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ON A DIFFERENTIAL PRINCIPLE OF HIGHER ORDER FOR NON-HOLONOMIC MECHANICAL SYSTEMS

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1. In the case where the mechanical system is acted upon by ideal non-holomic constraints of the most general form, a function is formed by applying the Gauss function of constraint, the variation of this function, in virtue of the principle of m-th order, being equal to zero. Equations of motion of the mechanical system under considerat.on were also obtained.

2. The differential principle of the m-th order*

$$\sum_{i=1}^{N} (m_i \ddot{\vec{r}}_i - \vec{F}_i)\, \delta_m \overset{(m)}{\vec{r}_i} = 0,$$

where

$$\vec{r}_i = \vec{r}_i (q^1, \ldots, q^n), \qquad n = 3N,$$

$$\overset{(m)}{\vec{r}_i} = \frac{d^m \vec{r}_i}{dt^m}, \quad \delta_m \overset{(m)}{\vec{r}_i} = \delta \overset{(m)}{\vec{r}_i} (\delta \vec{r}_i = \delta \dot{\vec{r}}_i = \cdots = \delta \overset{(m-1)}{\vec{r}_i} = 0),$$

can be written also in the form of

$$\delta_m \overset{(m-2)}{Z} = 0, \qquad \overset{(m-2)}{Z} = \frac{d^{m-2} z}{dt^{m-2}},$$

or

$$\frac{\partial \overset{(m-2)}{z}}{\partial \overset{(m)}{q^K}} \delta \overset{(m)}{q^K} = 0, \tag{1}$$

where Z iz Gauss's function of constraint.

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* The indices take the following values:

$j, k, r, s = 1, \ldots, n; \quad \mu = 1, \ldots, p; \quad \nu, \rho = p+1, \ldots, p+l = n.$

As shown in (2), we have

$$Z = \frac{1}{2} g_{Kr} (f^K - Q^K)(f^r - Q^r), \tag{2}$$

and, after introducing the Appell function of acceleration

$$S = \frac{1}{2} g_{Kp} f^K f^r,$$

and bearing in mind that

$$\overset{(m-2)}{Z} = \overset{(m-2)}{S} - Q_K \overset{(m)}{q}{}^K + \psi (q^j, \overset{\circ}{q}{}^j, \ldots, \overset{(m-1)}{q^j}),$$

the expression (1) can be rewritten in the well known form of

$$\left(\frac{\partial \overset{(m-2)}{S}}{\partial \overset{(m)}{q}{}^K} - Q_K \right) \delta \overset{(m)}{q}{}^K = 0. \tag{3}$$

Provided the expressions for the Appell function and the kinetic energy of the system involved are written in the form of

$$S = \frac{1}{2} \underset{\sim}{a} \circ \underset{\sim}{a}, \qquad T = \frac{1}{2} \underset{\sim}{v} \circ \underset{\sim}{v},$$

where

$$\underset{\sim}{v} = \overset{\circ}{q}{}^k \underset{\sim}{g}_k, \qquad \underset{\sim}{a} = f^k \underset{\sim}{g}_k, \qquad g_{kr} = \underset{\sim}{g}_k \circ \underset{\sim}{g}_v,$$

then

$$\overset{(m)}{T} = \overset{(m)}{\underset{\sim}{v}} \circ \underset{\sim}{v} + \binom{m}{1} \overset{(m-1)}{\underset{\sim}{v}} \circ \overset{\circ}{r} + \alpha (\overset{(m-1)}{q^j}, \ldots, q^j),$$

$$\overset{(m-2)}{S} = \overset{(m-2)}{\underset{\sim}{a}} \circ \underset{\sim}{a} + \beta (\overset{(m-1)}{q^j}, \ldots, q^j),$$

so that, because of

$$\frac{\partial}{\partial \overset{(m)}{q}{}^j} (\overset{(m)}{\underset{\sim}{v}} \circ \underset{\sim}{v}) = (m+1) g_{rS} \Gamma^r_{jk} \overset{\circ}{q}{}^K \overset{\circ}{q}{}^S,$$

$$\frac{\partial \overset{(m-2)}{S}}{\partial \overset{(m)}{q}{}^j} = \frac{1}{m} \frac{\partial \overset{(m)}{T}}{\partial \overset{(m)}{q}{}^j} - \frac{m+1}{m} \frac{\partial T}{\partial q^j},$$

the expression (1) obtains its well known (3)

$$\left[\frac{1}{m} \frac{\partial \overset{(m)}{T}}{\partial \overset{(m)}{q}{}^j} - \frac{m+1}{m} \frac{\partial T}{\partial q^j} - Q_j \right] \delta \overset{(m)}{q}{}^j = 0. \tag{4}$$

3. Let the system be subject to constraints of the most general form ([1], [3], [4])

$$\varphi^{\nu}(q^j, \overset{\circ}{q}{}^j, \ldots, \overset{(m-1)}{q}{}^j t) = 0,$$

which can be rewritten in the following way, too,

$$\frac{\partial \varphi^{\nu}}{\partial q^j}\overset{\circ}{q}{}^j + \frac{\partial \varphi^{\nu}}{\partial \overset{\circ}{q}{}^j}\overset{\circ\circ}{q}{}^j + \cdots + \frac{\partial \varphi^{\nu}}{\partial \overset{(m-1)}{q}{}^j}\overset{(m)}{q}{}^j + \frac{\partial \varphi^{\nu}}{\partial t} = 0,$$

whence, in virtue of

$$\frac{\partial \varphi^{\nu}}{\partial \overset{(m-1)}{q}}\delta \overset{(m)}{q}{}^j = 0$$

we can write

$$\delta \overset{(m)}{q}{}^{\nu} = b^{\nu}_{\mu}\,\delta \overset{(m)}{q}{}^{\mu}, \qquad b^{\nu}_{\mu} = b^{\nu}_{\mu}(q^j, \overset{\circ}{q}{}^j, \ldots, \overset{(m-1)}{q}{}^j).$$

By taking into consideration the last equation, (1) can be given the form of

$$\delta_m \overset{(m-2)}{Z} = \left(\frac{\partial \overset{(m-2)}{Z}}{\partial \overset{(m)}{q}{}^{\mu}} + b^{\nu}_{\mu}\frac{\partial \overset{(m-2)}{Z}}{\partial \overset{(m)}{q}{}^{\nu}}\right)\delta \overset{(m)}{q}{}^{\mu} = 0,$$

where from there follow the equations of motion of the system under consideration:

$$\frac{\partial \overset{(m-2)}{Z}}{\partial \overset{(m)}{q}{}^{\mu}} + b^{\nu}_{\mu}\frac{\partial \overset{(m-2)}{Z}}{\partial \overset{(m)}{q}{}^{\nu}} = 0.$$

If

$$\det\left[\frac{\partial \varphi^{\nu}}{\partial \overset{(m-1)}{q}{}^{\rho}}\right] \neq 0,$$

then it is possible to write

$$\overset{(m)}{q}{}^{\rho} = \psi^{\rho}(q^j, \overset{\circ}{q}{}^j, \ldots, \overset{(m-1)}{q}{}^j, \overset{(m)}{q}{}^{\mu}, t)$$

so that it is quite obvious that

$$\psi^{\rho} = b^{\rho}_{\mu}\overset{(m)}{q}{}^{\mu} + \varphi^{\rho}(q^j, \ldots, \overset{(m-1)}{q}{}^j, t).$$

If we put

$$\overset{(m-2)}{Z}{}^{*} = \overset{(m-2)}{Z}(\overset{(m)}{q}{}^{\rho} = \psi^{\rho})$$

then the differential principle of the *m*-the order can be written in the form of

(5) $$\delta_m \overset{(m-2)}{Z}{}^{*} = 0,$$

out of which there follow the equations of motion of the system involved,

(6) $$\frac{\partial \overset{(m-2)}{Z}{}^{*}}{\partial \overset{(m)}{q}{}^{\mu}}=0,$$

or (see, also [5])

(7) $$\frac{\partial \overset{(m-2)}{S}{}^{*}}{\partial \overset{(m)}{q}{}^{\mu}}=Q_{\mu}+b_{\mu}^{\rho}Q_{\rho},$$

where

$$\overset{(m-2)}{S}{}^{*}=\overset{(m-2)}{S}(\overset{(m)}{q}{}^{\rho}=\psi^{\rho}).$$

For $m=2$, the equations (7) are transformed into so-called Appell's equations. It is obvious that the Appell equations are a direct consequence of the Gauss principle. In fact, Appell divided the Gauss function of constraint into three parts: the part which represented the quadratic form of variables $\overset{\circ\circ}{q}{}^{j}$ was designated by S; the part which is the linear form of the variables $\overset{\circ\circ}{q}{}^{j}$ provides the generalized force; and, finally, the third part was discarded since it is not subject to variations.

From the preceding expostulations, we can see that the Gauss function of constraint (z) is a unique function in Analytical Mechanics with reference to the shortness of forms in which differential principles and equations of the mechanical systems are expressed.

4. In [4], the equations of motion of the mechanical system under consideration, that was subject to constraints

$$\overset{\circ}{q}{}^{\rho}=a_{\mu}^{\rho}(q^{1},\ldots,q^{n},t)\,\overset{\circ}{q}{}^{\mu}+a^{\rho}(q^{1},\ldots,q^{n},t)$$

were derived in the following way. The functions

$$K_{m}=\frac{1}{m}\left[\overset{(m)}{T}-(m+1)\frac{\partial T}{\partial q^{j}}\overset{(m)}{q}{}^{j}\right]-Q_{j}\overset{(m)}{q}{}^{j}$$

and

(8) $$K_{m}^{*}=K_{m}(\overset{(m)}{q}{}^{\rho}=a_{\mu}^{\rho}\overset{(m)}{q}{}^{\mu}+\cdots),$$

were first introduced, and then, from the condition of extremum of the function K_{m}^{*}, the differential equations of motion

(9) $$\frac{\partial K_{m}^{*}}{\partial \overset{(m)}{q}{}^{\mu}}=0.$$

were formed.

These equations are exact, but they are derived by wrong considerations. This is obvious if one bears in mind that

$$K_m = \overset{(m)}{\mathscr{L}_s q^s} + \cdots, \qquad L_s = \frac{d}{dt}\frac{\partial T}{\partial \overset{\circ}{q}^s} - \frac{\partial T}{\partial q^s} - Q_s,$$

where in the expression for K_m, terms which were not important for the expressions (9), were omitted. By considering (8) and

$$L_s = g_{ks} \overset{\circ\circ}{q}^k + \Gamma_{s,kr} \overset{\circ}{q}^k \overset{\circ}{q}^r - Q_s$$

it is quite obvious that K_m^* has its extremum only for $m=2$, and this is a well known fact. When $m>2$, the equations (8) are not proved by a correct method.

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SUR LE PRINCIPE DIFFERENTIEL D'ORDRE SUPERIEUR DES SYSTEMES NON-HOLONOMES

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Résumé

On formule un principe d'ordre supérieur dont l'essentiel est que la variation de la fonction, formée à l'aide de la fonction de Gauss, est égal à zéro.

О ДИФЕРЕНЦИЈАЛНОМ ПРИНЦИПУ ВИШЕГА РЕДА ЗА НЕХОЛОНОМНЕ МЕХАНИЧКЕ СИСТЕМЕ

Вукман Човић

Резиме

У раду се разматра механички систем подвргнут дејству идеалних нехолономних веза.

Полази се од принципа m-тога реда који се, користећи Гаусову функцију најмање принуде, трансформише на облик (1). Даље се показује да су са (1) еквивалентни познати облици (3) и (4).

У случају да је механички систем подвргнут дејству идеалних нехолономних веза вишега реда, диференцијални принцип (1) може да се напише у форми (5), из које је очигледно да Гаусова функција најмање принуде омогућава да се једначине кретања разматраног нехолономног система изразе у најкондензованијем облику (6).

На крају дат је осврт на решење овог проблема и радовима [3] и [4]. Показано је да се у тим радовима садрже извесне непрецизности.

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