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ONE-FREQUENCY NONLINEAR FORCED VIBRATIONS OF UNIFORM BEAMS

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I. Introduction

In this paper is applied the asymptotic method of Krylov, Bogoljubov, Mitropolskij for solution of partial differential equations of transversal forced vibrations of uniform continuous beams in terms of the action to a beam of one-frequency force with slowchanging frequency in linear and nonlinear conditions of vibrations of beam, the origin of which is in the nonlinearity of material of beam. By means of the differential equations of first approximation for amplitude and phase of one-frequency regime of vibration which depends on the initial conditions and exciting referent resonant frequency of an external force, there are composed the amplitude-frequency resonant curves of stationary and nonstationary regime of vibrations for linear and nonlinear cases, and for the case of different velocities of growth and fall of circular frequency of an external force which are shown on the graph. Using them, one can make analysis and comparison of the corresponding regimes of vibrations.

II. Application of the asymptotic method for finding solutions and differential equations of the first approximation for beams vibrations in the one-frequency regime

Now we can study nonstationary transversal vibrations of homogeneous rectilinear of uniform continuous beam in terms of expressed physical nonlinearities at the action of an external one-frequency force which is continuously distributed on the span of a beam. This force has slowchanging frequency in the resonant range of a proper natural circular frequency. Should the initial conditions be such that they enable the appearance of a one-frequency regime of beam vibrations? Therefore we will study twoparameter approximation of a family of twoparameter solutions for a partial equation of transversal forced vibrations of uniform continuous beam, which is homogenous, rectangular and has a matrix form:

$$\frac{\partial^2}{\partial t^2}\{v\}+c^2\frac{\partial^4}{\partial z^4}\{v\}+e\{v\}=\varepsilon\{f^*\}+\varepsilon\{h\}\sin\theta \tag{1}$$

In previous equation the marks are the following: $\{v\}$ matrix of columns, the elements of which are the flexures along the corresponding span with the form of:

$$(2)\qquad \{v\}=\begin{Bmatrix} v^{(1)}\\ v^{(2)}\\ v^{(3)}\\ \vdots\\ v^{(n_1)}\end{Bmatrix};\quad v^{(1)}=\begin{cases} v_1(z,t) & 0\leqslant z\leqslant l_1\\ 0 & l_1\leqslant z\leqslant L=\sum\limits_{i=1}^{n_1} l_i\end{cases};\quad v^{(k)}=\begin{cases} 0 & 0\leqslant z\leqslant \sum\limits_{i=1}^{k-1} l_i\\ v_k(z,t) & \sum\limits_{i=1}^{k-1} l_i\leqslant z\leqslant \sum\limits_{i=1}^{k} l_i\\ 0 & \sum\limits_{i=1}^{k} l_i\leqslant z\leqslant l=\sum\limits_{i=1}^{n_1} l_i\end{cases}$$

n_1 is number of spans,
ε is small parameter, $\varepsilon\{f^*\}$ is a matrix of columns of nonlinear perturbance due to proper nonlinearities of a continuous uniform beam, the elements of which are the proper perturbing forces on the corresponding spans of beam; $c^2=\mathcal{B}/\rho A$; e is the coefficient of elasticity of a substratum onto which the beam is vibrating. We add below the following boundary conditions to the partial differential equation (1);

$$(3)\qquad \left\{L_{ij}\left(\{v\},\ \frac{\partial^3}{\partial z^3}\{v\}\right)\right\}\Bigg|_{\substack{z=\sum\limits_{i=1}^{k-1} l_i\\ z=\sum\limits_{i=1}^{k} l_i}}=0\qquad \begin{matrix} i=1,\ 2\\ j=0,\ 1,\ 2,\dots,\ k,\dots,\ n_1;\end{matrix}$$

where L_{ij} are linear operators depending on the form of beam leaning, as well as the initial conditions:

$$(4)\qquad \{v(0,\ z)\}=a_0\{Z_n(z)\};\quad \left\{\frac{\partial v}{\partial t}(z,\ 0)\right\}=p_0\{Z_n(z)\};$$

which enable us to attain a one-frequency regime of vibrations. $\{Z_n(z)\}$ is a matrix of columns, the elements of which are proper functions $Z_n(z)$ of a homogenous uniform continuous beam for the case of unperturbed vibrations, but for n-th proper value. $Z_n(z)$ are satisfying the linear boundary conditions (3), for which it is shown in the literature that they fulfill the conditions of orthogonality. We suppose that $\varepsilon(f^*\}$ is a matrix of columns the elements of which are monotonous functions of the coordinate z, and of whole rational functions of the other arguments, $v,\ \frac{\partial v}{\partial t},\ \frac{\partial^2 v}{\partial t^2},\ \frac{\partial v}{\partial z},\ \frac{\partial^2 v}{\partial z^2},\dots$. Matrix of columns $\varepsilon\{h\}$ has the elements which are the amplitudes of one-frequency force to the corresponding spans of a continuous beam. $\frac{d\theta}{dt}=\nu(\tau)\approx\omega_n$ is circular frequency of a compelling force and is a slowchanging function of parameter τ, in the range of n-th proper natural circular frequency of unperturbed vibrations. According to the assumptions introduced into the works of Mitropolskij [1] for composing of one-frequency asymptotic approximations we can look for the first approximation of solutions in the form

$$\{v(z, t) = \{Z_n\} a_n \cos(\theta + \varphi_n) \tag{5}$$

where φ_n is the phase, while the amplitude a_n and phase φ_n are the functions of the time t and are calculated from the differential equations system of first approximation which have the following form in the energy interpretation:

$$\begin{aligned} \frac{d a_n}{dt} &= -\frac{\omega a_n}{2\omega_n m_n} \frac{d(\omega_n m_n)}{d\tau} + \frac{2\varepsilon}{m_n} \sum_{\sigma=-\infty}^{+\infty} \frac{i\sigma(q\omega_n - p\nu)\dfrac{\delta \overline{W}_\sigma}{\delta a_n} + 2\omega_n \dfrac{1}{a_n}\dfrac{\delta \overline{W}_\sigma}{\delta \varphi_n}}{4\omega_n^2 - \sigma^2 (q\omega_n + p\nu)^2} \\ \frac{d\varphi_n}{dt} &= \omega_n - \frac{p}{q}\nu + \frac{2\varepsilon}{m_n} \sum_{\sigma=-\infty}^{+\infty} \frac{i\sigma(q\omega_n - p\nu)\dfrac{\delta \overline{W}_\sigma}{\delta \varphi_n}\dfrac{1}{a_n} - 2\omega_n \dfrac{\delta \overline{W}_\sigma}{\delta a_n}}{a_n\{4\omega_n^2 - \sigma^2(q\omega_n + p\nu)^2\}} \end{aligned} \tag{6}$$

where

$$m_n = \int_0^L (Z_n)\{Z_n\}\, dz \tag{6.1}$$

and p and q are reciprocally simple numbers; $\dfrac{\delta \overline{W}_\sigma}{\delta \varphi_n}$ and $\dfrac{\delta \overline{W}_\sigma}{\delta a_n}$ are "the partial derivatives" of the mean value of virtual work.

To compose all these equations for the definite case, it is necessary to define for definite case in question the mean value of virtual work done by perturbing forces

$$\varepsilon\{f\} = \varepsilon\left\{f^*\left(v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial z}, \frac{\partial^2 v}{\partial z^2}, \ldots, \frac{\partial^4 v}{\partial z^4}, z\right)\right\} + \varepsilon\{h\}\sin\theta \tag{7}$$

in the regime of cosine vibrations on the virtual displacements, which correspond to amplitude and phase variations of n-th harmonic of proper unperturbed vibrations:

$$\{\delta v\} = \{Z_n\}[\delta a_n \cos(\theta + \varphi_n) - a_n \sin(\theta + \varphi_n)\delta\varphi_n].$$

The mean value of virtual work for this case is:

$$\begin{aligned} \delta \overline{W} = \frac{1}{2\pi}\int_0^{2\pi} \delta W \delta\psi_n &= \frac{\varepsilon \delta a_n}{2\pi} \int_0^{2\pi}\int_0^L (f)\{Z_n\}\cos\psi_n\, d\psi_n\, dz - \\ &- \delta\psi_n \frac{\varepsilon a_n}{2\pi} \int_0^{2\pi}\int_0^L (f)\{Z_n\}\sin\psi_n\, d\psi_n\, dz - \\ &- \varepsilon\left[\frac{\delta a_n}{2}\sin\varphi_n + \frac{a_n \delta\varphi_n}{2}\cos\varphi_n\right]\int_0^L (h)\{Z_n\}\, dz. \end{aligned} \tag{8}$$

where (f) is a row matrix whose elements are proper nonlinear perturbances.

Now using the mean value of virtual work the "partial derivative" of which are:

$$\frac{\delta \overline{W}_\sigma}{\delta a_n}=0 \quad \frac{\delta \overline{W}_\sigma}{\delta \varphi_n}=0 \quad \text{for} \quad \sigma \neq \pm 1,\, 0$$

$$\frac{\delta \overline{W}_0}{\delta a_n}=\frac{\varepsilon}{2\pi}\int_0^L\int_0^{2\pi}(f^*)\{Z_n\}\cos\psi_n\, d\psi_n\, dz$$

$$\frac{\delta \overline{W}_0}{\delta \varphi_n}=-\frac{\varepsilon a_n}{2\pi}\int_0^L\int_0^{2\pi}(f^*)\{Z_n\}\sin\psi_n\, d\psi_n\, dz$$

$$(9) \qquad \frac{\delta \overline{W}_{-1}}{\varepsilon a_n}=\frac{\varepsilon}{4i}e^{-i\varphi_n}\int_0^L (h)\{Z_n\}\, dz$$

$$\frac{\delta \overline{W}_{-1}}{\delta \varphi_n}=-\frac{\varepsilon a_n}{4}e^{-i\varphi_n}\int_0^L (h)\{Z_n\}\, dz$$

$$\frac{\delta \overline{W}_1}{\delta a_n}=-\frac{\varepsilon}{4i}e^{i\varphi_n}\int_0^L (h)\{Z_n\}\, dz$$

$$\frac{\delta \overline{W}_1}{\delta \varphi_n}=-\frac{\varepsilon a_n}{4}e^{i\varphi_n}\int_0^L (h)\{Z_n\}\, dz \qquad \psi_n=\theta+\varphi_n$$

we can compose the equations of first approximations for amplitude and phase of n-th perturbed form of homogenous uniform rectilinear continuous beam in one-frequency forced regime of vibrations and using the system (6) we have:

$$\frac{da_n}{dt}=-\frac{\varepsilon}{2\pi\omega_n}\frac{\int_0^L\int_0^{2\pi}(f^*)\{Z_n\}\cos\psi_n\, d\psi_n\, dz}{\int_0^L (Z_n)\{Z_n\}\, dz}-\frac{\varepsilon\int_0^L (h)\{Z_n\}\, dz}{(\omega_n+\nu)\int_0^L (Z_n)\{Z_n\}\, dz}\cos\varphi_n$$

$$(10) \qquad \frac{d\varphi_n}{dt}=\omega_n-\nu(\tau)-\frac{\varepsilon}{2\pi\omega_n a_n}\frac{\int_0^L\int_0^{2\pi}(f^*)\{Z_n\}\sin\psi_n\, d\psi_n\, dz}{\int_0^L (Z_n)\{Z_n\}\, dz}+$$

$$+\frac{\varepsilon\int_0^L (h)\{Z_n\}\, dz}{a_n(\omega_n+\nu)\int_0^L (Z_n)\{Z_n\}\, dz}\sin\varphi_n .$$

Let us study now this system of differential equations of first approximation for the case when the law of elasticity of beam material is nonlinear and takes the ratio between stress and deformation according to the technical theory of bending of Kauderer [4]. In that case perturbing force due to proper nonlinearity of material has the form:

$$\varepsilon f^* \left(\frac{\partial^2 v}{\partial z^2}, \frac{\partial^3 v}{\partial z^3}, \frac{\partial^4 v}{\partial z^4}\right) = \lambda \left[2 \left(\frac{\partial^3 v}{\partial z^3}\right)^2 + \frac{\partial^2 v}{\partial z^2} \frac{\partial^4 v}{\partial z^4} \right] \frac{\partial^2 v}{\partial z^2}. \tag{11}$$

The system of differential equations of first approximation for amplitude a_n and phase φ_n (10) of n-th one-frequency perturbed proper form of vibrations is now:

$$\begin{aligned} \frac{da_n}{dt} &= -\frac{\varepsilon H(Z_n, h)}{\omega_n + \nu(\tau)} \cos \varphi_n \\ \frac{d\varphi_n}{dt} &= \omega_n - \nu(\tau) + \frac{3\lambda}{8\omega_n} a_n^2 \chi(Z_n) + \frac{\varepsilon H(Z_n, h)}{[\omega_n + \nu(\tau)] a_n} \sin \varphi_n . \end{aligned} \tag{12}$$

where we have introduced the marks:

$$\chi(Z_n) = \frac{\sum\limits_{i=1}^{n_1} \int\limits_0^{l_i} [2(Z_n^{\mathrm{III}})^2 + Z_n^{\mathrm{II}} Z_n^{\mathrm{IV}}] Z_n^{\mathrm{II}} Z_n \, dz}{\sum\limits_{i=1}^{n_1} \int\limits_0^{l_i} Z_n^2 \, dz} \tag{13}$$

and

$$H(Z_n, h) = \frac{\sum\limits_{i=1}^{n_1} \int\limits_0^{l_i} h(z) Z_n(z) \, dz}{\sum\limits_{i=1}^{n_1} \int\limits_0^{l_i} Z_n^2(z) \, dz} \tag{14}$$

where n_1 is the number of spans. The coefficient (13) will be named as the coefficient of influence of nonlinearity of material of a beam to the change of proper circular frequency of vibrations, but another (14) will be named as the coefficient of influence of compelling force to velocities of amplitude and phase changes of the studied perturbed form of vibrations.

If vibrations are done in terms where resistive forces are in linear p oportion to the velocity, differential equations of first approximation would become of the form:

$$\begin{aligned} \frac{da_n}{dt} &= -\frac{a_n}{2}(\delta + \mu k_n^4) - \frac{\varepsilon H(Z_n, h)}{\omega_n + \nu(\tau)} \cos \varphi_n \\ \frac{d\varphi_n}{dt} &= \omega_n - \nu(\tau) + \frac{3\lambda}{8\omega_n} a_n^2 \chi(Z_n) + \frac{\varepsilon H(Z_n, h)}{[\omega_n + \nu(\tau)] a_n} \sin \varphi_n . \end{aligned} \tag{15}$$

III. Testing of stability of stationary resonant state

For the case of stationary resonant state it will be necessary that the following terms are satisfied $\frac{da_n}{dt}=0$ and $\frac{d\varphi_n}{dt}=0$ at the constant value of parameter of slowchanging time τ. This condition is applied to the system of differential equations of first approximation (12) and after removing the angle φ_n from the system of equations one can get the following algebraic equation of the third degree for calculation of stationary amplitudes in function of the perturbing force frequency:

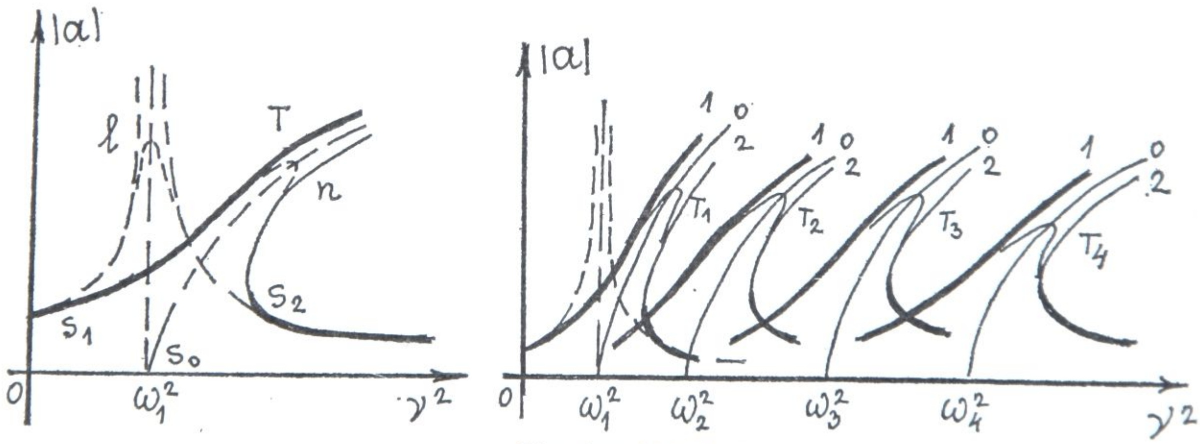
$$\cos\varphi_n=0 \quad \varphi_n=\frac{\pi}{2}+2k\pi \tag{16}$$

$$(\omega_n^2-\nu^2)\,a_n+\frac{3\lambda}{8\omega_n}a_n^3\chi(Z_n)(\omega_n+\nu)+\varepsilon H(Z_n, h)=0.$$

By solving this equation for change of frequency of the external force in the resonant range of n-th proper circular frequency, one can get the values of amplitude by means of which it would be possible to compose the amplitude frequency curve of stationary resonant state, the skeleton curve of which has the following form;

$$a_n^* = \pm\sqrt{\frac{8\omega_n}{3\lambda\chi(Z_n)}(\nu-\omega_n)} \tag{17}$$

On the sketch No. 1 a is shown the characteristic of that skeleton curve, which is a parabola and represents the asymptotic curve of amplitude--frequency graph of stationary resonant state.



Sketches No. 1

If we neglect nonlinear influences we may get the amplitude curve of stationary resonant state for the case of linear vibrations. For the linear case the skeleton curve of amplitude-frequency graph is straight line $\nu^2=\omega_n^2$. By comparison of those curves one can arrive to the conclusion that resonant curve of stationary amplitudes for nonlinear vibrations is bent to right, but at linear vibrations it is a hyperbola the asymptota of which is straight line.

Applying the Ljapunov's theorem of stability to the system of equations of first approximation and using those equations written by means of variations, we come to the conclusion that a certain part of amplitude-frequency curve representing stationary resonant state a_n is described by a thin line with amplitudes corresponding to unstable vibrations, so that on that part has appeared the so-called jump of amplitudes, which is characteristic for the nonlinear systems.

If we take into consideration the influence of external resistive forces from the conditions for stationary resonant state, we obtain the following equations.

$$\cos\varphi_n = -\frac{a_n(\delta+\mu k_n^4)(\omega_n+\nu)}{\varepsilon H(Z_n, h)} = -\frac{a_n(\bar{\delta}+\bar{\mu} k_n^4)(\omega_n+\nu)}{H(Z_n, h)}$$

(18)

$$(\omega_n^2-\nu^2)a_n + \frac{3\lambda}{8\omega_n}a_n^3\chi(Z_n)(\omega_n+\nu) \pm \varepsilon H(Z_n, h)\sqrt{1-\left[\frac{a_n(\bar{\delta}+\bar{\mu}k_n^4)(\omega_n+\nu)}{H(Z_n, h)}\right]^2} = 0.$$

The skeleton amplitude-frequency curve of the graph is the same as in the case of neglecting the resistive forces. In this case the amplitude-frequency curves are finite and cut the skeleton curve in the point that is obtained as a cross section of the skeleton curve

$$a_n^* = \pm\sqrt{\frac{8\omega_n}{3\lambda\chi(Z_n)}(\nu-\omega_n)} \tag{19}$$

and the curve

$$a_n(\bar{\delta}+\bar{\mu}k_n^4)(\omega_n-\nu) - H^2(Z_n, h) = 0. \tag{20}$$

At the cross section of these curves the coordinates of T point is marked in the sketches №. 1.

IV. Analysis of amplitude-frequency curves of nonstationary resonant state

In the example of vibrations of the homogenous uniform rectilinear continuous beam with two equal distances of length (spans) $l=2\,[m]$, with rectangular cross section $a\times b = 5\times 6$ [cm × cm], from steel, with module of elasticity $E=19{,}62\cdot 10^{10}$ [N/m^2], density of material of beam $\rho = 7{,}8\cdot 10^3$ [kg/m^3], coefficient of nonlinearity of elasticity law of material of beam $a_3 E^3 = 10^{12}$ [N/m^2], or $\lambda = 996{,}92\cdot 10^4$ [m^6/sec^2], loaded by equally distributed loading on the first span of amplitude $F_0 = 981$ [N/m] and the second span of amplitude $F_0 = -\,-981$ [N/m], circular frequency $\nu(\tau) = 180+\alpha t$ for the case of growth of frequency and $\nu(\tau) = 260-\alpha t$ for the decreasing case of circular frequency of compelling force. Let vibrations be done in terms of external and internal linear resistance of the coefficient $\mu = 0{,}8$ [m^4/sec] for external one and $\delta =$ [sec^{-2}] for internal one.

First proper circular frequency of unperturbed vibrations of two-spans beams is $\omega_1 = 216{,}3396$ [sec^{-1}], but its corresponding vibrating period is $T_1 = 0{,}02931414$ [sec]. Circular frequencies of the compelling force in increase and decrease pass through the value of the first proper circular frequency, so that we may study vibrations in the one-frequency basic regime of vibrations.

Equations of the first approximation (15) for given numerical data will obtain the following form:

$$
(21) \qquad
\begin{aligned}
\frac{da_1}{dt} &= -4{,}4\,a_1 - \frac{78{,}81538}{214{,}34 + \nu(\tau)}\cos\varphi_1 \\
\frac{d\varphi_1}{dt} &= 214{,}34 - \nu(\tau) + 32323{,}6\,a_1^2 + \frac{78{,}81538}{[214{,}34 + \nu(\tau)]\,a_1}\sin\varphi
\end{aligned}
$$

while the skeleton curve of amplitude-frequency graph has the form

$$
(22) \qquad \omega_1(a_1) = 214{,}34 + 32323{,}6\,a_1^2.
$$

and is marked in the Figure №. 2 by s_0.

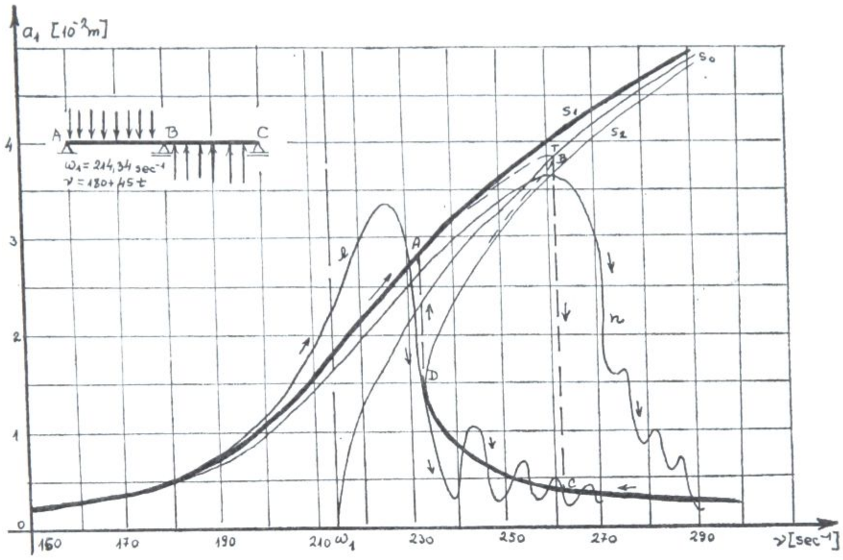


Fig. No. 2

For the case of stationary resonant state if we neglect the infuences of external and internal linear resistances we can obtain the curves s_1 and s_2 from the Figure №, 2 are obtained which continue infinitely. If we take into consideration the influence of resistive forces, we can obtain the amplitude--frequency curve drawn by a dotted line which is finite and approaches to smaller amplitudes if the coefficients of resistive forces are increased. Amplitudes on the eurve s_1 distinguish stable stationary vibratory regimes, while on the curve s_2 are appeared the amplitudes which by their side distinguish unstable vibratory regimes — thin drawn part of the curve s_2. In terms of influence of resistive forces in the growth of frequency of external force, the stationary regime of vibrations are distinguished by amplitudes on the curve s_1 in the direction of growth to the point $B\,(T\,(3{,}77^{28} \cdot 10^{-2}\,[m];\ 260{,}349\,[\sec^{-1}]))$, then has appeared the jump of amplitudes to the stable stationary regime of vibrations which is now distinguished by amplitudes from both sides of the curve BC, instead of the part of the curve TDC, in the direction of the growth of frequency and decrease of amplitudes.

In terms of influence of resistive forces, at decrease of frequency the stationary regime of oscillations is distinguished by amplitudes from the part of the graph s_2 up to the point D in the direction of increase of amplitude, in which, has appeared the jump of amplitude to the point A and amplitudes of the stationary regime of vibrations take the values from the curve s_1 in the direction of decrease of frequency and amplitude.

Equations of first approximation for amplitude and phase (21) are numerically integrated by means of the method of Runge-Kute on the *IBM* 1130 computer for mamy different velocities of change of the external force frequency. For each curve 1 500 points were calculated and 75 points were printed. As initial values of amplitudes and phases were adopted the stationary values for corresponding initial value of the external force frequency. On the Figures are shown only the amplitude-frequency curves for the cases of velocities of change of frequency of the external force: $\alpha_1 = 45\ \mathrm{sec}^{-2}$; $\alpha_2 = 90\ \mathrm{sec}^{-2}$; $\alpha_3 = 450\ \mathrm{sec}^{-2}$ and for the cases where it is taken into consideration the influence of nonlinear law of elasticity of material of a beam and where it should be neglected on purpose to make some comparisons.

On Figures №. 2 and №. 3 by l are denoted amplitude-frequency curves of the nonstationary resonant state in the linear case when the nonlinearity

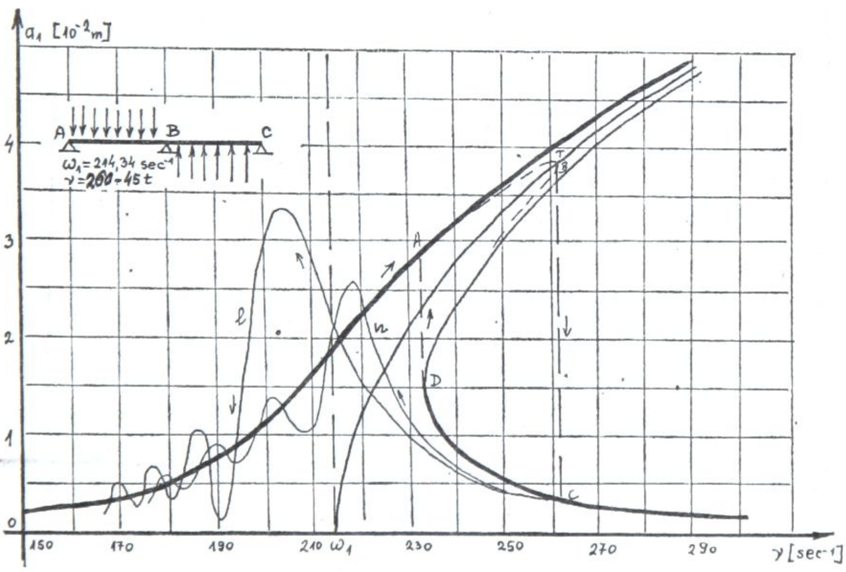


Fig. No. 3

of material of beam is neglected in Fig. №. 2 for increase of frequency, and in Fig. 3 for decrease of frequency when the frequency change of velocity is $45\ \mathrm{sec}^{-2}$. By the same Figure are denoted amplitude-frequency curves of the nonstationary resonant state-pass through resonant state, if we take into consideration the influence of nonlinearity of material of beam, that is, in Fig. No. 2 for the case of increase, and in Fig. No. 3 for the case of decrease of circular frequency of perturbing force.

In the case of increase of frequency of perturbing force it may be seen on the graph in Fig No. 2 that the first maximum of amplitude-frequency curve for the nonlinear case moves towards higher frequencies and amplitudes and it is more dangerous than linear case of pass through resonant state.

In the case of decrease of frequency it may be seen from the graph, in Fig. No. 3 that the linear case is more dangerous because the maximum amplitude is larger and moves towards lower frequencies closer to proper frequency of unperturbed proper vibrations than the maximum value of amplitude for the nonlinear case of nonstationary resonant state. Here the nonlinearity of law of elasticity of material of beam influences so that it moves again the first maximum towards higher frequencies, but not towards higher amplitudes too' as it was in the previous case, but towards lower amplitudes. We come to the conclusion that the influence of nonlinearity of material of beam is more dangerous only in the case of increase of frequency of the external force.

Also we come to similar conclusions and analysing amplitude-frequency curves of the nonstationary resonant state for the cases of change of circular frequency of external force at increase and decrease by velocities fo $\alpha_2 = 90\ \mathrm{sec}^{-2}$ and $\alpha_3 = 450\ \mathrm{sec}^{-2}$.

On Fig. No. 4 are shown amplitude-freqeuncy curves for the linear case of vibrations of twospan beam and this in terms of decrease and increase of frequency of the external force for three denoted velocities of

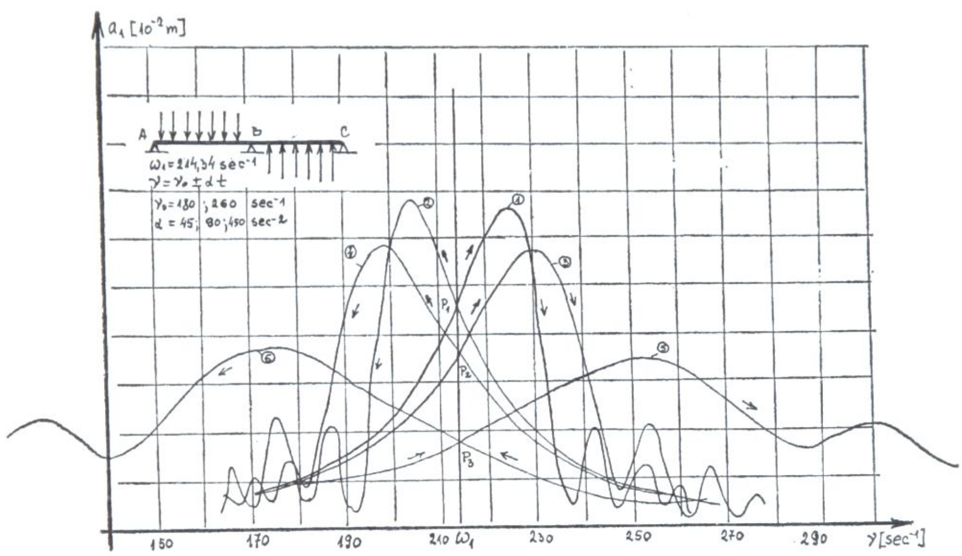


Fig. No. 4

change of frequency of the external force. It may be seen on the graph that the largest maximums of amplitudes are largest at lower velocities of change of frequency of the external force, namely that they are the smallest at highest velocities of change of fiequency of the external force. The conclusion is that from the view of development of the maximum amplitudes the dangerous cases are the slow passes through resonant state of vibrations. On these graphs it may be seen that amplitude-frequency curves of the nonstationary resonant state for the increase and decrease of the frequency by same velocities for the linear case vibrations cut on the line $\nu = \omega_1 = 214{,}34\ \mathrm{sec}^{-1}$, but the graphs are symmetrical if the initial conditions are conjugated. For higher velocities of the pass through resonant state of the linear system the maximums of amplitude-frequency curves for the increase and decrease of the trequencies of compelling force approach the line $\nu = \omega_1 = 214{,}34\ \mathrm{sec}^{-1}$.

In Fig. No. 5 are shown amplitude-frequency curves for the case of the twospan beam in terms of the influence on the nonlinear law of elasticity of material of a beam and when the frequency of the external force increases or

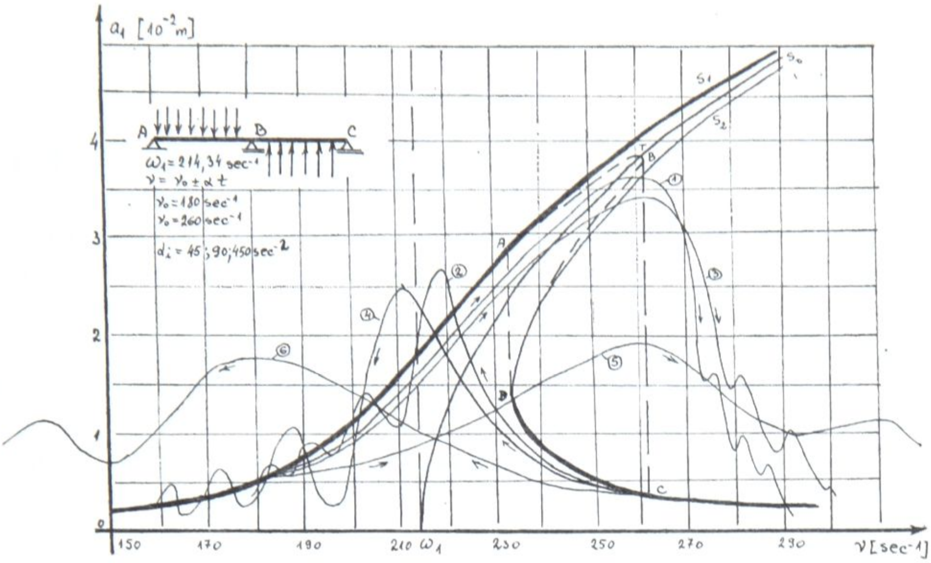


Fig. No. 5

decreases for three velocities of the change of the frequency $\alpha_i = 45$; 90 and 450 sec^{-2}. As in the linear case, the first and highest maximums of amplitude-frequency curves move towards higher amplitudes and for lower velocities of the change of frequency of the external force, but they move towards lower amplitudes for higher velocities of the change of the frequency. The same happens also in the case of decrease of the frequency of the external force In the case of the change of frequency of the external force by velocity of 45 sec^{-2} the resonant range will be passed for about 100 periods of unperturbed proper linear viabrations of twospan beam, but for other two velocities of the change of frequency of the external force, it would be passed for about 50 (for $\alpha_2 = 90$ sec^2), namely for 10 (for $\alpha_3 = 450$ sec^{-2}) periods of unperturbed proper vibrations.

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ОДНОЧАСТОТНЫЕ НЕЛНЕЙНЫЕ ВЫНУЖДЕННЫЕ КОЛЕБАНИЯ НЕПРЕРЫВНОЙ БАЛКИ

Катица (Стеванович) Хедрих

Резюме

В работе применяется энергетический метод Крюлова-Боголюбова--Митроиольского для построения асимптотического решения и системы дифференцциальных уравнений первого приближения для амплитуды и фазы одночастотного режима поперечных вынужденых колебаний непрерывной балки. Построены общие выражения для случая произвольных краевых условиях непрерывной балки, если нам известны фундаментальные функции и собственные частоты невозмущеной формы поперчниых колебаний непрерывной балки.

Построено выражение для величину средней выртулальной работы которую совершает внешная возмущающая вынужденая сила которая действует на непрерывную балку.

Частота внешней вынужденой силы медленноизменяющаяся функция времени. С помощью построеных уравнений изучаются линейне и нелинейние вынужденые поперечные одночатотные колебания двух пролётных балок для случая стационарного и нестационарного резонанса, и для случая растения и уманьшения частоты силы.

ЈЕДНОФРЕКВЕНТНЕ НЕЛИНЕАРНЕ ПРИНУДНЕ ОСЦИЛАЦИЈЕ КОНТИНУАЛНЕ ГРЕДЕ

Катица (Стевановић) Хедрих

Резиме

У овом раду искоришћена је енергијска интерпретација асимпототске методе Крилова-Богољубова — Митропољског за састављање прве апроксимације решења и система диференцијалних једначина прве апроксимације за амплитуду и фазу једнофреквентног режима принудних трансверзалних осцилација континуалних греда. Дати су општи обрасци за произвољан случај линеарних граничних услова континуалне греде, ако је познат систем сопствених функција и сопствених кружних фреквенција непоремећеног облика осциловања континуалног носача.

Дат је израз за средњу вредност виртуалног рада поремећајне принудне силе која дејствује иа греду. Тренутна кружна фреквенција спољашње силе је споро-промењљива функција времена. Помоћу изведених једначина анализирају се линеарне и нелинеарне принудне трасверзалне једнофреквентне осцилације континуалних греда за случај стационарног и нестационарног резонантног стања. Дати су амплитудно-фреквентни графици за случај стационарног и нестационарног резонантног стања, а у условима раста и опадања фреквенције спољашње снле.