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AN EQUIVALENCE THEOREM IN POINCARÉ-ČETAEV VARIABLES

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In Lagrangian dynamics the equivalence theorem for a conservative holonomic system is based upon the equivalence of Hamilton's equations to a certain pfaffian equation.

In this paper a generalisation of the mentioned theorem to Poincaré-Četaev variables has been done and the generalised equivalence theorem is further used to prove the Hamilton-Jacobi theorem.

1. Introduction

Consider a conservative holonomic dynamical system with n degrees of freedom and whose position at any time t is defined by the n parameters $x_1, x_2, \ldots, x_n$ known as Poincaré-Četaev variables. Let T and U be the kinetic and potential energies of the system respectively.

Various indices along with their ranges of variation, which have been employed in the sequel, are

$$i, j, k, \alpha, \beta, \gamma = 1, 2, \ldots, n; \qquad s = 1, 2, \ldots, 2n.$$

and summation over a repeated index is implied.

In what follows we use the Poincaré-Četaev method [1, 5] to write the equations of motion of the system.

Let $\eta_1, \eta_2, \ldots, \eta_n$ be the parameters of real displacement and $X_0, X_1, \ldots, X_n$ be the corresponding displacement operators which are expressed by the relations

$$X_0 = \frac{\partial}{\partial t} + \underset{0}{\overset{i}{\xi}} \frac{\partial}{\partial x_i}, \quad X_i = \underset{i}{\overset{j}{\xi}} \frac{\partial}{\partial x_j}, \tag{1}$$

where $\underset{0}{\overset{i}{\xi}}$ and $\underset{i}{\overset{j}{\xi}}$ are functions of $x_1, x_2, \ldots, x_n$ and t. Since these operators form a closed system we have the commutation relations

$$(X_0, X_i) = X_0 X_i - X_i X_0 = C_{oij} X_j, \quad (X_i, X_j) = X_i X_j - X_j X_i = C_{ijk} X_k. \tag{2}$$

Here C_{oij} and C_{ijk} are functions of $x_1, x_2, \ldots, x_n$ and t and depend upon the choice of displacement parameters.

If $f(x_1, x_2, \ldots, x_n; t)$ is an arbitrary function of position then corresponding to an infinitesimal real displacement of the system the change in f is defined by the relation

$$df = X_0(f) + \eta_i X_i(f)\, dt. \tag{3}$$

Putting $f = x_j$ in (3), we obtain

$$\dot{x}_j = \frac{dx_j}{dt} = \overset{j}{\underset{0}{\xi}} + \eta_i \overset{j}{\underset{i}{\xi}}. \tag{4}$$

Since the operators $X_1, X_2, \ldots, X_n$ are independent it follows that the matrix $\left\| \overset{i}{\underset{j}{\xi}} \right\|$ is non-singular therefore the relations (4) yield

$$\eta_i = A_{ij} \dot{x}_j + A_i \tag{5}$$

where

$$A_i = -A_{ij} \overset{j}{\underset{0}{\xi}} \tag{6}$$

Let $L = T - U$ be the Lagrangian of the system then using (4) we can express it as a function of $x_1, x_2, \ldots, x_n, \eta_1, \ldots, \eta_n$ and t. Consequently the Poincaré-Četaev equations of motion of the system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \eta_i}\right) - C_{oij} \frac{\partial L}{\partial \eta_j} - \eta_j C_{jik} \frac{\partial L}{\partial \eta_k} - X_i(L) = 0. \tag{7}$$

If we introduce the momenta y_i by the relations

$$y_i = \frac{\partial L}{\partial \eta_i}. \tag{8}$$

The canonical equations of the system and Hamilton's differential equation as obtained in [2] are

$$\eta_i = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = C_{oij} y_j + \eta_j C_{jik} y_k - X_i(H), \tag{9}$$

$$X_0(V) + H(x_1, x_2, \ldots, x_n;\ X_1(V), \ldots, X_n(V), t) = 0, \tag{10}$$

where H is the Hamiltonian of the system defined by the relation

$$H = \eta_i y_i - L$$

and can be expressed as a function of $x_1, x_2, \ldots, x_n, y_1, \ldots, y_n$ and t by means of (8).

2. The Equivalence Theorem

Let $\omega_1, \omega_2, \ldots, \omega_{2n}$ be any $2n$ independent parameters which define the position of the system in the phase space then the functions

$$\begin{aligned} x_i &= x_i(\omega_1, \omega_2, \ldots, \omega_{2n}, t) \\ y_i &= y_i(\omega_1, \omega_2, \ldots, \omega_{2n}, t) \end{aligned} \tag{11}$$

are $2n$ independent functions of class C_2 in a domain D of $(\omega_1, \omega_2, \ldots, \omega_{2n})$ and an interval I of t such that the Jacobian

$$\frac{\partial(x_1, \ldots, x_n, y_1, \ldots, y_n)}{\partial(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{2n})} \neq 0 \tag{12}$$

for $(\omega_1, \omega_2, \ldots, \omega_{2n}) \in D$ and $t \in I$. We now prove the theorems:

Direct Theorem: If $H(x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, t)$ *is a given function of its* $2n+1$ *arguments and x's, y's satisfy identically for all values of* ω*'s in D the equations* (9) *then*

$$(\eta_i y_i - H)\, dt = d\psi + K_s\, d\omega_s$$

where ψ *is a function of* ω*'s and t, of class* C_2 *and coefficients* K_s *are functions of* ω*'s only.*

Converse Theorem: If there exists a function $H(x_1, \ldots, x_n;\ y_1, \ldots, y_n;\ t)$ *such that the pfaffian form* $(\eta_i y_i - H)\, dt$ *when expressed in terms of* ω*'s and t, has the form* $d\psi + K_s\, d\omega_s$ *then x's and y's satisfy the equations* (9).

Proof of Direct Theorem: Using (5) and (11), we get

$$(\eta_i y_i - H)\, dt = A_{ij} y_i \frac{\partial x_j}{\partial \omega_s} y_i\, d\omega_s + A_{ij} y_i \frac{\partial x_j}{\partial t} dt + A_i y_i\, dt - H\, dt =$$

$$= U_s\, d\omega_s + U\, dt,$$

where

$$U_s = A_{ij} y_i \frac{\partial x_j}{\partial \omega_s}, \tag{13}$$

$$U = A_{ij} y_i \frac{\partial x_j}{\partial t} + A_i y_i - H. \tag{14}$$

We shall now prove that

$$\frac{\partial U}{\partial \omega_s} = \frac{\partial U_s}{\partial t}.$$

Using (4) and (14), we have

$$\frac{\partial U}{\partial \omega_s} = \frac{\partial}{\partial \omega_s}\left[\left(A_{ij}\frac{\partial x_j}{\partial t} + A_i\right) y_i - H\right] = \frac{\partial \eta_i}{\partial \omega_s} y_i + \eta_i \frac{\partial y_i}{\partial \omega_s} - \frac{\partial H}{\partial x_i}\frac{\partial x_i}{\partial \omega_s} - \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial \omega_s},$$

or using (9), we obtain

$$\frac{\partial U}{\partial \omega_s} = y_i \frac{\partial \eta_i}{\partial \omega_s} - \frac{\partial H}{\partial x_i}\frac{\partial x_i}{\partial \omega_s}. \tag{15}$$

Now

$$\frac{\partial U_s}{\partial t} = \frac{\partial y_i}{\partial t} A_{ij} \frac{\partial x_j}{\partial \omega_s} + y_i \frac{\partial (A_{ij})}{\partial x_K}\frac{\partial x_K}{\partial t}\frac{\partial x_j}{\partial \omega_s} + y_i \frac{\partial (A_{ij})}{\partial t}\frac{\partial x_j}{\partial \omega_s} + y_i A_{ij} \frac{\partial 2 x_j}{\partial \omega_s\, \partial t},$$

which, in view of (1), (4) and (9), becomes

$$\begin{aligned}\frac{\partial U_s}{\partial t} &= [C_{oi\alpha} y_\alpha + \eta_\alpha C_{\alpha ik} y_k - X_i(H)]\, A_{ij}\frac{\partial x_j}{\partial \omega_s} + y_i \frac{\partial (A_{ij})}{\partial x_K}\frac{\partial x_j}{\partial \omega_s}\left[\eta_\alpha \overset{k}{\underset{\alpha}{\xi}} + \overset{k}{\underset{0}{\xi}}\right] + \\ &+ y_i \frac{\partial (A_{ij})}{\partial t}\frac{\partial x_j}{\partial \omega_s} + y_i A_{ij}\frac{\partial}{\partial \omega_s}\left[\eta_\alpha \overset{j}{\underset{\alpha}{\xi}} + \overset{j}{\underset{0}{\xi}}\right].\end{aligned} \tag{16}$$

Since (1) and (2) give

$$C_{oi\alpha}=\frac{\partial}{\partial t}\begin{pmatrix}\overset{k}{\xi}\\ i\end{pmatrix}A_{\alpha K}+\overset{k}{\underset{0}{\xi}}\frac{\partial}{\partial x_K}\begin{pmatrix}\overset{\beta}{\xi}\\ i\end{pmatrix}A_{\alpha\beta}-\overset{k}{\underset{i}{\xi}}\frac{\partial}{\partial x_K}\begin{pmatrix}\overset{\beta}{\xi}\\ 0\end{pmatrix}A_{\alpha\beta}=$$

$$=-\overset{k}{\underset{i}{\xi}}\frac{\partial}{\partial t}(A_{\alpha K})-\overset{k}{\underset{0}{\xi}}\overset{\beta}{\underset{i}{\xi}}\frac{\partial}{\partial x_K}(A_{\alpha\beta})-\overset{k}{\underset{i}{\xi}}\frac{\partial}{\partial x_K}\begin{pmatrix}\overset{\beta}{\xi}\\ 0\end{pmatrix}A_{\alpha\beta},$$

and

$$C_{\alpha iK}=\frac{\partial(A_{K\beta})}{\partial x_\gamma}\left[\overset{\gamma}{\underset{i}{\xi}}\overset{\beta}{\underset{\alpha}{\xi}}-\overset{\gamma}{\underset{\alpha}{\xi}}\overset{\beta}{\underset{i}{\xi}}\right],$$

therefore, after some simple manipulations, the relation (16) yields

$$\frac{\partial U_s}{\partial t}=y_i\frac{\partial\eta_i}{\partial\omega_s}-\frac{\partial H}{\partial x_j}\frac{\partial x_j}{\partial\omega_s}. \tag{17}$$

From (15) and (17) we get the required result

$$\frac{\partial U}{\partial\omega_s}=\frac{\partial U_s}{\partial t}. \tag{18}$$

We now introduce a function $\psi(\omega_1, \ldots, \omega_{2n}; t)$ such that

$$\frac{\partial\psi}{\partial t}=U,$$

therefore (18) gives

$$\frac{\partial U_s}{\partial t}=\frac{\partial U}{\partial\omega_s}=\frac{\partial_2\psi}{\partial\omega_s\,\partial t}=\frac{\partial_2\psi}{\partial t\,\partial\omega_s}.$$

Integrating we get

$$U_s=\frac{\partial\psi}{\partial\omega_s}+K_s,$$

where K_s is independent of t and consequently

$$(\eta_i y_i-H)\,dt=U_s\,d\omega_s+U\,dt=\frac{\partial\psi}{\partial t}dt+\left(\frac{\partial\psi}{\partial\omega_s}+K_s\right)d\omega_s$$

or

$$(\eta_i y_i-H)\,dt=d\psi+K_s\,d\omega_s$$

which is the required result.

Proof of Converse Theorem: Since

$$(\eta_i y_i-H)\,dt=d\psi+K_s\,d\omega_s,$$

it follows that

$$y_i A_{ij}\frac{\partial x_j}{\partial t}+A_i y_i-H=\frac{\partial\psi}{\partial t},\quad y_i A_{ij}\frac{\partial x_j}{\partial\omega_s}=\frac{\partial\psi}{\partial\omega_s}+K_s.$$

Now

$$\frac{\partial}{\partial t}\left(y_i A_{ij}\frac{\partial x_j}{\partial\omega_s}\right)-\frac{\partial}{\partial\omega_s}\left(y_i A_{ij}\frac{\partial x_j}{\partial t}+A_i y_i\right)=-\frac{\partial H}{\partial\omega_s}=-\left(\frac{\partial H}{\partial x_i}\frac{\partial x_i}{\partial\omega_s}+\frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial\omega_s}\right)$$

where we have used the relations

$$\frac{\partial_2 \psi}{\partial t\,\partial \omega_s} = \frac{\partial_2 \psi}{\partial \omega_s\,\partial t}, \quad \frac{\partial K_s}{\partial t} = 0,$$

or

$$\frac{\partial y_i}{\partial t} A_{ij} \frac{\partial x_j}{\partial \omega_s} + y_i \frac{\partial (A_{ij})}{\partial x_K} \frac{\partial x_K}{\partial t} \frac{\partial x_j}{\partial \omega_s} + y_j \frac{\partial (A_{ij})}{\partial t} \frac{\partial x_j}{\partial \omega_s} + y_i A_{ij} \frac{\partial_2 x_j}{\partial t\,\partial \omega_s} -$$

$$-\frac{\partial y_i}{\partial \omega_s} A_{ij} \frac{\partial x_j}{\partial t} - y_i \frac{\partial (A_{iK})}{\partial x_j} \frac{\partial x_j}{\partial \omega_s} \frac{\partial x_K}{\partial t} - y_i A_{ij} \frac{\partial_2 x_j}{\partial \omega_s\,\partial t} - \frac{\partial y_i}{\partial \omega_s} A_i - y_i \frac{\partial A_i}{\partial x_j} \frac{\partial x_j}{\partial \omega_s} =$$

$$= -\frac{\partial H}{\partial x_j} \frac{\partial x_j}{\partial \omega_s} - \frac{\partial H}{\partial y_j} \frac{\partial y_j}{\partial \omega_s}.$$

Using (4), the last relation takes the form

$$\frac{\partial x_j}{\partial \omega_s} \left[A_{ij} \frac{\partial y_i}{\partial t} + y_i \frac{\partial (A_{ij})}{\partial x_K} \left(\eta_\alpha \underset{\alpha}{\overset{k}{\xi}} + \underset{0}{\overset{k}{\xi}} \right) + y_i \frac{\partial (A_{ij})}{\partial t} - y_i \frac{\partial (A_{iK})}{\partial x_j} \left(\eta_\alpha \underset{\alpha}{\overset{k}{\xi}} + \underset{0}{\overset{k}{\xi}} \right) - \right.$$

$$\left. - y_i \frac{\partial A_i}{\partial x_j} + \frac{\partial H}{\partial x_j} \right] + \frac{\partial y_j}{\partial \omega_s} \left[-A_{jK} \left(\eta_\alpha \underset{\alpha}{\overset{k}{\xi}} + \underset{0}{\overset{k}{\xi}} \right) - A_j + \frac{\partial H}{\partial y_j} \right] = 0$$

or using (6), we get

$$\frac{\partial x_j}{\partial \omega_s} \left[A_{ij} \frac{\partial y_i}{\partial t} + y_i \left\{ \frac{\partial (A_{ij})}{\partial t} + \underset{0}{\overset{k}{\xi}} \frac{\partial (A_{ij})}{\partial x_K} + A_{iK} \frac{\partial \left(\underset{0}{\overset{k}{\xi}} \right)}{\partial x_j} \right\} + \right.$$

$$\left. + \eta_\alpha y_i \left\{ \underset{\alpha}{\overset{k}{\xi}} \frac{\partial (A_{ij})}{\partial x_K} - \underset{\alpha}{\overset{k}{\xi}} \frac{\partial (A_{iK})}{\partial x_j} \right\} + \frac{\partial H}{\partial x_j} \right] + \frac{\partial y_j}{\partial \omega_s} \left[-\eta_j + \frac{\partial H}{\partial y_j} \right] = 0.$$

There are $2n$ such relations one corresponding to each ω and consequently (12) yields

$$\eta_i = \frac{\partial H}{\partial y_i} \tag{19}$$

$$\frac{\partial y_i}{\partial t} = -y_j \left[\underset{i}{\overset{k}{\xi}} \frac{\partial (A_{jK})}{\partial t} + \underset{0}{\overset{k}{\xi}} \underset{i}{\overset{\beta}{\xi}} \frac{\partial}{\partial x_K} (A_{j\beta}) + \underset{i}{\overset{k}{\xi}} A_{j\beta} \frac{\partial}{\partial x_K} \left(\underset{0}{\overset{\beta}{\xi}} \right) \right] +$$

$$+ \eta_j y_K \left[\underset{i}{\overset{\gamma}{\xi}} \underset{j}{\overset{\beta}{\xi}} - \underset{j}{\overset{\gamma}{\xi}} \underset{i}{\overset{\beta}{\xi}} \right] \frac{\partial (A_{K\beta})}{\partial x_\gamma} - \underset{i}{\overset{j}{\xi}} \frac{\partial H}{\partial x_j},$$

or using the expressions for C's and X's we finally get the second of equations (9). Hence x's and y's satisfy (9). This completes the proof of the equivalence theorem.

We now deduce Hamilton-Jacobi theorem in Poincaré-Četaev variables by the application of equivalence theorem.

3. Hamilton-Jacobi Theorem

If $V(x_1, \ldots, x_n; a_1, \ldots, a_n; t)$ is a complete integral of Hamilton's differential equation (10) then the integrals of Hamilton's equations (9) are given by the relations

$$y_i = X_i(V), \quad b_i = -\frac{\partial V}{\partial a_i} \tag{20}$$

where the b's n are new arbitrary constants.

Proof: From (10) and (20) we have

$$(\eta_i y_i - H)\, dt = \eta_i X_i(V)\, dt + X_0(V)\, dt = dV - \frac{\partial V}{\partial a_i} d a_i$$

or

$$(\eta_i y_i - H)\, dt = d\psi + b_i\, d a_i,$$

where ψ is V expressed in terms of a's, b's and t. Now x's and y's are independent functions of a's, b's and t as given by (20). Therefore by the converse theorem it follows that x's and y's satisfy Hamilton's equations of motion. Hence the theorem is proved.

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