed by Minkowski. Hence, the Minkowskian form of $\tau_{\alpha\beta}$ did not lose its interest.

Total energy momentum tensor of the perfect charged fluid has the form

$$T_{\alpha\beta} = \mathcal{T}_{\alpha\beta} + \tau_{\alpha\beta}$$

It was the idea in this paper to express first the energy momentum tensor of the electromagnetic field in terms of the vectors of electric field, $e_\alpha$, electric induction, $d_\alpha$, magnetic field, $h_\alpha$ and magnetic induction, $b_\alpha$. Having energy momentum tensor in such a form we could use a suitable coordinate system which makes more clear the physical situation, and makes the calculation more simple. Then we form a matrix of this tensor and calculate the eigen values and principal directions.

To express $\tau_{\alpha\beta}$ in terms of $e_\alpha$, $d_\alpha$, $h_\alpha$ and $b_\alpha$, we use following formulas connecting $H_{\alpha\beta}$ and $G_{\alpha\beta}$ with these vectors

$$H_{\alpha\beta} = u_\alpha e_\beta - u_\beta e_\alpha - \varepsilon_{\alpha\beta\gamma\delta} u^\gamma b^\delta$$

$$G_{\alpha\beta} = u_\alpha d_\beta - u_\beta d_\alpha - \varepsilon_{\alpha\beta\varphi\psi} u^\varphi h^\psi$$

Tensor $\tau_{\alpha\beta}$ has now the form

$$\tau_{\alpha\beta} = -\frac{1}{2}(\lambda e^2 - \mu h^2) g_{\alpha\beta} - e_\rho d^\rho u_\alpha u_\beta - e_\alpha d_\beta + u_\beta \varepsilon_{\alpha\xi\gamma\delta} d^\xi u^\gamma b^\delta +$$

$$+ u_\alpha \varepsilon_{\beta\xi\varphi\gamma} e^\xi u^\varphi h^\gamma - \varepsilon_{\alpha\xi\gamma\delta} g^{\rho\xi} \varepsilon_{\beta\rho\varphi\psi} u^\gamma b^\delta u^\varphi h^\psi$$

We shall express the components of tensors $\tau_{\alpha\beta}$ and $T_{\alpha\beta}$, relative to a special coordinate system. We take a small part of a fluid at some definite event and fix a coordinate system with it. We choose the space axes of that coordinate system so that one axis, say $0x^3$, is perpendicular to both vectors $e_\alpha$ and $h_\alpha$ in the event in question. Then we shall have

$$u_i = 0, \; u_4 = 1, \; e_3 = 0, \; h_3 = 0 \qquad (i = 1, 2, 3)$$

This implies

$$d_3 = 0 \qquad b_3 = 0$$

Using this simplification we shall calculate the components of the tensor $T_{\alpha\beta}$. We shall present it in the form of matrix

$$(1) \left\{ \begin{matrix} p + \frac{1}{2}(\lambda e^2 + \mu h^2) + b_2 h_2 - e_1 d_1 & -e_1 d_2 - h_1 b_2 & 0 & 0 \\ -e_1 d_2 - h_1 b_2 & p + \frac{1}{2}(\lambda e^2 + \mu h^2) - e_2 d_2 + b_1 h_1 & 0 & 0 \\ 0 & 0 & p + \frac{1}{2}(\lambda e^2 - \mu h^2) & -d_1 b_2 + d_2 b_1 \\ 0 & 0 & -e_1 h_2 + e_2 h_1 & p + \frac{1}{2}(\lambda e^2 - \mu h^2) \end{matrix} \right\|$$

In this coordinate system the only term showing nonsymmetry is $\tau_{43}$.

The eigen values and principal directions of the tensor $T_{\alpha\beta}$ are by definition, those scalars $\mathcal{N}$ and those vectors $l_\alpha$ which satisfy the relation

$$T_{\alpha\beta}\, l^\beta = \mathcal{N}\, g_{\alpha\beta}\, l^\beta$$

The eigen values are the solutions of the equation

$$| T_{\alpha\beta} - \mathcal{N} g_{\alpha\beta} | = 0$$

This equation of the fourth order can be split up in two equations of the second order, on account of its special form.

$$(2) \qquad \begin{vmatrix} T_{11} + \mathcal{N} & T_{12} \\ T_{21} & T_{22} + \mathcal{N} \end{vmatrix} = 0 \qquad \begin{vmatrix} T_{33} + \mathcal{N} & T_{34} \\ T_{43} & T_{44} - \mathcal{N} \end{vmatrix} = 0$$

The notation $T_{\alpha\beta}$ refers to the matrix (1) and is used only for clearness.

The first block of the matiix is symmetric, so quadratic equation which follows by the evaluation of the first determinant (2) will have real roots, i. e.

$$\mathcal{N}_{I/II} = -p - \lambda e^2 \pm \sqrt{(\lambda e^2 - \mu h^2)^2 + 4 \lambda\mu e^2 h^2 \cos^2 \vartheta}$$

where $\vartheta$ is the angle between $e_\alpha$ and $h_\alpha$.

Next two eigen values could be obtained from the equation derived from the second block. For the simplification of calculation we shall introduce the following notation

$$\frac{1}{2}(\lambda e^2 - \mu h^2) = D \quad e_2 h_2 - e_2 h_1 = P \quad P_\alpha = \varepsilon_{\alpha\beta\gamma\delta}\, u^\beta e^\gamma h^\delta$$

$P_\alpha$ is the Poynting vector. Using this notation we get the next two eigen values in the form

$$\mathcal{N}_{III/IV} = \frac{1}{2}\{\rho - p \pm \sqrt{(\rho - p)^2 + 4(p + D)(\rho + D) - 4 \lambda\mu P^2}\}$$

In this case the discriminant is not always positive and we shall find the condition making the roots real.

We shall restrict our consideration to the case of a "gas without pressure" putting $p = 0$. Then we have the eigen values in the form

$$\mathcal{N}_{III/VI} = \frac{1}{2}\{\rho \pm \sqrt{(2D + \rho)^2 - 4 \lambda\mu P^2}\}$$

The discriminant will be positive if the following condition is satisfied

$$(2D + \rho)^2 - 4 \lambda\mu P^2 > (2D)^2 - 4 \lambda\mu P^2 \geqslant 0$$

and can be expressed in terms of the vectors of the electric and magnetic field.

$$\lambda e^2 < (1 + 2\sin^2\vartheta - \sqrt{(1 + 2\sin^2\vartheta)^2 - 1})\,\mu h^2$$

$$\lambda e^2 > (1 + 2\sin^2\vartheta + \sqrt{(1 + 2\sin^2\vartheta)^2 - 1})\,\mu h^2$$

Under these circumstances the eigen values will be real.

Next we shall determine the principal directions as the solutions of the following system of equations

$$T_{\alpha\beta}\, l^\beta = \mathcal{N}\, g_{\alpha\beta}\, l^\beta$$

Due to the special form of a matrix of the tensor $T_{\alpha\beta}$ the first two coordinates of the principal vectors appear only in the first two equations, and the components $l^3$ and $l^4$ into the third and fourth equations only.

Now we substitute the first two eigen values in the system. Thus the first two equations form a homogeneous system of linear equations whose determinant is equal to zero (that was the condition for the determination of $\mathcal{N}_{I}$ and $\mathcal{N}_{II}$). Solutions are linearly dependent. However, the next two equations form a homogeneous system with a determinant different from zero so permitting only trivial solutions. Hence, the first two eigen vectors have the form

$$\vec{l}_{I/II} = \left\{ l^1,\ -\frac{T_{11} + \mathcal{N}_{I/II}}{T_{12}}\, l^1,\ 0,\ 0 \right\}$$

For the determination of the third and fouıth eigen vector we substitute $\mathcal{N}_{III}$ and $\mathcal{N}_{IV}$ into the system. Now the subsystem determining $l_2$ and $l_2$ has only trivial solutions and the components $l^3$ and $l^4$ are linearly dependent. The corresponding eigen vectors have the form

$$\vec{l}_{III/IV} = \left\{ 0,\ 0,\ -\frac{T_{33} - \mathcal{N}_{III/IV}}{T_{43}}\, l^4,\ l^4 \right\}$$

The first two eigen vectors are spacelike as could be seen ırom their form.

To determinate whether the third and fourth vectors are timelike or spacelike, we have to investigate the sign of the following expression

$$g_{\alpha\beta}\, l^\alpha l^\beta = \left[ -\left( \frac{T_{44} - \mathcal{N}}{T_{43}} \right)^2 + 1 \right] (l_4)^2$$

When we substitute the amounts of the corresponding components of the tensor $T_{\alpha\beta}$, we have

$$4\,(1 + \lambda\mu)\, P^2 - 2\,(2\,D + \rho)^2 \pm (2\,D + \rho) \sqrt{(2\,D + \rho)^2 - 4\,\lambda\mu\, P^2}$$

Upon the condition making eigen values real, we conclude that the third and fourth vector are spacelike also.

There exists a possibility of a double solution when

$$(2D+\rho)^2 = 4\lambda\mu P^2$$

In that case there exsist only three different eigen vectors and they are spacelike.

We emphasize that each one of the first two vectors is perpendicular on each one of the other two vectors.

In the case of a double solution the third eigen vector is perpendicular to both the first and the second one.

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## SUR LES VALEURS PROPRES ET LES VECTEURS PRINCIPAUX DU TENSEUR D'IMPULSION-ENERGIE D'UN FLUIDE PARFAIT CHARGE CONDUCTEUR

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Résumé

On considère ici les valuers propres et les vecteurs principaux du tenseur d'impulsion-énergie d'un fluide parfait chargé conducteur. On obtient quatre valeurs propres différentes. Deux valeurs propres sont réelles et les vecteurs propres correspondantes sont orientés dans l'espace. Pour les deux autres valeurs propres on obtient une inégalité qui donne une condition pour qu'ils soient réelles. Les vecteurs propres correspondants sont orienté dans l'espace.

# О СОПСТВЕНИМ ВРЕДНОСТИМА И СОПСТВЕНИМ ВЕКТОРИМА ТЕНЗОРА ЕНЕРГИЈЕ-ИМПУЛСА ИДЕАЛНОГ НАЕЛЕКТРИСАНОГ ФЛУИДА

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Резиме

У раду се разматра идеалан наелектрисан флуид коначне проводљивости електрицитета по шеми коју је предложио Лишнеровић. Тензор енергије-импула оваквог флуида није симетричан због појаве електричне и магнетне индукције, па зато није унапред извесно да ће сопствене вредности бити реалне. У раду се долази до услова да решења буду реална, а онда се израчунавају сопствени вектори. Сва четири сопствена вектора су просторног типа

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