

## MICROPOLAR THEORY OF AN INTERFACE

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### Abstract

It is well known that surfaces of bodies and interfaces between pairs of bodies exhibit properties quite different from those associated with their interiors. The problem of surface phenomena is based, for the most part, on molecular considerations, because it was thought that such phenomena could be explained properly only by properties of molecules.

Recently this problem has been considered by Scriven, Moeckel, Gurtin and Murdoch from the point of view of mechanics of continua. Their model of interfaces is a classical model of two-dimensional continuum. In this paper we consider an interface as a two-dimensional micropolar continuum. This point of view much better corresponds to the physical problem, because the micropolar theory takes into account the structure of material. We derived all field equations for interface and bulk material and, at the end, the constitutive equations for an interface.

### 1. Introduction

Mathematical model of an interface, from the standpoint of mechanics of continua, is a two-dimensional manifold in a body within which the parameters, deciding phenomenological properties of body surrounding this interface, have been defined. In such a case, an interface is a singular surface of a body and, from the general point of view, should not necessarily be material, i. e. within the process of body motion should not be composed from the same material points. Also, the material surfaces are two-dimensional manifolds, their material properties being affected by the corresponding parameters of surroundings.

The importance of an interface has been known so far, but it has been generally taken that this problem should be considered from the point of view of molecular theory. Those considerations have been undertaken from case to case. However, common basis of these problems could not be found. Mechanics of continua, as a method of studying phenomenological phenomena, offers this basis for the problems of interface. Each and every of those separate problems is derived by corresponding material properties of interface and surroundings which, within the mechanics of continua, are derived by respective constitutive relations.

Theories in mechanics of continua, relating to certain physical phenomena, have been neglecting influence of interface upon said processes. The crack theory, for instance, dealing with surface stress, has been overlooking the influence of surface stress upon the field of deformation within a body. Hence, certain contraversies between experiment and theoretical results.

This paper deals with micropolar theory of interface. The reason for such an approach lies in physical basis of this problems, since we do believe that an approach like this is the nearest to the molecular theories. It is well known that micropolar theory is such a theory that takes into consideration structure of material.

## 2. Geometry of an interface

Since material of an interface possesses properties essentially different from those of three-dimensional continuum, we shall suppose that an interface could be considered as a two-dimensional continuum. Therefore, we shall at the first stage, give the geometry of an interface. To achieve this, we shall use the most general parametrisation avoiding to introduce any specific restrictions, to make difference from papers [2] and [4], since we believe that any restrictions in parametrisation also restrict our considerations.

Let  $x^i$  be rectangular Cartesian coordinates of a point. Then a moving surface  $s(t)$  has the representation

$$(2.1) \quad s(t): \quad x^i = x^i(u^\alpha, t), \quad i = 1, 2, 3; \quad \alpha = 1, 2$$

where  $u^\alpha$  are surface parameters and  $t$  parameter that represent time. By the relation

$$(2.2) \quad u^\alpha = u^\alpha(U^\Gamma, t); \quad \Gamma = 1, 2$$

the motion of material particles is determined on the surface  $s(t)$ , where  $U^\Gamma$  are the convective coordinates.

In each point of surface a unit normal vector  $\nu^i = \nu^i(u^\alpha, t)$  exists, as well as tangent vectors  $x^i;_\alpha$ , so that

$$(2.3) \quad \nu^i \nu_i = 1, \quad x^i;_\alpha \cdot \nu_i = 0.$$

A square of element of length in the surface is given by

$$(2.4) \quad (dl)^2 = g_{\alpha\beta} du^\alpha du^\beta$$

where  $g_{\alpha\beta} = x^i;_\alpha x^i;_\beta$  the represents first metric tensor of the surface. This metric tensor has the inverse  $g^{\alpha\beta}$  such that

$$(2.5) \quad g^{\alpha\beta} \cdot g_{\alpha\gamma} = \delta_\gamma^\beta$$

where  $\delta_\gamma^\beta$  is the Kronecker delta symbol.

The second metrical tensor of the surface  $b_{\alpha\beta}$  is

$$(2.6) \quad b_{\alpha\beta} = x^i;_\alpha \nu_i,$$

that is,

$$(2.7) \quad x^i;_\alpha \nu_i = a_{\alpha\beta} \nu^\beta, \quad \nu^i;_\alpha = -b_\alpha^\beta x^i;_\beta$$



and satisfies the following conditions

$$(2.8) \quad b_Y^\alpha b_\beta^\gamma - 2 K_M b_\beta^\alpha + K_G \delta_\beta^\alpha = 0;$$

where  $K_M = 1/2 b_\alpha^\alpha$ ,  $K_G = \det b_\beta^\alpha = b/g$  ( $b = \det b_{\alpha\beta}$ ,  $g = \det g_{\alpha\beta}$ ).

The velocity of displacement of surface point with given parametrisation, is given by relation

$$(2.9) \quad \dot{x}^i = \frac{dx^i}{dt} \Big|_{U^r} = \frac{\partial x^i}{\partial t} \Big|_{u^\alpha, U^r} + x^i_{;\alpha} V^\alpha$$

where

$$(2.10) \quad V^\alpha = \frac{du^\alpha}{dt} \Big|_{U^r} = \frac{\partial u^\alpha}{\partial t} \Big|_{U^r}$$

represents the velocity of a point of the surface.

On the other hand, we have

$$(2.11) \quad \frac{\partial x^i}{\partial t} = A^\alpha x^i_{;\alpha} + U_n v^i$$

where

$$(2.12) \quad U_n = \frac{\partial x^i}{\partial t} v_i$$

is called the velocity of displacement of surface, and

$$(2.13) \quad A_\alpha = \frac{\partial x^i}{\partial t} x_{i;\alpha}, \quad A^\alpha = g^{\alpha\beta} A_\beta.$$

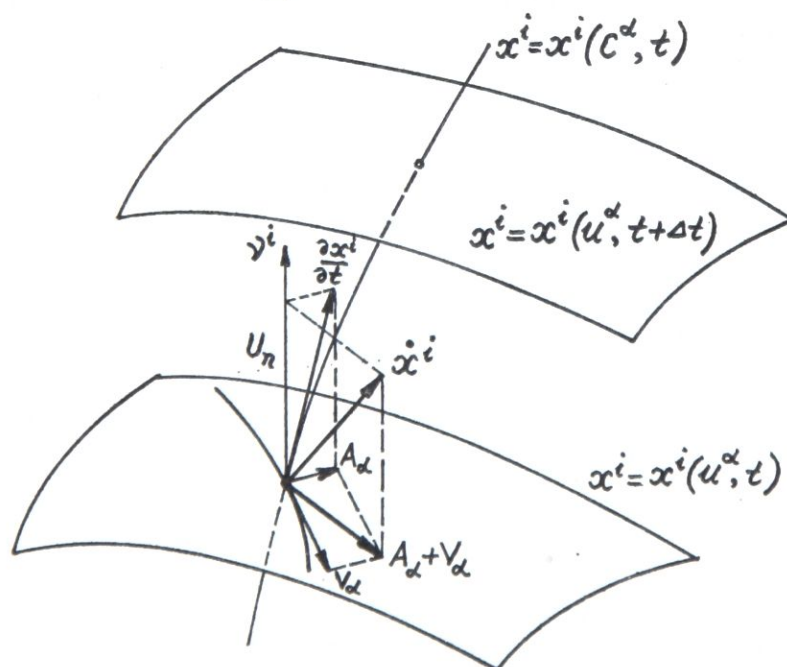


Fig. 1

From (2.9) and (2.11) it follows

$$(2.14) \quad V_\beta + A_\beta = \dot{x}^i x_{i;\beta}$$

where from, taking into account that

$$(2.15) \quad \dot{x}^i;_{\gamma} = \left( \frac{\delta x^i}{\delta t} \right);_{\gamma} = \frac{\partial x^i;_{\gamma}}{\partial t} + v^i b_{\alpha\gamma} V^{\alpha} + x^i;_{\alpha} V^{\alpha};_{\gamma}$$

we get through covariant differentiation

$$(2.16) \quad A_{(\beta; \gamma)} = 1/2 \frac{\partial g_{\alpha\beta}}{\partial t} + u_n b_{\beta\gamma}.$$

If we now differentiate the relation (2.6), and taking into account (2.7)<sub>2</sub>, we get

$$(2.17) \quad \frac{\partial v^i}{\partial t} = - (u_n;_{\beta} + b^{\alpha\beta} A_{\alpha}) x^i;_{\beta}$$

and

$$(2.18) \quad \frac{\partial b_{\alpha\beta}}{\partial t} - A^{\gamma} b_{\alpha\beta};_{\gamma} - b_{\gamma\beta} A^{\gamma};_{\alpha} - b_{\alpha\gamma} A^{\gamma};_{\beta} = U_n;_{\alpha\beta} - U_n b_{\beta}^{\gamma} b_{\gamma\alpha}.$$

The relations (2.16) and (2.18) are also called compatibility conditions. In this paper, they are of more general form than in [2] and [4], since we have introduced parametrisation without any restrictions. However, if certain restrictions, [2], [4] are taken into consideration, all the expressions could be treated as earlier obtained respective expressions, in which the Truesdell parametrisation obtains for  $A_{\alpha} + V_{\alpha} = 0$ , and Moeckel parametrisation for  $A_{\alpha} = 0$ .

### 3. General balance laws

In thermodynamics continuum, the general balance law for any body is of the following form

$$(3.1) \quad \frac{d}{dt} \psi = - \Phi(\psi) + P(\psi)$$

where  $\psi$ ,  $\Phi(\psi)$  and  $P(\psi)$  are time dependent measures of thermodynamical quantities defined in body  $b(t)$ . The measure  $\Phi$  is the efflux of  $\psi$  through the boundary of the body, and  $P$  is the production of  $\psi$ , [2].

To obtain local balance laws we shall utilize results of paper [2], taking into account that our paper observes the general form of surface parametrisation.

Local form of general balance law for those points of body that are not lying on an interface is as follows

$$(3.2) \quad \frac{\partial \psi}{\partial t} + (\psi \dot{\xi}^k + E^k),_{k} = p,$$

and local form of general balance law for those points of body that are lying on an interface is given by the following relation

$$(3.3) \quad \begin{aligned} \frac{\partial \psi_s}{\partial t} + (\psi_s V^{\alpha} + E_s^{\alpha}),_{\alpha} + \psi_s A^{\alpha};_{\alpha} - 2 \psi_s U_n K_M = \\ = [|\psi (U_n - \dot{\xi}_n)|] - [|E^k|] v_k^* + p_s \end{aligned}$$

$\xi_v$  is projection of velocity of surroundings points towards direction  $v^k$ .  $\psi = \psi(\xi^k, t)$  and  $\psi_s = \psi_s(u^\alpha, t)$  are the volume and surface density of quantity  $\Psi(t)$ ,  $p = p(\xi^k, t)$  and  $p_s = p_s(u^\alpha, t)$  are the volume and surface density of quantity  $P(\Psi)$ , and  $E^k = E^k(\xi^k, t)$  also  $E_s^\alpha = E_s^\alpha(u^\alpha, t)$  are the material flux or specific surface flux of quantity  $\Phi(\Psi)$ .

#### 4. Balance laws

On derivation of separate balance laws, for particular physical quantities, functions  $\psi$ ,  $E^k$  and  $p$ , as well as  $\psi_s$ ,  $E_s^\alpha$  and  $p_s$  become:

Table I

$\Psi$	$\psi$	$E^k$	$p$
Mass	$\rho$	0	0
Inertial tensor	$j^{kl}$	0	$-2 v_m^{(k} j^{l)m}$
Momentum	$\xi^k$	$-t^{kl}$	$-\rho f^k$
Moment of momentum	$\varepsilon_{ijk} x^j \xi^k + \sigma_i$	$-\varepsilon_{ijk} x^j t^{kl} - m_i^{.l}$	$\rho (\varepsilon_{ijk} x^j f^k + l_i)$
Energy	$1/2 (\xi^i \xi_i + \sigma^i v_i) + \varepsilon$	$-t^{kl} \xi_l - m^{kl} v_l - q^k$	$\rho (f^k \xi_k + l^k v_k + h)$
Entropy	$\eta$	$-q^k/T$	$\rho h/T$

Table II

$\Psi_s$	$\psi_s$	$E_s^\alpha$	$p_s$
Mass	$\gamma$	0	0
Inertial tensor	$J^{kl}$	0	$-2 \mu_m^{(k} J^{l)m}$
Momentum	$V^i$	$-S^{i\alpha}$	$\gamma F^i$
Moment of momentum	$\varepsilon_{ijk} x^j V^k + \tau_i$	$-\varepsilon_{ijk} x^j S^{k\alpha} - M_i^{.\alpha}$	$\gamma (\varepsilon_{ijk} x^j F^k + L_i)$
Energy	$1/2 (V^i V_i + \tau^i \mu_i) + \varepsilon_s$	$-S^{i\alpha} V_i - M^{i\alpha} \mu_i - Q^\alpha$	$\gamma (F^i V_i + L^i \mu_i + H)$
Entropy	$\eta_s$	$-Q^\alpha/T_s$	$\gamma H/T_s$



Using equations (3.2) and (3.3), and also taking into account values of respective functions given in those expressions, from Tables I and II, we now obtain:

a) Equations of balance of mass, microinertia, momentum, moment of momentum, energy, as well as entropy inequality for those points of body that are not lying on an interface, in the form

$$\begin{aligned}
 (4.1) \quad & \frac{\delta \rho}{\delta t} + \rho \dot{\xi}_{;m}^m = 0; \\
 & \frac{\delta j^{kl}}{\delta t} = 2 v_{;m}^{(k} j^{l)m} \\
 & t_{;k}^{kl} + \rho (f^l - \dot{\xi}^l) = 0 \\
 & m_{;k}^{kl} + \varepsilon^{lmk} t_{mk} + \rho l^l = \rho \dot{\sigma}^l; \\
 & \rho \dot{\varepsilon} = t^{kl} (\dot{\xi}_{l,k} + v_{kl}) + m^{kl} v_{l,k} - q_{;k}^k + \rho h = 0 \\
 & \rho \dot{\eta} + (q^k/T)_{;k} - \rho h/T \geq 0,
 \end{aligned}$$

where  $\varepsilon$  is the specific internal energy,  $\eta$  is the specific entropy,  $T$  is the temperature and  $h$  is the heat source per unit mass.  $\sigma^i$  is the inertial spin,  $l^i$  is the body couple and  $q^i$  is the heat flux vector.  $v_{kl}$  is the skew-symmetric gyration tensor,  $t^{kl}$  is the skew-symmetric stress tensor, and  $m^{kl}$  is the couple stress tensor.

b) Equations of balance of mass, microinertia, momentum, moment of momentum, energy, as well as entropy inequality for those point of body that are lying on an interface, in the form

$$\begin{aligned}
 (4.2) \quad & \frac{\delta \gamma}{\delta t} + \gamma \left( V_{;\alpha}^\alpha + \frac{g}{2g} \right) = [|\rho (u_n - \dot{\xi}_v)|], \\
 & \gamma \frac{\delta J^{kl}}{\delta t} = [|\rho (j^{kl} - J^{kl}) (u_n - \dot{\xi}_v)|] \\
 & \gamma \frac{\delta V^i}{\delta t} - S_{;\alpha}^{i\alpha} - \gamma F^i = [|\rho (\dot{\xi}^i - V^i) (u_n - \dot{\xi}_v)|] + [|t^{ik}|] v_k \\
 & \gamma \frac{\delta \tau_i}{\delta t} - \varepsilon_{ijk} x_{;\alpha}^j S^{k\alpha} - M_{i;\alpha}^{i\alpha} - \gamma L_i = [|\rho (\sigma_i - \tau_i) (u_n - \dot{\xi}_v)|] + [|\dot{m}_i^l|] v_l \\
 & \gamma \dot{\varepsilon}_s - S^{i\alpha} (V_{i;\alpha} + \mu_{ij} x_{;\alpha}^j) - M^{i\alpha} \mu_{i;\alpha} - Q_{;\alpha}^\alpha - \gamma H = [|\rho (\varepsilon - \varepsilon_s) (u_n - \dot{\xi}_v)|] + \\
 & \quad + [|\frac{1}{2} \rho (\dot{\xi}^i - V^i)^2 (u_n - \dot{\xi}_v)|] + [|t^{kl} (\dot{\xi}_l - V_l)|] v_k + \\
 & \quad + [|\rho \mu_i (\sigma^i - \tau^i) (u_n - \dot{\xi}_v)|] + [|\dot{q}^k|] v_k \\
 & \gamma \frac{\delta \eta_s}{\delta t} + (Q^\alpha/T_s)_{;\alpha} + \gamma H/T_s \geq [|\rho (\eta - \eta_s) (u_n - \dot{\xi}_v)|] + [|\dot{q}^k/T|] v_k
 \end{aligned}$$

where  $\varepsilon_s$  is the specific internal surface energy,  $\eta_s$  is the specific surface entropy and  $T_s$  is the surface temperature.  $\tau_i$  is the inertial surface spin,  $F_i$  is

the surface force,  $L_i$  is the surface couple and  $q_s^\alpha$  is the surface heat flux vector.  $S^{i\alpha}$  is the surface stress,  $M^{i\alpha}$  is the surface gyration tensor,  $\mu_{ij}$  is the surface couple stress and  $V_i$  is the particle velocity in the surface.

Given expressions show the influence of the surroundings upon the balance of certain quantities of an interface. It should be underlined that these equations are valid for surface, irrespective if it is material or not, and as such are of general character. When specific kind of material is concerned, constitutive equations characterizing properties of material surface, must satisfy these equations. In this case, entropy represents criterion of thermodynamically admissible processes on an interface.

In special cases, when  $\dot{\xi}_v^+ = u_n = \dot{\xi}_v^-$ , in other words when diffusion does not exist, balance equations are simplified and influence of surroundings upon an interface is essentially changed. If the tangent velocity components of the surroundings points are equal with tangent velocity components of an interface points then the velocities of an interface points and surroundings points are also equal, and the material surface is moving as integral part of the surroundings. The influences of the surroundings are, in that case, the least. If there are no heat effects either, an interface could be observed independently from the surroundings, and that is a well known problem of shells and plates. Then, from the thermodynamical point of view, an interface is considered as an insulated system.

## 5. Constitutive equations

We suppose that an interface is elastic. In that case the specific internal surface energy  $\epsilon_s$  is a function of the form

$$(5.1) \quad \epsilon_s = \epsilon_s(x_i^i; \Delta, \mu_i, \mu_i; \Delta, \eta_s)$$

and, the free surface energy function  $\varphi_s$  has the following form

$$(5.2) \quad \varphi_s = \varphi_s(x_i^i; \Delta, \mu_i, \mu_i; \Delta, T_s, T_s; \Delta)$$

since,

$$(5.3) \quad \varphi_s = \epsilon_s - T_s \eta_s$$

Substituting (5.2) and (5.3) into (4.2)<sub>5</sub> and (4.2)<sub>6</sub>, we obtain

$$(5.4) \quad \begin{aligned} & \gamma \varphi_s + \gamma \dot{T}_s \eta_s - S^{i\alpha} (V_{i,\alpha} + \mu_{ij} x_{,\alpha}^j) M^{i\alpha} \mu_{i,\alpha} = \\ & = [|\rho[(\varphi - \varphi_s) + (T - T_s)](u_n - \dot{\xi}_v)|] + [|\frac{1}{2} \rho (\dot{\xi}^i - V^i)(u_n - \dot{\xi}_v)|] + \\ & + [|\dot{t}^{kl} (\dot{\xi}_l - V_l)|] v_k + [|\rho \mu_i (\sigma^i - \tau^i)(u_n - \dot{\xi}_v)|] + [|\dot{q}^k|] v_k \\ & - \gamma (\dot{\varphi}_s + \eta_s \dot{T}_s) + S^{i\alpha} (V_{i,\alpha} + \mu_{ij} x_{,\alpha}^j) + M^{i\alpha} \mu_{i,\alpha} - Q^\alpha (\log T_s), \alpha \geq \\ & - [|\dot{t}^{kl} (\dot{\xi}_l - V_l) + m^{kl} (v_l - \mu_l) + \dot{q}^k (1 + T_s/T)|] v_k, \end{aligned}$$

where  $\varphi$  is the free energy function for bulk materials.

As we have seen, [5],

$$\mu_i = -\frac{1}{2} \epsilon_{ijk} \mu^{jk} = -\frac{1}{2} \epsilon_{ijk} \dot{\chi}^j_{\cdot k} \chi^{kK}_{\cdot}$$

so that

$$(5.5) \quad \varphi_s = \varphi_s(x_{;\Delta}^i, \chi_{\cdot K}^k, \chi_{\cdot K;\Delta}^k, T_s, T_{s;\Delta}).$$

With respect to (5.2), (5.4)<sub>1</sub> and (5.4)<sub>2</sub>, we get

$$(5.6) \quad \begin{aligned} \eta_s &= -\frac{\partial \varphi_s}{\partial T_s}; \\ S^{i\alpha} &= \gamma \frac{\partial \varphi_s}{\partial x_{i;\Delta}} u_{;\Delta}^\alpha \\ M^{i\alpha} &= -\gamma \varepsilon^{ijk} \frac{\partial \varphi_s}{\partial \chi_{\cdot K;\Delta}^k} u_{;\Delta}^\alpha \chi_{jk} \\ \frac{\partial \varphi_s}{\partial T_{s;\Delta}} &= 0, \end{aligned}$$

and the condition of the form

$$(5.7) \quad S_j^\alpha \chi_{\cdot k}^K x_{;\alpha}^k - \gamma \frac{\partial \varphi_s}{\partial \chi_{\cdot K}^j} - M^{i\alpha} \varepsilon_{ijk} \chi_{;\alpha}^{kK} = 0.$$

Relations (5.6) are the non-linear constitutive relations for an non-isotropic micropolar elastic interface.

If we introduce following measures of deformation

$$(5.8) \quad B_{K\Delta} = x_{;\Delta}^I \chi_{IK}, \quad \Gamma_{KL\Delta} = \chi_{IK} \chi_{\cdot L;\Delta}^I,$$

we obtain

$$(5.9) \quad \varphi_s = \varphi_s(B_{K\Delta}, \Gamma_{KL\Delta}, T_s),$$

and the non-linear constitutive relations (5.6) take the form

$$(5.10) \quad \begin{aligned} S^{i\alpha} &= \gamma \frac{\partial \varphi_s}{\partial B_{K\Delta}} \chi_{\cdot K}^i u_{;\Delta}^\alpha \\ M^{i\alpha} &= -\gamma \varepsilon^{ijk} \frac{\partial \varphi_s}{\partial \Gamma_{KL\Delta}} \chi_{jk} \chi_{kL} u_{;\Delta}^\alpha \\ \eta_s &= -\frac{\partial \varphi_s}{\partial T_s} \end{aligned}$$

The condition (5.7) is satisfied identically.

The constitutive relations for bulk materials are already derived, [5], [6].

#### REFERENCES

- [1] I. E. Scriven, Chemical Engineering Science, 2, 1960.
- [2] G. P. Moekel, Arch. Rational Mech. Anal., 57, 3, 1975.
- [3] M. E. Gurtin and Murdoch, Arch. Rational Mech. Anal., 57, 4, 1975.
- [4] C. A. Truesdell and R. A. Tupin, Handbuch der Physik, vol. III/1, Berlin—Heidelberg—New York, Springer 1960.
- [5] A. C. Eringen, Int. J. Engng. Sci., 9, 1966.
- [6] M. Plavšić and J. Jarić, Engineering Transactions, 22, 2, 181—211, 1974  
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## МИКРОПОЛЯРНАЯ ТЕОРИЯ МЕЖДУПОВЕРХНОСТЕЙ

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## Резюме

В работе рассматривается упругое поведение материальной между-поверхности с точки зрения микрополярной механике сплошной среды. При этом получаются уравнения сохранения массы, микроинерции, количества движения, момента количества движения, энергии, второй закон термодинамики и определяющие уравнения.

## МИКРОПОЛАРНА ТЕОРИЈА МЕЋУПОВРШИ

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## Резиме

У раду се разматра еластична материјална међуповрш са становишта микрополарне механике континуума. Изводе се једначине поља и конститутивне релације. Дискутује се утицај околине на међуповрш и посматра специјалан случај.

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