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THERMOELASTIC DIPOLAR CONTINUUM

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1. Introduction

The theory of finite thermal deformations was derived by R. Stojanović et al [1], who separated the total compatible deformation into two incompatible ones-thermal and elastic. However, in that paper no constitutive assumptions were made, but they were taken into account in [2], where a stress relation was formed for nonlinear thermoelasticity. In [2], an analysis was also made of constitutive equations for isotropic media. A stress-strain relation for nondissipative processes is given in [3], with stress couples included. After a detailed analysis of a geometrical treatment of the problem of thermoelastic materials without structural defects, the compatibility conditions for nonlinear thermoelasticity were developed by M. Mićunović [4].

At the same time appeared some new theories of the mechanics of continuous media. A general theory of microelastic materials, considered as a generalized Cosserat continuum was presented by M. Plavšić and J. Jarić [5]. Nonlinear constitutive equations were derived by means of the principle of virtual work. In the case of isotropic materials these equations were linearized. Consequently, micropolar, dipolar and polar theories were derived.

The present paper analyses thermoelastic deformations of a dipolar continuum from a geometrical point of view. The continuum deformation will be completely determined by deformation gradiants, since in the dipolar theory the vectors joined to continuum points are material vectors. Besides, the temperature field is assumed to be known.

2. The structure of a geometrical model of the thermoelastic deformation process

Let $\mathscr{B}$ be the part of space filled up with continuously distributed material. At the moment t_0, the body $\mathscr{B}$ is at its initial Euclidian configuration K_0, with some constant reference temperature ϑ_0. Let us first assume that the body $\mathscr{B}$ is constituted of material volume elements which can be independently deformed. The body $\mathscr{B}$ in its reference confiquration is heated without external forces in such a way that the temperature changes from element to element, but is constant in each of them all the time. Then the body $\mathscr{B}$, from the reference configuration K_0 comes to the unstressed, thermally deformed configuration K_1, composed of variously deformed volume elements. As a consequence, we get a non-Euclidian configuration K_1, because thermal deformations induced by unequal heating are incompatible. However, since any non-Euclidian configuration of the body $\mathcal{B}$ is only imaginary and has no physical meaning, the body $\mathcal{B}$ in configuration K_1 must undergo an addicnal incompatible deformation, which causes stresses inside the body and transforms the non-Euclidian configuration K_1 into the Euclidian deformed configuration K.

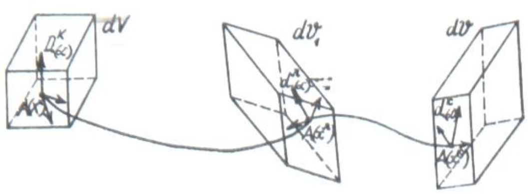

Fig 1

Referring to a system of coordinates X^K, let us consider an arbitrary point $A(X^k)$ inside an arbitrary volume element dV of the reference configuration K_0, and at this point three noncomplanar vectors $D^K_{\cdot(\alpha)}$, $\alpha=1, 2, 3$.

The described thermal deformation transforms element dV into dv_1, point $A(X^K)$ into $A(x^\lambda)$, and a vector triad $D^K_{\cdot(\alpha)}$ into $d^\lambda_{\cdot(\alpha)}$. Since K_0 is a Euclidian and K_1 non-Euclidian configuration only a local correspodence is established between them through the following thermal distortions $\theta^\lambda_{\cdot K}$:

$$d^\lambda_{\cdot(\alpha)}=\theta^\lambda_{\cdot(K)}D^K_{\cdot(\alpha)} \tag{2.1}$$

However, the deformation of the given vector triad is not independent of the continuum points displacement, and hence the deformation of dV (i.e deformation of the body $\mathcal{B}$ as a whole) in the transformation $K_0\to K_1$ is completely defined by the thermal distortions:

$$\theta^\lambda_{\cdot K}=d^\lambda_{\cdot K}D^{(\alpha)}_{\cdot K} \tag{2.2}$$

for, knowing $\theta^\lambda_{\cdot K}$, we also know the deformation of the arbitrary fourth vector:

$$d^\mu=\theta^\mu_{\cdot K}D^K \tag{2.3}$$

If we look at the coordinates D^K, which refer to the point $A(X^K)$, as differentials:

$$D^K=d\,\xi^K \tag{2.4}$$

it is evident that the linear differential expressions $\theta^\lambda_{\cdot K}d\,\xi^K$ are not exact differentials. If we introduce:

$$d^\lambda=dn^\lambda \tag{2.5}$$

then, the equation (2.3) takes the form:

$$dn^\lambda=\theta^\lambda_{\cdot K}d\,\xi^K \tag{2.6}$$

These conrdinates n^λ are not true, but non-holonomic.

The transformation of K_1 into K by an elastic incompatible deformation will be now referred to the system of curvilinear coordinates x^k. Then the element dv_1 is transformed into dv, the point $A(x^\lambda)$ into $A(x^k)$, and the vector triad $d^\lambda_{\cdot\alpha}$ into $d^k_{\cdot(\alpha)}$:

$$d^k_{\cdot(\alpha)} = \Phi^k_{\cdot\lambda}\, d^\lambda_{\cdot(\alpha)} \tag{2.7}$$

Since these vectors are material ones, the transformation $K_1 \to K$ is completely defined by elastic distortions:

$$\Phi^k_{\cdot\lambda} = d^K_{\cdot(\alpha)}\, d^{(\alpha)}_{\cdot\lambda} \tag{2.8}$$

From (2.1) and (2.7) it is evident that:

$$x^k_{;K} = \Phi^k_{\cdot\lambda}\, \theta^\lambda_{\cdot K}; \qquad X^K_{;k} = \theta^K_{\cdot\lambda}\, \Phi^\lambda_{\cdot k} \tag{2.9}$$

because:

$$d^k_{\cdot(\alpha)} = \Phi^k_{\cdot\lambda}\, \theta^\lambda_{\cdot K}\, D^K_{\cdot(\alpha)}; \qquad D^K_{\cdot(\alpha)} = \theta^K_{\cdot\lambda}\, \Phi^\lambda_{\cdot k}\, d^k_{\cdot(\alpha)} \tag{2.10}$$

where $\underset{\sim}{X}$ are material coordinates and $\underset{\sim}{x}$ are spatial Euclidian ones.

Since the difference between K_0 and K_1 arises only from different temperatures v_0 and v_1, thermal distortions must be functions of the local temperature. That dependence is supposed to be known:

$$\theta^\lambda_{\cdot k} = \theta^\lambda_{\cdot K}[\theta(X^K)] \tag{2.11}$$

Then it follows from (2.1):

$$\dot{d}^\lambda_{\cdot(\alpha)} = 0 \tag{2.12}$$

while from (2.7), (2.12), (2.1), $(2.9)_2$, $(2.10)_2$ we get:

$$\dot{d}^k_{\cdot(\alpha)} = \dot{\Phi}^k_{\cdot\mu}\, \Phi^\mu_{\cdot e}\, d^e_{\cdot(\alpha)}; \qquad \dot{d}^k_{\cdot(\alpha)} = v^k_{,l}\, d^l_{\cdot(\alpha)} \tag{2.13}$$

where:

$$\dot{\Phi}^k_{\cdot\mu}\, \Phi^\mu_{\cdot l} = v^k_{\cdot l} \tag{2.14}$$

Let us further assume that the mass of the body remains constant during the deformation:

$$dm = \rho_0\, dV = \rho\, dv; \qquad \dot{\overline{dm}} = 0 \tag{2.15}$$

where ρ_0 and ρ are mean densities of dV and dv.

Kinetic energy can be expressed as in [5]:

$$2T = \int_v \rho\,(v^i v_i + I^{\alpha\beta}\, \dot{d}^i_{\cdot(\alpha)}\, \dot{d}_{i(\beta)}\, dv, \tag{2.16}$$

where the coefficients $I^{\alpha\beta}$ are defined by:

$$I^{\alpha\beta} = I^{\beta\alpha} = I^{KL}\, D^{(\alpha)}_{\cdot K}\, D^{(\beta)}_{\cdot L}, \tag{2.17}$$

while the coefficients of inertia I^{KL} refer to the center of mass of the given element. Then:

$$\dot{T} = \int_v \rho\,(\dot{v}^i v_i + I^{\alpha\beta}\, \ddot{d}^i_{\cdot(\alpha)}\, \dot{d}_{i(\beta)})\, dv \tag{2.18}$$

which, making use of (2.13), can be written as:

(2.19) $$\dot{T} = \int_v \rho \left(\dot{v}^i v_i + \Gamma^{ij} \dot{\Phi}_{i\mu} \Phi^{\mu}_{\cdot j}\right) dv$$

i.e., by using (2.15):

(2.20) $$\dot{T} = \int_v \rho \left(\dot{v}^i v_i + \Gamma^{ij} v_{i,j}\right) dv$$

where the inertial spin is:

(2.21) $$\Gamma^{ij} = I^{\alpha\beta} \ddot{d}^{i}_{\cdot(\alpha)} d^{j}_{\cdot(\beta)}$$

Using (2.13) and (2.21), we get:

(2.22) $$\Gamma^{ij} = i^{kj} \left(\ddot{\Phi}^{i}_{\cdot\mu} \Phi^{\mu}_{\cdot k} + \dot{\Phi}^{i}_{\cdot\mu} \Phi^{\mu}_{\cdot k} + \dot{\Phi}^{i}_{\cdot\mu} \Phi^{\mu}_{\cdot l} \dot{\Phi}^{l}_{\cdot\lambda} \Phi^{\lambda}_{\cdot k}\right)$$

where:

(2.23) $$i^{kj} = I^{\alpha\beta} d^{k}_{\cdot(\alpha)} d^{j}_{\cdot(\beta)}$$

From this it can be seen that the inertial spin is completely determined by the elastic distortions and quantities i^{kj}.

3. Equations of energy balance and motion

We suppose that the body $\mathscr{B}$ is acted upon by the surface forces T^i and H^{ij}, as well as the volume forces f^i and l^{ij}. Then the effect of work of the surface and volume forces is:

(3.1) $$\dot{A} = \oint_s \left(T^i v_i + H^{i(\alpha)} \dot{d}_{i(\alpha)}\right) ds + \int_v \rho \left(f^i v_i + l^{i(\alpha)} \dot{d}_{i(\alpha)}\right) dv$$

where s is the closed surface which envelops the volume v, while the quantities $H^{i(\alpha)}$ and $l^{i(\alpha)}$ are defined by the following relations:

(3.2) $$H^{i(\alpha)} = H^{ij} d^{(\alpha)}_{\cdot j}; \qquad l^{i(\alpha)} = l^{ij} d^{(\alpha)}_{\cdot j}$$

where:

H^{ij} — is the first surface moment

l^{ij} — first volume moment.

Using (2.13) and (3.2), (3.1) can be written in the form:

(3.3) $$\dot{A} = \oint_s \left(T^i v_i + H^{ij} v_{i,j}\right) ds + \int_v \rho \left(f^i v_i + l^{ij} v_{i,j}\right) dv$$

The effect of nonmechanical work is given by:

(3.4) $$Q = \oint_s q\, ds + \int_v \rho\, h\, dv$$

where q is the heat flux and h — specific energy production. Taking into account (2.15), the time derivation of specific internal energy ε can be written in the form:

$$\dot{E}=\int_{v}\rho\dot{\varepsilon}\,dv \tag{3.5}$$

Using (2.20), (3.3), (3.4) and (3.5), the first law of thermodynamics can be written as:

$$\begin{aligned}&\int_{v}\rho\,(\dot{v}\,v_i+\Gamma^{ij}v_{i,j})\,dv+\int_{v}\rho\dot{\varepsilon}\,dv=\oint_{s}T^i v_i\,ds+\oint_{s}H^{ij}v_{i,j}\,ds+\\&+\int_{v}\rho f^i v_i\,dv+\int_{v}\rho\,l^{ij}v_{i,j}\,dv+\int_{s}q\,ds+\int_{v}\rho\,h dv\end{aligned} \tag{3.6}$$

If the surface integrals in (3.6) are transformed into the volume ones, because the equation is valid for an arbitrary element, we get:

$$\begin{aligned}&\rho\,\dot{v}^i v_i+\rho\,\Gamma^{ij}v_{i,j}+\rho\dot{\varepsilon}=t^{ij}_{,j}v_i+t^{ij}v_{i,j}+h^{ijk}_{,k}v_{i,j}+\\&+h^{ijk}v_{i,jk}+q^i_{,i}+\rho f^i v_i+\rho\,l^{ij}v_{i,j}+\rho\,h.\end{aligned} \tag{3.7}$$

which represents a local equation of the total energy balance, where t^{ij} is nonsymmetric stress tensor, and h^{ijk} — the first stress moment.

We shall now postulate the invariance of equation (3.7) in respect to superposed rigid motions. If at a position corresponding to the moment t the translation velocity, which is constant, is superposed on the velocities of the body points, then it could be shown that all quantities in (3.7) remain unchanged with the exception of v_i, which should be replaced by v_i+a_i, where $a_i=\text{const}$. In order that equation (3.7) may stay invariant with regard to such a superposition, the following relation must be satisfied:

$$\rho\,\dot{v}_i=t^{ij}_{,j}+\rho f^i. \tag{3.8}$$

This is the first Cauchy law of motion. Thus, (3.7) becomes:

$$\rho\,\Gamma^{ij}v_{i,j}+\rho\dot{\varepsilon}=t^{ij}v_{i,j}+h^{ijk}_{,k}v_{i,j}+h^{ijk}v_{i,jk}+q^i_{,i}+\rho\,l^{ij}v_{i,j}+\rho\,h. \tag{3.9}$$

If, now, at the considered position which corresponds to the moment t we superpose the angular velocity of rotation, which is constant, upon the velocities of the points of the body, then it could be shown that all quantities in (3.9) remain unchanged with the exception of quantities $v_{i,j}$ and $\Gamma^{ij}v_{i,j}$. The quantities $v_{i,j}$ and $\Gamma^{ij}v_{i,j}$ in (3.9) should be replaced by $v_{i,j}+\Omega_{ij}$ and $\Gamma^{ij}(v_{i,j}+\Omega_{ij})$, where Ω_{ij} represents an arbitrary constant antisymmetric tensor. In order that (3.9) may be invariant on this superposition, the following relation must be fulfilled:

$$\tau^{ij}=t^{ij} \tag{3.10}$$

where τ^{ij} denotes:

$$\tau^{ij}=t^{ij}+h^{ijk}_{,k}+\rho\,(l^{ij}-\Gamma^{ij}) \tag{3.11}$$

Thus, (3,7) becomes:

$$\rho\dot{\varepsilon} = \tau^{ij} v_{i,j} + h^{ijk} v_{i,jk} + q^{i}_{,i} + \rho h. \tag{3.12}$$

The equation obtained in that way, which describes the rate of change of specific internal energy, is invariant in respect to the superposed rigid motions. For reversible processes, the equation of specific entropy production becomes:

$$\rho\theta\dot{\eta} = q^{k}_{,k} + \rho h \tag{3.13}$$

where θ is temperature and η specific entropy. So (3.12) can be written as:

$$\rho\dot{\varepsilon} = \tau^{ij} v_{i,j} + h^{ijk} v_{i,jk} + \rho\theta\dot{\eta} \tag{3.14}$$

which, by means of (2.15), may be expressed in the form:

$$\rho\dot{\varepsilon} = \tau^{ij} \dot{\Phi}_{i\mu} \Phi^{\mu}_{\cdot j} + h^{i(jk)} (\dot{\Phi}_{i\mu;k} \Phi^{\mu}_{\cdot j} + \dot{\Phi}_{i\mu} \Phi^{\mu}_{\cdot j;k}) X^{k}_{;k} + \rho\theta\dot{\eta}. \tag{3.15}$$

This equation speaks about the quantities which could be determined from the constitutive equations. These are τ^{ij}, $h^{i(jk)}$ and θ. It is evident that from the constitutive equations it is impossible to find stress tensor t^{ij} directly, but it must be determined from the system (3.11). However, since the temperature field is assumed to be known, the constitutive equation for θ is quite needless. Consequently, the dependence of ε on η does not play an important role, and (3.15) becomes:

$$\rho \dot{W} = \tau^{ij} \dot{\Phi}_{i\mu} \Phi^{\mu}_{\cdot j} + h^{i(jk)} \left(\dot{\Phi}_{i\mu;k} \Phi^{\mu}_{\cdot j} + \dot{\Phi}_{i\mu} \Phi^{\mu}_{\cdot j;k}\right) X^{k}_{;k} \tag{3.16}$$

4. Constitutive equations

From (3.16) we conclude that the specific energy of deformation w is a function of elastic distortions only:

$$W = W\left(\Phi^{l}_{\cdot\mu}; \Phi^{l}_{\cdot\mu;k}\right) \tag{4.1}$$

so that:

$$\dot{W} = \frac{\partial W}{\partial \Phi^{l}_{\cdot\mu}} \dot{\Phi}^{l}_{\cdot\mu} + \frac{\partial W}{\partial \Phi^{l}_{\cdot\mu;k}} \dot{\Phi}^{l}_{\cdot\mu;k} \tag{4.2}$$

By comparing the coefficients of independent velocities $\dot{\Phi}_{i\mu}$ and $\dot{\Phi}_{i\mu;k}$ in equations (3.15) and (4.2), we get the constitutive equations:

$$\begin{aligned} \tau^{ij} &= \rho\, g^{il} \left(\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu}} \Phi^{j}_{\cdot\mu} + \frac{\partial W}{\partial \Phi^{l}_{\cdot;\mu k}} \Phi^{j}_{\cdot\mu;k} \right) \\ h^{i(jk)} &= \rho\, g^{il} \frac{\partial W}{\partial \Phi^{l}_{\cdot\mu;k}} \Phi^{(j}_{\cdot\mu} x^{k)}_{;k} \end{aligned} \tag{4.3}$$

which, by means of $(2.9)_1$, can be written as:

$$\tau^{ij}=\rho\, g^{il}\left(\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu}}\Phi^{j}_{\cdot\mu}+\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu\lambda}}\Phi^{j}_{\cdot\mu\lambda}\right)$$

(4.4)

$$h^{i(jk)}=\rho\, g^{il}\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu\lambda}}\Phi^{(j}_{\cdot\mu}\Phi^{k)}_{\cdot\lambda}$$

where:

(4.5) $$\Phi^{l}_{\cdot\mu\lambda}=\Phi^{l}_{\cdot\mu;\,k}\,\theta^{k}_{\cdot\lambda}$$

is introduced as a formal notation in order to write constitutive equations as concisely as possible. If the second Cauchy law (3.10) is to be satisfied, the condition:

(4.6) $$\left(g^{il}\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu}}\Phi^{j}_{\cdot\mu}+g^{il}\frac{\partial W}{\partial \Phi^{l}_{\cdot\mu\lambda}}\Phi^{j}_{\cdot\mu\lambda}\right)_{[ij]}=0$$

must be fulfilled, which for the specific energy of deformation and constitutive equations represents an invariance condition as related to the superposed rigid motions.

From (4.4) we see that the specific energy of deformation is a function of the form:

(4.7) $$W=W(\Phi^{l}_{\cdot\mu};\quad \Phi^{l}_{\cdot\mu\lambda})$$

wherefrom it can be seen that it depends on $9+27=36$ variables. However we have $3+9$ partial differential equations τ^{ij} and $h^{i(jk)}$, and these will have $36-12=24$ fundamental integrals. We shall take as fundamental integrals:

$$C^{E}_{\lambda\mu}=g_{ij}\,\Phi^{i}_{\cdot\lambda}\,\Phi^{j}_{\cdot\mu}$$

(4.8)

$$D^{E}_{\lambda\mu\nu}=g_{ii}\,\Phi^{i}_{\cdot\lambda}\,\Phi^{j}_{(\mu\nu)}=\frac{1}{2}\,g_{ij}\,\Phi^{i}_{\cdot\lambda}\left(\Phi^{j}_{\cdot\mu\lambda}+\Phi^{j}_{\cdot\lambda\mu}\right)=D^{E}_{\lambda\mu\nu}$$

so that the general solution of system of partial differential equations reads:

(4.9) $$W=W(C^{E}_{\lambda\mu};\quad D^{E}_{\lambda\mu\nu})$$

Substituting (4.9) in (4.4), and using (4.8), we get the constitutive equations:

$$\tau^{ij}=2\,\rho\,\frac{\partial W}{\partial C^{E}_{\lambda\mu}}\Phi^{i}_{\cdot\lambda}\,\Phi^{j}_{\cdot\mu}$$

(4.10)

$$h^{i(jk)}=\rho\,\frac{\partial W}{\partial C^{E}_{\lambda\mu\nu}}\Phi^{i}_{\cdot\lambda}\,\Phi^{i}_{\cdot\mu}\,\Phi^{k}_{\cdot\nu}$$

They are also invariant with regard to the rigid motions.

5. Isotropy

For isotropic materials the following space tensor elastic deformation can be introduced:

$$(5.1)\qquad \begin{aligned} c^{E}_{kp} &= g_{\mu\nu}\,\Phi^{\mu}_{\cdot k}\,\Phi^{\nu}_{\cdot p} = \Phi_{\lambda k}\,\Phi^{\lambda}_{\cdot p} \\ d^{E}_{kpu} &= \frac{1}{2}\,g_{rk}\,\Phi^{r}_{\cdot\lambda\mu}\left(\Phi^{\lambda}_{\cdot p}\,\Phi^{\nu}_{\cdot m} + \Phi^{\lambda}_{\cdot m}\,\Phi^{\nu}_{\cdot p}\right) \end{aligned}$$

In this case, the constitutive equations (4.4) could be written as:

$$(5.2)\qquad \begin{aligned} \tau^{ij} &= -2\rho\,\frac{\partial W}{\partial c^{E}_{kj}}\,c^{E\cdot i}_{k} - 2\rho\,\frac{\partial W}{\partial d^{E}_{kpj}}\,d^{E\cdot\cdot i}_{kp} + \rho\,\frac{\partial W}{\partial d^{E}_{ipm}}\,d^{Ej}_{\cdot pm} \\ h^{i(jk)} &= \frac{\partial W}{\partial d^{E}_{ijk}} \end{aligned}$$

where the condition of objectivity must be satisfied. Since the spatial tensor of elastic deformation c^{E}_{kp} may be replaced by a corresponding tensor of relative deformation:

$$(5.3)\qquad 2\,e^{E}_{kp} = g_{kp} - c^{E}_{kp}$$

the constitutive equations (5.2) could be written in the form:

$$(5.4)\qquad \begin{aligned} \tau^{ij} &= \rho\left(\frac{\partial W}{\partial e^{E}_{ij}} - 2\,\frac{\partial W}{\partial e^{E}_{kj}}\,e^{E\cdot i}_{k} - 2\,\frac{\partial W}{\partial d^{E}_{kpj}}\,d^{E\cdot\cdot i}_{kp} + \frac{\partial W}{\partial d^{E}_{ipm}}\,d^{Ej}_{\cdot pm}\right) \\ h^{i(jk)} &= \frac{\partial W}{\partial d^{E}_{ijk}} \end{aligned}$$

where the condition of objectivity must be satisfied:

$$(5.5)\qquad \left(-2\,\frac{\partial W}{\partial e^{E}_{kj}}\,e^{E\cdot i}_{k} - 2\,\frac{\partial W}{\partial d^{E}_{kpj}}\,d^{E\cdot\cdot i}_{kp} + \frac{\partial W}{\partial d^{E}_{ipm}}\,d^{Ej}_{\cdot pm}\right)_{[ij]} = 0$$

Using (5.3), (5.1) and (4.9), as well as the relations which correlate the deformation and displacement gradients:

$$(5.6)\qquad X^{K}_{;p} = g^{K}_{p} - u^{K}_{;p}; \qquad x^{r}_{;M} = g^{r}_{M} + u^{r}_{;M}$$

we can express the deformation tensors e^{E}_{kp} i d^{E}_{kpm} in linear approximations:

$$(5.7)\qquad \begin{aligned} e^{E}_{kp} &= \frac{1}{2}\,(u_{k,p} + u_{p,k}) - \alpha\theta\,g_{kp} \\ d^{E}_{kpm} &= u_{k,pm} - \alpha\,g_{k(p}\theta_{,m)}. \end{aligned}$$

If the temperature increment is sufficiently small, $\theta^{\alpha}_{\cdot k}$ can be approximated by:

$$\theta^{\alpha}_{\cdot k} = (1 + \alpha\theta)\,\delta^{\alpha}_{k} \tag{5.8}$$

where α is the coefficient of thermal propagation. However, by analogy with [2] we can write:

$$\overset{T}{e}_{kp} + \overset{E}{e}_{kp} = e_{kp}; \qquad \overset{T}{d}_{kpm} + \overset{E}{d}_{kpm} = d_{kpm} \tag{5.9}$$

so that:

$$\begin{aligned} &\overset{T}{e}_{kp} = \alpha\theta\, g_{kp};\ e_{kp} = \frac{1}{2}(u_{k,p} + u_{p,k}); \\ &d_{kpm} = u_{k,pm};\ \overset{T}{d}_{kpm} = \alpha\, g_{k(p}\theta_{,m)} \end{aligned} \tag{5.10}$$

If nonlinear terms in constitutive equations (5.2) are neglected we obtain:

$$\tau^{ij} = \rho\frac{\partial W}{\partial \overset{E}{e}_{ij}}; \qquad h^{i(jk)} = \rho\frac{\partial W}{\partial \overset{E}{d}_{ijk}} \tag{5.11}$$

Therefrom we see that the specific energy deformation function assumed the form:

$$W = W\left(\overset{E}{e}_{ij};\ \overset{E}{d}_{ijk}\right) \tag{5.12}$$

and, because it is an isotropic function in the linear theory, it can be approximated by the quadratic polynomial:

$$\rho W = \frac{1}{2}A^{ijkl}\overset{E}{e}_{ij}\overset{E}{e}_{kl} + \frac{1}{2}B^{ijklmn}\overset{E}{d}_{ijk}\overset{E}{d}_{lmn} \tag{5.13}$$

where A^{ijkl} and B^{ijklmn} are isotropic material tensors given by material constants ([5]).

Using (5.13) in constitutive equations (5.11), and bearing in mind the form A^{ijkl} and B^{ijklmn}, we get:

$$\tau^{ij} = \lambda\,\overset{E}{e}_{I}\,g^{ij} + 2\mu\, e^{Eij} \tag{5.14}$$

$$\begin{aligned} h^{i(jk)} = {} & h_1 g^{jk} d^{Eil}_{\cdot\cdot l} + h_2(d^{Ejki} + d^{Ekji}) + h_3 d^{Eijk} + \\ & + h_4\left(2g^{jk} d^{El\cdot i}_{\cdot l} + g^{ki} d^{Ejl}_{\cdot\cdot l} + g^{ij} d^{Ekl}_{\cdot\cdot l}\right) + h_5\left(g^{ij} d^{El\cdot k}_{\cdot l} + g^{ik} d^{El\cdot j}_{\cdot l}\right) \end{aligned} \tag{5.15}$$

where:

$$\gamma_1 = h_1;\quad 2\gamma_2 = h_2;\quad 2\gamma_3 = h_3;\quad \gamma_4 = h_4;\quad 2\gamma_5 = h_5.$$

If in constitutive equations (5.14) and (5.15) the space tensors of elastic deformation $\overset{E}{e}_{ij}$ and $\overset{E}{d}_{ijk}$ are expressed by means of (5.9), we obtain:

$$\tau^{ij} = \lambda\, e_I\, g^{ij} + 2\mu\, e^{ij} - (3\lambda + 2\mu)\,\alpha\theta\, g^{ij} \tag{5.17}$$

$$
\begin{aligned}
(5.18)\qquad h^{i(jk)} = {} & h_1\, g^{jk}\left(d^{il}_{\cdot\cdot l} - \alpha\theta^{,i}\right) + h_2\left[d^{jki} + d^{kji} - \frac{\alpha}{2}\left(2\, g^{jk}\,\theta^{,i} - g^{jk}\,\theta^{,k} - g^{ki}\,\theta^{,j}\right)\right] + \\
& + h_3\left[d^{ijk} - \frac{\alpha}{2}\left(g^{ij}\,\theta^{,k} + g^{ik}\,\theta^{,j}\right)\right] \\
& + h_4\left[2\, g^{jk}\left(d^{l\cdot i}_{\cdot l} - 2\,\alpha\theta^{,i}\right) + g^{ki}\left(d^{jl}_{\cdot\cdot l} - \alpha\theta^{,j}\right) + g^{ij}\left(d^{kl}_{\cdot\cdot l} - \alpha\theta^{,k}\right)\right] + \\
& + h_5\left[g^{ij}\left(d^{l\cdot k}_{\cdot l} - 2\,\alpha\theta^{,k}\right) + g^{ik}\left(d^{l\cdot j}_{\cdot l} - 2\,\alpha\theta^{,j}\right)\right].
\end{aligned}
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ТЕРМОУПРУГИЙ ДИПОЛЯРНЫЙ КОНТИНУУМ

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Резюме

Предметом рассмотрения является теория с учетом температуры. Исходной предпосылкой является известное температурное поле. Деформация континуума вполне определяется деформацией в каждой точке замоченного триедра директора. Используя геометрический подход, выведены уравнения баланса энергии и дифференциальные уравнения движения. Выведены также нелинейные конститутивные уравнения, линеаризованные для изотропных материалов.

ТЕРМОЕЛАСТИЧНИ ДИПОЛАРНИ КОНТИНУУМ

Зоран Голубовић

Резиме

Предмет разматрања у овом раду је понашање диполарних материјала у познатом температурном пољу. Користећи геометријски прилаз, изведене су једначине баланса енергије и диференцијалне једначине кретања. Такође су изведене нелинеарне конститутивне једначине, које су за изотропне материјале линеаризоване.