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HOMOGENEOUS DEFORMATION PROCESS WITH MINIMAL STRESS WORK

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1. Introduction

For simple elastic materials, for which the stress tensor is not derivable from a potential (Elastic but not Hyper-elastic materials), the stress work in a homogeneous deformation process depends on the deformation path, not only on the initial $\tilde{F}_0$ and final $\tilde{F}_1$ value of the deformation gradient. In this note we analyse the conditions that the homogeneous deformation process must satisfy in order that the stress work in this deformation process is minimal, when compared with other homogeneous deformation processes connecting $\tilde{F}_0$ and $\tilde{F}_1$. The analysis presented will reveal an interesting property of the elasticity tensor.

2. Formulation of the Problem

Consider a homogeneous simple elastic body $\mathscr{B}$ that is subjected to such system of forces that it deforms homogeneously from an initial configuration $\varkappa_0$ to another configuration $\varkappa$. Let $\tilde{F}_0$ denote the value of the deformation gradient in $\varkappa_0$. If $\tilde{F}_1$ is the value of the deformation gradient in $\varkappa$ and if $[t_0, t_1]$ denote the time interval in which the deformation takes place, then the stress work is given by [1]

$$(1) \qquad W_{01} = V \int_{\tilde{F}_0}^{\tilde{F}_1} \mathrm{tr}\,(\tilde{T}^T\, d\tilde{F}) = V \int_{t_0}^{t_1} f_{ij}\, \mathring{F}_{ij}\, dt$$

Here $\tilde{T}$ is the first Piola-Kirchhoff stress tensor determined by the constitutive equation of the material

$$(2) \qquad T_{ij} = f_{ij}\,(F_{km}),$$

V is the volume of $\mathscr{B}$ in $\varkappa_0$, superimposed dots denote the derivative taken with respect to time, $(\cdot)^T$ denotes transpose and tr $(\cdot)$ the trace operation.

Every homogeneous deformation process can be characterized by a one-parameter family of deformation gradients $\tilde{F}(t)$, $t \in [t_0, t_1]$. While $\mathscr{B}$ is moving from $\varkappa_0$ to $\varkappa$, $\tilde{F}(t)$ describes a path in 9-dimensional space $\mathscr{L}$ of all second order tensors [2, p. 305]. We state now the following problem: Find a path in $\mathscr{L}$ connecting $\tilde{F}_0$ and $\tilde{F}_1$ such that on it the expression (1) reaches its minimum. We note that a problem of this type was treated in [3] for one-dimensional linear viscoelastic material.

In solving the problem just posed, we will distinguish two special cases:

1. The material of the body $\mathcal{B}$ is Hyper-elastic with $\sigma(F_{ij})$ as strain energy function. Then (1) reduce to

$$W_{01} = V\,[\sigma(\underset{\sim}{F}_1) - \sigma(\underset{\sim}{F}_0)], \tag{3}$$

that is the stress work is independent of the path in $\mathcal{L}$ that connects $\underset{\sim}{F}_0$ and $\underset{\sim}{F}_1$.

2. The material of the body is simple elastic material without strain energy function. We want to find the path that minimizes (1) by use of the Pontryagin principle [4]*. In this setting, the problem becomes: minimize the functional

$$W_{01} = V \int_{t_0}^{t_1} f_{ij}(F_{km})\, U_{ij}\, dt \tag{4}$$

where U_{ij} is the "control", in terms of the Optimal Control Theory, subject to the following differential constraint

$$\dot{F}_{ij} = U_{ij} \qquad U_{ij} \in \mathcal{M}. \tag{5}$$

The control vector U_{ij} is assumed to belong to the set $\mathcal{M}$ of all admissible control vectors. Restrictions that we may impose (such as boundness of $\dot{F}_{ij}(t)$, incompressibility ect.) on paths in $\mathcal{L}$ that connect F_0 and $\underset{\sim}{F}_1$ determine the structure of $\mathcal{M}$. The maximum principle of Pontryagin requests construction of the Hamiltonian [4]

$$H = f_{ij}\, U_{ij} + \lambda_{ij}\, U_{ij}, \tag{6}$$

where λ_{ij} are the constate variables, determined, by the system of adjoint differential equations

$$\dot{\lambda}_{ij} = -\frac{\partial H}{\partial F_{ij}}. \tag{7}$$

The optimal U_{ij} should be chosen so that the Hamiltonian H given by (6) is minimized

$$H(\hat{U}_{ij} = U_{ij} \text{ optimal}) \leqslant H(U_{ij}) \qquad U_{ij} \in \mathcal{M}. \tag{8}$$

Thus, the equation (5), (6) and (8) characterize the regular optimal path completely. However, since the Hamiltonian (6) is linear function of the control U_{ij}, the problem we are dealing with is singular [6]. The optimality condition for the singular optimal path reads

$$f_{ij} + \lambda_{ij} = 0. \tag{9}$$

Using (6) in (7) we obtain for the adjoint system of differential equations

$$\dot{\lambda}_{ij} = -\frac{\partial f_{rs}}{\partial F_{ij}}\, U_{rs} = -\frac{\partial f_{rs}}{\partial F_{ij}}\, \dot{F}_{rs}, \tag{10}$$

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* Alternatively the problem could be solved by the methods of the Classical Variational Calculus in which case it belongs to the class of problems with degenerate functions [5].

where the last identity follows after the use of (5). Differentiating (9) with respect to time and using (10) to eliminate $\dot{\lambda}_{ij}$ we get

$$U_{rs}\left[\frac{\partial f_{rs}}{\partial F_{ij}}-\frac{\partial f_{ij}}{\partial F_{rs}}\right]=0. \tag{11}$$

From (11) and the condition that on the singular optimal path the Hamiltonian does not depend on U_{ij} [6] we obtain the following:

Theorem: *A necessary condition that a homogeneous deformation process with the deformation gradient* $\underset{\sim}{F}^*(t)$ *is a singular solution to the optimization problem* (4), (5) *is that* $\underset{\sim}{F}^*(t)$, $t\in[t_0, t_1]$ *satisfies the following equation*:

$$\dot{F}^*_{rs}[A_{rsij}(F^*_{km})-A_{ijrs}(F^*_{km})]=0, \tag{12}$$

where $A_{ijrs}(\cdot)$ *are components of the elasticity tensor* [2].

Note that the equation (12) is an identity satisfied for any F_{km} in the case of Hyper-elastic material, due to the symmetry property of the elasticity tensor $A_{rsij}=A_{ijrs}$. Since A_{ijrs} is the fourth order tensor in the space of dimension 3, there are at most 9 non-trivial equations (12). If they are such that $M<9$ components of the deformation gradient could be expressed in terms of time then there is possibility for the existence of a singular solution to the problem (4), (5). Of course the question that still remains open is how singular and regular arcs should be combined so that they connect in advance the given points $\underset{\sim}{F}_0$ and $\underset{\sim}{F}_1$. However, this and the question of determining the sufficient conditions for the existence of local minimum will not be treated here.

3. The controllability of the optimal solutions

Let us suppose that the optimal deformation path $\underset{\sim}{F}^*(t)$, $t\in[t_0, t_1]$ that connects points $\underset{\sim}{F}_0$ and $\underset{\sim}{F}_1$ in the space $\mathscr{L}$ is obtained. If the motion $\chi^*(t)$ with the deformation gradient $\underset{\sim}{F}^*(t)$ could be produced by application of surface tractions only, then the optimal solution $\underset{\sim}{F}^*(t)$ is controllable. Thus in order to be able to deform the body $\mathscr{B}$ along the optimal path by application of surface tractions only $\chi^*(t)$ must satisfy the equations of motion with zero body force. For compressible elastic materials this condition is that the motion $\chi^*(t)$ must be accelerationless [1, p. 193], while for incompressible elastic materials the motion $\chi^*(t)$ must be such that the acceleration $\ddot{\chi}^*(t)$ is lamellar field [7, p. 318]. Therefore we conclude that:

i. the optimal solution $\underset{\sim}{F}^*(t)$ is controllable for compressible elastic materials if it is of the form $\underset{\sim}{F}^*(t)=\underset{\sim}{F}_0(1+t\underset{\sim}{F})$, where $\underset{\sim}{F}_0$ and $\underset{\sim}{F}$ are constant tensors.

ii. the optimal solution $\underset{\sim}{F}^*(t)$ is controllable for incompressible elastic materials if the tensor $\underset{\sim}{F}^*(t)$ is unimodular for all $t\in[_0, t_1]$ and if

$$(\ddot{\underset{\sim}{F}}^*)^T\underset{\sim}{F}^*=(\underset{\sim}{F}^*)^T\ddot{\underset{\sim}{F}}^*. \tag{13}$$

4. Conclusion

The analysis presented above shows that the possibility of existence of a homogeneous deformation process $\tilde{F}^*(t)$, $t \in [t_0, t_1]$ for which the stress work is in minimum, when compared with other homogeneous deformation processes that connect two in advance given points $\tilde{F}_0$ and $\tilde{F}_1$, is the property that distinguish elastic simple materials from hyperelastic ones. The candidates for the singular optimal paths are the solutions of the equation (12). This is a new property of the elasticity tensor.

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DIE HOMOGENE DEFORMATIONSPROZESS MIT MINIMAL DEFORMATIONSARBEIT

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Zusammenfassung

Die Arbeit analisiert jene Bedingungen die ein homogenes Deformationsprozess erfüllen soll damit die Deformationsarbeit während des Deformationsprozess minimal wird.

Es wurde auf eine neue Eigenschaft des Elastizitätstensors hingewiesen die mit den optimalen Singularlösungen verbunden ist.

ХОМОГЕНИ ДЕФОРМАЦИОНИ ПРОЦЕС СА МИНИМАЛНИМ ДЕФОРМАЦИОНИМ РАДОМ

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Резиме

У раду су анализирани услови које треба да задовољи хомогени деформациони процес да би деформациони рад током тог деформационог процеса био минималан.

Указано је на једну нову особину тензора еластичности која је повезана са сингуларним оптималним решењима.

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