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FROM DEFORMATION TO MIXED AND HYBRID FORMULATION OF THE FINITE ELEMENT METHOD

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1. Introduction

In the development of the finite element models there are many approaches, which are based on several variational principles. The most often used model is the so-called compatible model, from which the displacement or direct stiffness method is developed. This method is based on the principle of minimum potential energy. The method gives lower bound and convergency is monotonic. That is the main advantage of the method. However, the stresses which this method gives are not continuous along the interelement boundaries, and the accuracy of the stresses is lower than the accuracy of the deformations.

The mixed and hybrid methods are based on variational principle which represent some modifications of the minimum potential energy and minimum complementary energy principles, or Reissner's variational principle. The stresses computed by these methods usually are better than the stresses computed by the stiffnes method. The stresses and deformations by these methods are assumed independently. Due to that assumption, there are some imcompatibilities in the deformations within the element and along the interelement boundaries. By such an assumption it is not clear which violations are made. Also, the variational principle applied, for an engineer not deeply involved in the problem, are not clear. The results derived by these methods could be higher or lower than the exact one. One can not be sure whether they represent upper or lower bound. Those are some of the disadvantages, which probably are the main cause that these methods have not yet found wide practical application they deserve.

However one could show that deformations and stresses are not necessarry to be assumed independently. The deformations, as in the stiffness method, can be expressed as polynomial expansion, continuous within the element. As a difference from the deformation approach, the polynomial coefficients should be expressed not only by the nodal deformations, but by the nodal deformations and nodal stresses. In that way some improvemen.s in the accuracy and monotonic convergence of the results can be gained. Also, instead of application of any variational principle, only the principle of virtual work could de applied.

This paper is an attempt to show how those improvements could be achieved. The approach is illustrated on the problem of bending of plates, on which the author of this paper has been working.

2. The stiffness method

The stiffness method is a well known method, on which a lot of work has been done. Here we will not give details of the method, but only some general remarks, which are necessary for development of the mixed and hybrid methods, in a way as was stated. The problem which here will be treated is bending of plates, although the same approach can be applied on other types of problems.

The governing differential equation of the problem of bending of plates is,

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}=p\,(x,y)/D. \tag{1}$$

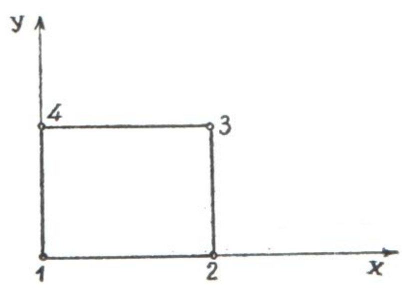
For an approximate numerical solution, long ago, Kantorovich [1] developed following functional of that differential equation,

$$\Pi=\iint\left\{D\left[\frac{1}{2}(\Delta w)^2-(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right)\right]-pw\right\}$$
$$-\int M\,(s)\frac{\partial w}{\partial n}\,ds+\int P\,(s)\,w\,ds\ldots \tag{2}$$

This functional can be developed in an engineering way. The surface integral gives the potential energy, first line integral gives the work of the boundary normal moments, and the last integral gives the work of the boundary shear forces. The moment M and the shear force P in this expression are in terms of the second and third derivatives of w.

The solution of the problem usually is derived by assuming the displacement field within the element as a polynomial expansion. Such a polynomial could be as follows:

$$w=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2+a_7x^3$$
$$+a_8x^2y+a_9xy^2+a_{10}y^3+a_{11}x^3y+a_{12}xy^3\ldots\,. \tag{3}$$



DEGREES OF FREEDOM:

- Stiffness and Hybrid I, $w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

- Mixed and Hybrid II, $w, M_x, M_y (M_{xy})$

Fig. 1

This is a 12 term polynomial, which corresponds to a 12 degree of freedom element, such as the rectangular element on Fig. 1. That is $4 \times 3 = 12$ degrees of freedom element. The deformation distribution given by Eq. 3 in the high order terms could be different. Here that distribution was chosen so that the second derivative, or the bending moments, have distribution commonly assumed in the mixed and hybrid methods.

The coefficients $a_1, \ldots, a_{12}$ in the deformation approach are expressed in terms of the nodal displacements, rotations and coordinates. In matrix form these coefficients would be

$$[a] = [c]\,\{\delta\} \tag{4}$$

where $[c]$ is a 12×12 matrix depending on nodal coordinates, $\{\delta\}$ — matrix of the nodal displacements and rotations

$$\{\delta\}^T = [w_1, w_{1,x}, w_{1,y} \ldots]\tag{5}$$

and comma (,) means differentiation.

Now the distribution of the displacements within the element can be expressed as follows

$$w = [P]\,[C]^{-1}\{\delta\}\tag{6}$$

where

$$[P] = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3].\tag{7}$$

By substitution of the assumed displacement field given by Eq. 6 into Eq. 2 a functional is obtained, in which subject of variation are the nodal displacements and rotations.

In this method the particular problem represents the first line integral in the functional

$$\int M_n \frac{\partial w}{\partial n}\, ds.$$

The moments M_n and $\partial w/\partial n$ in this case are functions of same variables. The moments are external forces of the element and they give work equal to the work of the internal forces, given by the surface integral. Therefore, the first line integral can not be simply introduced into functional. In order to decrease the influence of that integral, on the interelement boundary there should be continuity of the slopes $\partial w/\partial n$. However, the assumed displacement distribution given by Eq. 6 does not provide continuity of the normal slopes. To provide such continuity in some works on this problem an additional node on the middle of the sides has been taken. Some authors have suggested application of a "corrective" function along the interelement boundaries, of additional constraint, by inforcing linear variation of the normal slopes for instance.

The stiffness method gives discontinuous stresses on the boundary. Thus, although the continuity of the slopes along the boundaries is provided, there is an amount of work which is not taken into account. That work should be accounted by a line integral in the system [4]. It means that for the formulation of the system stiffness matrix additional computations should de carried out, which represents some complications.

That line integral, which is a problem in the stiffness method, in the mixed and hybrid methods is not a problem. The discontinuity of the stresses in the stiffness method is another disadvantage. The stresses are not primary unknown. They are computed by use of the deformations, and consequetly, their accuracy is smaller than the accuracy of the deformations. This disadvantage in the mixed method and assumed displacement hybrid method is not present.

3. The mixed method

3.1. The mixed method is known with the main assumption that deformations along the boundaries and stresses within the element are independently assumed. However, such an assumption is not necessary. The deformations in the element and along the boundaries can be assumed as given by Eq. 3. But now the coefficiens a_i should be expressed in terms of the nodal coordinates, displacements and second derivatives or moments. In matrix form these coefficients and the displacement w shall have the same form of expressions (4) and (6), but the meaning of the submatrixes is different. The matrix $\{\delta\}$ shall be,

$$\{\delta\}^T = [w_1, M_{1x}, M_{1y} \ldots\ldots w_4, M_{4x}, M_{4y}] \tag{8}$$

The matrix [c] has different coefficients also. In explicit form the displacements for a rectangular element (Fig. 1) are expressed as follows,

$$\begin{aligned}
w = {} & w_1 + (w_2 - w_1)\frac{x}{a} + (w_4 - w_1)\frac{y}{b} + (w_1 - w_2 + w_3 - w_4)\frac{xy}{ab} \\
& - (\overline{M}_{2x} + 2\overline{M}_{1x})\frac{ax}{6D} - (\overline{M}_{4y} + 2\overline{M}_{1y})\frac{by}{6D} \\
& + \frac{x^2}{2D}\left[\overline{M}_{1x} + (\overline{M}_{2x} - \overline{M}_{1x})\frac{x}{3a} + (\overline{M}_{4x} - \overline{M}_{1x})\frac{y}{b}\right. \\
& \left. + (\overline{M}_{1x} - \overline{M}_{2x} + \overline{M}_{3x} - \overline{M}_{4x})\frac{xy}{3ab}\right] \\
& + \frac{y^2}{2D}\left[\overline{M}_{1y} + (\overline{M}_{2y} - \overline{M}_{1y})\frac{x}{a} + (\overline{M}_{4y} - \overline{M}_{1y})\frac{y}{3b}\right. \\
& \left. + (\overline{M}_{1y} - \overline{M}_{2y} + \overline{M}_{3y} - \overline{M}_{4y})\frac{xy}{3ab}\right] \\
& - \frac{xy}{6D}\left[(2\overline{M}_{4x} - \overline{M}_{2x} + \overline{M}_{3x} - 2\overline{M}_{1x})\frac{a}{b} + 2\overline{M}_{2y} - \overline{M}_{4y} + \overline{M}_{3y} - 2\overline{M}_{1y})\frac{b}{a}\right] \ldots
\end{aligned} \tag{9}$$

where by $\overline{M}$ are denoted moments when $\nu = 0$.

By differentiation of the so defined displacement one can get the following bending and twisting moment distribution:

$$M_x = [N]\{M_{ix}\} \tag{10}$$

$$M_y = [N]\{M_{iy}\} \tag{11}$$

$$M_{xy} = [N R_1 R_2] \begin{Bmatrix} \bar{w}_i \\ \bar{M}_{ix} \\ \bar{M}_{iy} \end{Bmatrix} \tag{12}$$

where

$$\left.\begin{aligned} &[N] = [1 - \xi - \eta + \xi\eta,\ \ \xi(1-\eta),\ \ \xi\eta,\ \ \eta(1-\xi)] \\ &[R_1] = \frac{1-\nu}{6}[2r(1 - 3\xi + 3\xi^2/2);\ \ r(1-\xi^2);\ \ r(-1+3\xi^2); \\ &\qquad 2r(-1+3\xi-3\xi^2/2)] \\ &[R_2] = \frac{1-\nu}{6}\left[\frac{2}{r}(1-3\eta+3\eta^2/2);\ \ \frac{2}{r}(-1+3\eta-3\eta^2/2);\right. \\ &\qquad \left.\frac{1}{r}(-1+3\eta^2);\ \ \frac{1}{r}(1-3\eta^2)\right] \\ &\xi = x/a,\ \ \eta = y/b,\ \ r = a/b;\ \ \bar{w}_i = D(1-\nu)\,w_i. \end{aligned}\right\} \tag{13}$$

As one can see, the displacements w given by Eq. 9, now are functions of two types of coefficients: nodal displacements w_i and nodal moments M_{ix} and M_{iy}. These coefficients in the functional (2) are independent variables and should be independently varied. Thus, as is common in the mixed method, one can say that displacements and moments are independently assumed as follows,

$$\begin{Bmatrix} W \\ M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} \begin{Bmatrix} W_i \\ M_{ix} \\ M_{iy} \end{Bmatrix} \tag{14}$$

where $[N]$ is submatrix defined by expression (13), and w_i, M_{ix} and M_{iy} are nodal displacements and nodal bending moments respectively.

As one can see, the displacement distribution and bending moment distribution are the same (Eq. 14). In the mixed method, besides that, the twisting moments are assumed with the same distribution also. That is,

$$\underline{M_{xy} = [N]\{M_{ixy}\}} \tag{15}$$

where M_{ixy} are nodal twisting moments.

However, the twisting moment distribution derived here, given by Eqs. 12 and 13, is completely different. What is important, the twisting moments are functions of the bending moments also. For simplicity of the expression, the twisting moments were given in termps of $\overline{M}$, but they can be easily expressed in terms of bending moments when $\nu \neq 0$.

3.2. In the case of a rectangular element, with axis orientation as given on Fig. 1, the assumption of the twisting moments according to Eq. 15, as was common in the mixed method, does not give any result. The twisting moments can not be determined, because there will be no equations with twisting moments. Actually the twisting moments have been neglected.

Here was clearly shown that the twisting moments can not be independently assumed. They depend on the bending moments, as given by Eqs. 12 and 13. If the nodal twisting moments were taken as additional degrees of freedom, the displacement distribution polynomial (Eqs. 3 and 9) would be with 16 terms, and the twisting moment distribution corresponding to such displacement distribution would depend on nodal twisting moments and bending moments again.

The influence of the twisting moments taken according to Eq. 12 and 13 was tested on a simple example, simply supported square plate loaded uniformly. The results of the analysis show increase of the accuracy of the moments, and decrease of the accuracy of displacements. And what is important, the convergence of both, moments and displacements, is monotonic, from above.

That is an improvement in the mixed method, due to the assumption of moment distribution which gives compatible displacement distribution within the element.

It is interesting to discuss the displacement distribution given by Eq. 14. In the case of triangular element that is a linear distribution, which gives rigid body motion only. In the literature such a distribution is described as quite rough. T. Oden says that such simple distribution is on the expense of the accuracy, and concerning that gives some theorems [12]. Derivations given here showed that such conclusions are wrong. The actual displacement distribution, as given by Eq 9, is of much higher order.

Here should be noted that the displacements described by Eq. 14, should be considered as given displacements not only on the boundaries, but throughout the complete element. If a unit nodal displacement $w_i = 1$ is given, in the element there will be a work in twisting deformations. Thus, in the functional there should be the following term

$$\iint f(w,_{xy})\, dx\, dy \tag{16}$$

which gives that work. In the case of triangular element this integral will not be present.

That the displacement distribution described by Eq. 14 is not rough, but good, can be understood in a simple engineering way. Such deformations could be considered as given possible deformations. By equating the work of the external load and the work of the internal forces (boundary normal moments and twisting moment in the element due to the term with xy in the given displacement w), on the given displacements, one can get the equilibrium equations. These equations would be exact! The approximation is only in the assumption of linear variation of the moments along the boundaries.

The simplest way to develop the equilibrium equations is as follows. Along the boundaries linear hinges are assumed, and continuity is provided by unknown boundary normal moments. By giving unit displacements and computing the work of the boundary normal moments, the work in twisting deformations in the element, and the work of the external load, the equilibrium equations are derived.

When the moment distribution is assumed linear in one direction and constant in the other direction, as is done in Ref. 7, 8 and 10, or constant moments in the element (Ref. 6), the twisting moments are independent of the bending moments. Thus, probably is the reason why such rough elements have given very good results, better than the elements with linear moment distribution.

In the case of 4 node element the twisting moments depend on nodal displacement and nodal moments (Eq. 12). So, they contributed to the equilibrium and compatibility equations. If there are no twisting degrees of freedom, there will be discontinuity of the twisting moments along the boundaries. The effect of this discontinuity can be neglected, but by the following integral it can be taken into account also,

$$\int M_{xy} \frac{\partial w}{\partial t} ds$$

where $\partial w/\partial t$ is boundary tangent slope of the given displacements. In this case this integral is possible, because the M_{xy} and $\partial w/\partial t$ are expressed in terms of different variables.

3.3. In the mixed method there are two sets of equations: equilibrium equations, which are derived by variation of the functional on the nodal displacements, and compatibility equations which are derived by variation of the functional on the nodal moments. In the first set of equation the influence of the external load p, by the surface integral in the functional (2) is taken into account. But in the second set of equation the external distributed load has not been taken into account. That was due to the "independent" assumption of the displacement and moment distribution. Now, because the displacements, as given by Eq. 9, are function of the moments also, the direct action of the distributed load in the compatibility equations can be introduced. In the tunctional following integral should be present,

$$\int p\, w(m)\, dx\, dy \tag{17}$$

where $w(m)$ is displacements distribution, function of the nodal moments (Eq. 9).

However, if this term is added, the line integrals in the functional (2), in terms of the moments only, should be added also. And now, with those integrals there appear some problems, similar to the problems in the stiffness method. Some terms of the line integral

$$\int M(m) \frac{\partial m(m)}{\partial n} ds \tag{18}$$

are present in the surface integral expressed in terms of moments. Here should be added line integrals in which the moment $M(m)$ and rotations $\partial w(m)/\partial n$ are functions of independent variables (M_{ix} and M_{iy}).

Instead of variation of the functional, the unit force theorem could be applied. By giving a unit moment, for instance $M_{1x}=1$, along a boundary of the element on Fig. 1 appears shear force $P(s)$ and boundary deflection w, giving additional work, which should be added. That is the integral

$$\int P(s)\, w(m)\, ds \tag{19}$$

where $P(s)$ and $w(m)$ are functions of the varied moment.

The direct action of the external distributed load was taken according to Eqs. 17—19 and applied on the problem of bending of a simply supported square plate. Preliminary results show similar influence of this action as was the influence of the twisting moments. The accuracy of the moments is increased, the accuracy of the displacements slightly decreased, and both results represent an upper bound.

That is another improvement in the mixed method, which comes out of the dependent assumption of the displacements and the moments also.

Should be noted that introduction of these improvements does not much complicate the method. The method remains simple. There is some additional work to be done, to develop element matrix, which should be done once only. And the number of unknowns is not increased, but could be decreased.

Now we know what violations have been made, and we are sure that convergence of the method will be as well as the convergence of the stiffness method.

A good example of application of so improved mixed method is the linear problem. In this case the method can give exact results.

4. The hybrid methods

4.1. There are several approaches in the hybrid formulation of the finite element method. Two of them are the most often applied:

— Hybrid I, with assumed stress distribution in the element,

— Hybrid II, with assumed deformation distribution in the element.

By both methods the line integral in the functional (2) is easily defined. That is their advantage over the stiffness method. In the line integral

$$\int M_n \frac{\partial w}{\partial n} ds$$

the moments and slopes are defined by independent variables. In hybrid I the moments are defined by assuming the moment distribution in the element, and the slopes are independently defined by the nodal slopes. In the hybrid II the moments are defined by an assumed moment distribution along the boundaries, and the slopes by independently assumed deformation distribution in the element.

Here shall be considered the model hybrid I only. However, some of the conclusions derived for that model will be good for the model hybrid II also.

4.2. In the assumed stress distribution model the moment distribution is assumed similar to the Eqs. 10, 11 and 15,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [P]\{\beta\} \tag{20}$$

where $\{\beta\}$ is an array of undetermined coefficients equivalent to the nodal moments in the mixed method. The matrix $[P]$ gives the moment distribution and can be assumed of different order. For an triangular element with 9 stress modes Cook [17] this matrix assumes as follows

$$[P] = \begin{bmatrix} 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & 0 & y & 0 \\ 0 & 0 & 1 & 0 & 0 & x & 0 & 0 & y \end{bmatrix}. \tag{21}$$

It means that all moments are independently assumed, with linear distribution. Here again the same question arises: as in the discussion of the mixed method. Is possible such an independent assumption of all 3 moments, or is there a possible continuous displacement field which corresponds to such moment distribution? As was shown before, when the bending moment distribution is linear, as provides (21), the twisting moments M_{xy} depend on the bending moments M_x and M_y. It means that, if for instance a unit moment, let say $M_{1x}=1$ is applied, besides the bending deformations there will be twisting deformations also. By assuming the stress distribution as Eqs. 20 and 21 provide, it means that work in twisting deformations is omitted. Therefore, the twisting moments should be given similar to Eqs. 12 and 13

$$M_{xy} = f(M_{ix}, M_{iy}, M_{ixy})$$

not independent, but as a function of the nodal bending moments and nodal twisting moments.

It is interesting to note that, it is not necessary to take the twisting degree of freedom. The work of the twisting moments in the element can be taken although the twisting moments are expressed in terms of bending moments only

$$M_{xy} = f(M_{ix}, M_{iy}). \tag{22}$$

In that case the total number of equations for solution of the problem is decreased, and the expected accuracy should be as good as the accuracy of the direct stiffness method.

R. Cook has applied 5 stress mode element also [17]. That is an element with constant moment distribution along one axis. In this case the twisting moments do not depend on the bending moments, and consequently, the work in twisting, mentioned above, does not appear. Probably that is one of the main reasons why the 5 stress mode elements have given the best results. However, such an element with constant moments in one direction, should not be recommended for a wide application.

4.3. The system equations in the mixed method, in matrix form, for triangular element loaded at the nodes only, are as follows

$$\begin{bmatrix} A & 0 \\ B & A^T \end{bmatrix} \begin{Bmatrix} M \\ W \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \tag{23}$$

where $[A]$ and $[B]$ are submatrices of the system, $\{M\}$ and $\{W\}$ are nodal moments and displacements, and $\{P\}$ is submatrix of the external nodal forces.

The fi·st row of this equation gives the equilibrium equations, and the second row the continuity equations. In the assumed stress hybrid method the element matrix equation is similar to this equation, and with some modifications and additional variation on the nodal slopes, can be developed starting from the equations of the mixed method. Now, in the second row equations, the slopes are not equal to zero, but equal to the nodal slopes, which are additional degrees of freedom.

The slopes usually are assumed with linear variation along the boundaries, as

$$\varphi = c_1 + c_2 s \tag{24}$$

where s is coordinate along the boundary, c_1 and c_2 are constants expressed in terms of the nodal slopes. The displacements in this method have been assumed with linear or cubic distribution along the boundaries.

By giving possible boundary rotations with linear variation, as (24) provides, there is a work in bending of the element. That work in the hybrid method is substituted by the work due to the assumed stress distribution in the element. However, these two works are quite different. In that way the system is made more flexible, and therefore the computed results are lower and sometimes better than the results obtained by the direct stiffness method. But sometimes, due to that discrepancy between the assumed boundary deformations and assumed stress distribution, the method can give unstable solution [12].

When there is agreement between the assumed deformations and stresses, i.e. when both, boundary deformations and stresses in the element are due to same continuous deformations in the element, the hybrid method should give the same stiffness matrix as the direct stiffness method. The assumed stress distribution of the rough elements better corresponds to the assumed linear variation of the boundary rotations than the refined elements do. Thus, the rough elements should give more reliable results than the refined elements give.

4.4. By variation of the moments compatibility equations are derived. The terms in these equations represent rotations. In the assumed stress approach, as given in the literature, there are terms due to moments and deformations, but there is no term due to direct action of an distributed load on the element. Thus, as in the mixed method, an additional term in the surface integral should be added. That is an integral of the same form as given by expression (17)

$$\int p w(m)\, ds$$

where $w(m)$ is displacement distribution corresponding to the assumed stress distribution. The displacement distribution should be assumed before assuming the stress distribution, so that a compatible displacement distribution is provided, and requirements shown before, are satified. Such a displacement distribution was given by Eq. 9, with the corresponding moment distribution given by Eqs. 10, 11 and 12. Like in the mixed method, besides that integral the line integrals (18) and (19) should be added,

$$\int M_n \frac{\partial w}{\partial n} ds$$

$$\int P(s)\, w(m)\, ds.$$

Of course the method gives results which converge towards the exact solution without taking the last three integrals. But with addition of these integrals the convergence should be improved. In the case of concentrated load at the nodes these integrals are not present. Probably that is the reason why Cook in the case of concentrated load got the best results [17].

With the additions developed under this chapter seems that the method loses its simplicity and becomes a complicate one. However, it is not true. Some additional work has to be done in the development of the element matrix. What is the most important, the number of degrees of freedom is not increased, but could be decreased.

5. A general functional

In the previous chapters of this paper was shown that starting from the functional given by Eq. 2, by application of the standard procedure in the development of the stiffness method, one can develop the mixed and hybrid methods also.

In the current practice it is common to develop a finite element model starting from a variational principle based on the minimum potential energy principle or minimum complementary energy principle and their combinations and modifications.

According to the development of the mixed method, given in chapter 3 of this paper, one can give the following functional

(25) $$\Pi_g = U + V + U^* + V^* + V_b + V_b^*$$

where, U is potential energy, V — work of the external load on the assumed deformations, U^* — complementary energy, V^* — work of the external load on the deformations corresponding to the assumed stresses, V_b — work of the boundary tractions on the assumed deformations, and V_b^* — work of the boundary tractions on the deformations corresponding to the assumed stress distribution [19].

The first term U, potential energy, for a rectangular element in the mixed method, with displacement distribution field as given by Eq. 14, or the first part of Eq. 9, shall be

$$U = \int\int D(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 \partial x\,\partial y \tag{26}$$

In the case of triangular element this term is zero because the assumed displacement describes rigid body motion only.

The second term in the general functional is,

$$V = -\int\int p\,w\,dx\,dy \tag{27}$$

where w is a function of the nodal deformations only (Eq. 14).

The complementary energy U^* is derived by computing the energy of the assumed displacement field, expressed in terms of nodal stresses, as given by Eq. 9, according to functional (2),

$$U^* = \int\int \frac{1}{D(1-\nu^2)}\left[\frac{1}{2}M_x^2 + \frac{1}{2}M_y^2 - \nu M_x M_y + (1+\nu) M_{xy}^2\right] dx\,dy \tag{28}$$

where M_x, M_y and M_{xy} are defined by Eqs. 10—12.

The term V^* was given by Eq. 17,

$$V^* = \int p\,w(m)\,dx\,dy \tag{29}$$

where $w(m)$ is displacement distribution, function of the nodal stresses.

The work of the boundary tractions V_b was given by the first line integral in the functional (2),

$$V_b = -\int M(s)\frac{\partial w}{\partial n}ds \tag{30}$$

where $M(s)$ are the boundary normal moments which come out of the assumed moment distribution, as given by Eqs. 10—12, and $\partial w/\partial n$ is the slope derived from the assumed displacement distribution, as given by Eq. 14. The last integral in the functional (2), when the displacements along the inter element boundaries are continuous, should be omitted.

The last term in the functional (25), in the case of the considered mixed method, was given hy the Eqs. 18 and 19,

$$V^* = -\int M(s) \frac{\partial w(m)}{\partial n} ds + \int P(s)\, w(m)\, ds \tag{31}$$

where $M(s)$ and $P(s)$ are boundary moments and shear forces, $\partial w(m)/\partial n$ and $w(m)$ slopes and displacements along the boundaries, corresponding to the assumed stresses.

Thus, in the problem of bending of plates, the functional (25), expressed in terms of deformations and moments shall be,

$$\Pi_g = \iint \left\{ D\left[\frac{1}{2}(\Delta w)^2 - (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2\right]\right] - pw \right\} dx\, dy$$

$$+ \iint \left\{ \frac{1}{D(1-\nu^2)}\left[\frac{1}{2} M_x^2 + \frac{1}{2} M_y^2 - \nu M_x M_y + (1+\nu) M_{xy}^2\right] - p\, w(m) \right\} dx\, dy$$

$$-\int M_n \frac{\partial w}{\partial n} ds + \int P w\, ds - \int M_n(m) \frac{\partial w(n)}{\partial n} ds + \int P(m)\, w(m)\, ds \ldots . \tag{32}$$

The functional given here by Eqs. 25 and 32, was developed by considering the mixed method. However, it contains all possible energy terms, and it is not difficult to see that all variational principles used in the finite element method can be developed out of this functional. For instance, if the solution of a problem is assumed in terms of deformations only, the functional becomes,

$$\Pi_p = U + V$$

and the minimum potential energy principle is developed.

In the functional for the assumed stress hybrid method all terms of the functional (25) should be present also. In the current practice the terms in the functional V^* and V_b^* are neglected. They should be taken as was given by the Eqs. 17, 18 and 19.

Because of the complex meaning of the functional given by Eqs, 25 and 32, it could be called a general functional.

6. Conclusions

On the base of the results presented in this paper some conclusions concerning the mixed and hybrid methods could be drawn.

1. By application of the standard procedure used in the development of the stiffness method, assuming deformation distribution in terms of nodal deformations and stresses, the mixed and hybrid methods can be developed.

2. Stress distribution in the mixed and hybrid methods should not be assumed completely independent. In the problem of bending of plates only M_x and M_y can be independently assumed. Distribution of the M_{xy} moments depends on the bending moments distribution.

With the rough elements that is not the case, and therefore such elements have given better results than the refined elements.

3. The direct action of a distributed load on the element has to be taken into account. That is the work of the applied load on the deformations corresponding to the assumed stress distribution. In that case line integrals in terms of the stresses should be added also.

4. Independent assumption of the deformations along the boundaries is not correct. The deformations should be considered as possible given deformations throughout the complete element. Such deformations should correspond to the assumed stress distribution, or vice versa.

5. By introducing the suggestions presented here in the development of the mixed and hybrid method, some improvements can be gained. The accuracy of the stresses can be increased and convergency made monotonic. A simple example shows that in the mixed method such improvements are evident. In the hybrid method, on some pratical problems yet such improvements should be confirmed. What is important, with introduction of these improvements the number of degrees of freedom is not increased, but could be decreased.

6. It seems 3 deqrees of freedom per node, as in the stiffness method, are enough to get good accuracy. Refined elements, with more degrees of freedom, have given worst results than the rough elements. With the improvements given here the refined elements should give good results also, but it seems the application of such elements is not necessary.

7. A general functional has been developed, from which all variational principles used in the finite element method can be developed.

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ОТ ДЕФОРМАЦИОННОГО К СМЕШОННОГО И ХИБРИДНОГО ФОРМУЛИРОВАНИЯ МЕТОДА КОНЕЧНИЙХ ЕЛЕМЕНТОВ

Апостол Поцески

Р е з ю м е

Применнением стандарного способа в развитии метода деформации, смешонны и хибридный методов можно развит. Таким способом ясно можно увидет какие апроксимации делаются, и можно получит увеличение точности. Точност увеличывается и конвергенция становится монотонна. Развитый опщий функционалъ из которого можно получит всех вариационных принципов применяющих в методе конечных елементов.

Примененние предложеного способа илистрированно на проблеме изгиба плыт. Деформации в елемента выражаются как функции узловых перемещений и моментов. Моменты в елементе не надо взят независиммо от деформациях, как это делается в смешонном и хибридным методам. Толъко моменты изгиба можно выразит независимо. Крутяющие моменты зависимие от изгибающих. Они должны быт соответствующие с непрерывним деформациям.

Выражение деформаций как функции узловых перемещений и моментов даёт возможностъ в уравнениях компатибилности взят в внимание директное влияние распределённой нагрузки. Таким образом получается улучщение точности расчитанных моментов.

ОД ДЕФОРМАЦИОНОГ КА МЕШОВИТОМ И ХИБРИДНОМ ФОРМУЛИРАЊУ МЕТОДА КОНАЧНИХ ЕЛЕМЕНАТА

Апостол Поцески

Резиме

Применом стандардног приступа у методу деформација могу се развити мешовити и хибридни методи.

Предложени приступ примењен је на проблему савијања плоча. Деформације у елементу изражавају се као функције померања и момента савијања (9). Моменти у елементу не треба да се усвајају потпуно независно, као што је то било уобичајено у мешовитим и хибридним методима. Само моменти савијања могу бити усвојени независно (10, 11), док торзиони моменти зависе од момената савијања (12, 13). Код грубих елемената (са константним напрезањима на пример), то није случај, због чега такви елементи су дали боље резултате. Прелиминарна анализа са претпостављеним деформацијама по изразу (9) показује да се код мешовитог метода добија боља таьност момената, док тачност померања је мања, али обе величине конвергирају са једне (горње) стране, и њихова тачност је истог реда. Оваква претпоставка деформација омогућује да се узме у обзир и директно дејство подељеног оптерећења Код тога број непознатих по чвору се не повећава, већ може бити и мањи.

Развијен је општи функционал дат изразима 25 и 32, из кога се могу извести сви варијациони принципи који се примењују у методу коначних елемената.