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A CONTRIBUTION TO UNIVERSAL SOLUTIONS OF THE BOUNDARY LAYER THEORY

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1. **Introduction**

The universalization of the governing differential equations represents one of the ways for improvement of modern analytical methods for the calculation of laminar boundary layers. As a result of this procedure, which consists of introducing a conveniently chosen set of parameters, quantities characterizing any special problem are eliminated from the governing set of equations and the corresponding boundary conditions. A numerical solution of the universal equation can be found once for all and then it can be used in any special problem of the boundary layeı theory.

The idea of universal solutions is old almost as the boundary layer theory itself. The progress of this theory is, therefore, inseparably connected with the gradual development of the idea of universalization. Very known researchers as Blasius, Howarth, Falkner, Karman, Pohlhausen, Görtler, Loitsianskii and others, every one in his own way, contributed to a gradual approach to the final goal. But the problem of universalization could not find, from the point of view of mathematical accuracy, as well as from the aspect of successful applications of universal multiparameter methods in engineering practice, its most optimal solution, until modern sophisticated computers had been used in applied mathematics and mechanics.

Since the universal equations contain, as it will be seen later, summs of terms of number of which is equal to the number of parameters, it is necessary to limit the number of parameters at numerical integration, due to:

— the volume and the complexity of the work on the developing the finite difference method and the corresponding of the algorithm obtained,

— the limitness of the memory of available computers and

— economy of computer-time, i. e. the duration of computers work.

That is why it is very important that the set of parameters chosen possesses the following two properties:

1. the first parameter is to be enough "strong", so that the one-parameter solution lies close to the exact solution and

2. the following parameters introduce in the solution small corrections only, and provide the convergence to be enough fast.

From many authors (Shkadov [6], Loitsianskii [3], Saljnikov &. Oka [8], who tried to satisfy those requirements by choosing various sets of parameters, Loitsianskii was the most successful one. His set of parameters, as Najfeld [5] showed, possesses both necessary properties in contrast with the methods developed by Shkadov and Saljnikov & Oka. The two-parameter approximation by Loitsianskii coincides almost exactly with exact solutions, excepting a small region in the immediate neighbourhood of the separation point.

However, this method possesses a property which makes the calculations of the boundary layer slightly difficult and absorbs much time. Namely, in order to determine quantities, characteristic for the boundary layer, an additional integration of momentum equation in every special case is necessary. That is why the calculation of characteristic quantities of the boundary layer by means of the Loitsianskii's method, carried out by Najfeld, had certain complications and consequently absorbed much time, in contrast with the method of Saljnikov & Oka. Namely, in the method of Saljnikov & Oka there is no need for additional integration of momentum equation. The results can be achieved faster, but with poorer convergence. Consequently, the aforementioned property of the Loitsianskii's method can in spite of certain advantages prevent its acceptance by engineers in practice, who very often like to obtain necessary results in a simple way, taking less care of the accuracy.

We think, therefore, that the further improvement of parametric methods should be directed to the simplification of the application of universal solutions, satisfying the following two conditions:

1. to use the experience gathered in the paper of Saljnikov & Oka, concerning the efficient practical application of universal solutions and

2. to keep, if possible, the Loitsianskii's set of parameters, providing an unexceeded convergence so far.

The way to this goal has been partly already traced in an earlier paper (Saljnikov [9], while in this one it has been finaly achieved.

In order to make our method as clear, and simple as possible we decided to demonstrate it on the example of the classical Prandtl model of the boundary layer. It is to be mentioned however, that from the papers cited at the end of the abstract which represent a continuation of our investigations, can be concluded that this method can be extended also without any principal difficulties to the more complex physical models of the boundary layer.

2. The universal equations of the problem considered

In an earlier paper (Saljnikov [9] it was shown that the satisfaction of all requirements cited in the introduction can be achieved if in the equation for the stream function $\psi(x, y)$ of the considered problem:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\,\partial y}-\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{\partial^3 \psi}{\partial y^3} \tag{2.1}$$

with the corresponding boundary conditions

$$y=0: \qquad \psi=\frac{\partial \psi}{\partial y}=0;$$

(2.2)
$$y\to\infty: \qquad \frac{\partial \psi}{\partial y}\to U(x);$$

$$x=x_0: \qquad \frac{\partial \psi}{\partial y}=u_0(y),$$

the following transforms are first introduced:

(2.3)
$$\varkappa=x; \quad ; \quad \eta=U^{b_0/2}\, y\left(a_0\,\nu\int_0^x U^{b_0-1}\,dx\right)^{-1/2};$$

$$\Phi(\varkappa,\eta)=U^{\frac{b_0}{2}-1}\,\psi(x,y)\left(a_0\,\nu\int_0^x U^{b_0-1}\,dx\right)^{-1/2},$$

with the usual notations:

x — the coordinate measured along the contour from the stagnation point,
y — the coordinate perpendicular to the contour
η — transformed nondimensional y
$U(x)$ — free stream velocity
$u(x, y)$ — the velocity into the direction of x
$u_0(y)$ — $u(x, y)$ for $x=x_0$
ν — kinematic viscosity
a_0, b_0 — arbitrary constants
$\Phi(x, \eta)$ — transformed nondimensional stream function

The universalization of the equation (2.1) can not be, however, achieved until the Loitsianskii's [3] set of parameters is additionally introduced:

(2.4)
$$f_k=U^{k-1}\frac{d^k U}{dx^k}\left(\frac{\delta^{**2}}{\nu}\right)^k; \quad k=1, 2, \ldots \infty,$$

which take over the role of the coordinate x in further considerations.

The first term of that set (for $k=1$):

(2.5)
$$f_1=\frac{dU}{dx}\left(\frac{\delta^{**2}}{\nu}\right),$$

represents the "form parameter" of the generalized Karman-Pohlhausen method and δ^{**} is the momentum thickness defined as:

(2.6)
$$\delta^{**}=\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy.$$

Making use of (2.3), (2.5) and (2.6), the parameter f_1 becomes

$$f_1 = \frac{a_0 B^2 U'}{U^{b_0}} \int_0^\infty U^{b_0 - 1}\, dx. \tag{2.7}$$

It is not difficult here to conclude that, since $\Phi = \Phi(x, \eta)$, the quantity B defined as:

$$B = \int_0^\infty \frac{\partial \Phi}{\partial \eta}\left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta, \tag{2.8}$$

and consequently the parameter f_1 also represents continuous function of x, that is: $B = B(x)$ and $f_1 = f_1(x)$.

It is interesting to note that for $a_0 = b_0 = 2$, the Görtler's ([1]) "principal function" will emerge from (2.7):

$$\beta(x) = \frac{f_1(x)}{B^2} = \frac{2U'}{U^2} \int_0^x U dx. \tag{2.9}$$

The same is valid for η and Φ, which after substituting $a_0 = b_0 = 2$ into (2.3) are reduced to the corresponding quantities of Görtler's type:

$$\eta = Uy\left(2\nu \int_0^x U dx\right)^{-1/2}; \qquad \Phi = \psi\left(2\nu \int_0^x U dx\right)^{-1/2}. \tag{2.10}$$

Due to the arbitrariness of the function $U(x)$, the set of parameters (2.4) represents a system of independent functions, that, as shown by Loitsianskii ([3]), satisfy the recursion ordinary differential equation:

$$\frac{U}{U'} f_1 f'_k = [(k-1) f_1 + kF] f_k + f_{k+1} = \theta_k. \tag{2.11}$$

One can easily derive it by differentiating (2.4) and introducing into the consideration the momentum equation in the following forms:

$$\frac{\delta^{**\prime}}{\delta^{**}} = \frac{U' F}{2 U f_1}; \qquad f_1 = \frac{U'}{U} F + \frac{U''}{U'} f_1. \tag{2.12}$$

Thereby the following, in the boundary layer theory, usual notations are used:

$$F = 2[\zeta - (2 + H) f_1]; \qquad \zeta = \left[\frac{\partial (u/U)}{\partial (y/\delta^{**})}\right]_{y=0};$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy; \qquad H^* = \delta^*/\delta^{**}. \tag{2.13}$$

From the considerations of the preceding connections, it is not difficult to come to a conclusion, drawn already in the paper by Loitsianskii, that is valid in this case also. Namely, since the quantities δ^*, δ^{**}, ζ, H^* and F are all functions of the set of parameters f_k, the right-hand side of the equation (2.11), denoted by θ_k, will be a function of the same kind.

Hence, we will adopt the parameters f_k as new independent variables instead of the coordinate x and perform the substitution of differentiation by means of the differential operator:

$$\frac{\partial}{\partial k}=\sum_{k=1}^{\infty}\frac{\partial}{\partial f_k}\frac{df_k}{dx}=\frac{U'}{Uf_1}\sum_{k=1}^{\infty}\theta_k\frac{\partial}{\partial f_k}, \tag{2.14}$$

derived from (2.11). The governing system of equations (2.1) and (2.2), transformed first by means of (2.3), reduces to the following universal form in this case:

$$\begin{aligned}&\frac{\partial^3\Phi}{\partial\eta^3}+\frac{1}{2B^2}[a_0B^2+f_1(2-b_0)]\,\Phi\frac{\partial^2\Phi}{\partial\eta^2}+\frac{f_1}{B^2}\left[1-\left(\frac{\partial\Phi}{\partial\eta}\right)^2\right]=\\&=\frac{1}{B^2}\sum_{k=1}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\,\partial f_k}-\frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\eta^2}\right);\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\eta=0:\quad \Phi=\frac{\partial\Phi}{\partial\eta}=0;\\&\eta\to\infty:\quad \frac{\partial\Phi}{\partial\eta}\to 1;\\&f_1=f_2=\cdots=0:\quad \Phi=\Phi_0.\end{aligned} \tag{2.16}$$

Namely, since the free stream velocity $U(x)$ characterizing every special case of flow is really not present in this system of equations, the solution of the differential equation (2.15), with the corresponding boundary conditions (2.16), can be considered as universal. Thus after the numerical integration of the system (2.15), (2.16) on a computer, solution can by used in every special case in a way, that will be outlined in the next sections.

Therefore, the additional integration of the momentum equation, necessary in the method due to Loitsianskii, becomes needless in the procedure developed here.

3. The generalized equation of "similar" solutions ($a_0=b_0=2$)

Apart from the indisputable benefit, that the solution and its application of the universal system (2.15), (2.16) would give, the dilemma concerning the appropriate choice of a_0 and b_0 is unresolved so far in this paper.

In connection with this, in an earlier paper (Saljnikov [9]) an attempt was made with $a_0=b_0=2$, what has taken our attention, at this stage of our investigations. It was namely mentioned in the previous Section that η and Φ, given by (2.3), are reduced for $a_0=b_0=2$ to the known transforms of Görtler's type (2.10). Thereby f_1/B^2 coincides with the "principal function" $\beta(x)$ (2.9), while x remains unaltered.

Since η (2.10) coincides for $U(x)=cx^m$ with the known variable of the "similar" solutions:

$$\eta=\sqrt{\frac{(m+1)\,cx^{m-1}}{2\nu}}\,y, \tag{3.1}$$

the quantity (2.10) can be considered as a generalized variable of these solutions. The generalized "similar" solutions theirselves, represented by the non-dimensional stream function $\Phi(x, \eta)$ (2.10), are determined with the corresponding differential equation:

$$\frac{\partial^3 \Phi}{\partial \eta^3}+\Phi\frac{\partial^2 \Phi}{\partial \eta^2}+\frac{f_1}{B^2}\left[1-\left(\frac{\partial \Phi}{\partial \eta}\right)^2\right]=\frac{1}{B^2}\sum_{k=1}^{\infty}\theta_k\left(\frac{\partial \Phi}{\partial \eta}\frac{\partial^2 \Phi}{\partial \eta\,\partial f_k}-\frac{\partial \Phi}{\partial f_k}\frac{\partial^2 \Phi}{\partial \eta^2}\right), \tag{3.2}$$

that can be obtained by the substitution of $a_0=b_0=2$ into the equation (2.15).

In the one-parameter approximation (for $f_1\neq 0;\ f_2=f_3=\cdots=0;\ \theta_1=F^{(1)} f_1$) from (3.2) follows:

$$\begin{aligned}&\frac{\partial^3 \Phi^{(1)}}{\partial \eta^3}+\Phi^{(1)}\frac{\partial^2 \Phi^{(1)}}{\partial \eta^2}+\frac{f_1}{B^2}\left[1-\left(\frac{\partial \Phi^{(1)}}{\partial \eta}\right)^2\right]=\\&=\frac{F(f_1)}{B^2}f_1\left(\frac{\partial \Phi^{(1)}}{\partial \eta}\frac{\partial^2 \Phi^{(1)}}{\partial \eta\,\partial f_1}-\frac{\partial \Phi^{(1)}}{\partial f_1}\frac{\partial^2 \Phi^{(1)}}{\partial \eta^2}\right),\end{aligned} \tag{3.3}$$

that after the "localization" in regard to f_1, that is after the application of the condition $\partial/\partial f_1=0$, reduces, taking into account (2.9), to the known Falkner-Skan equation of "similar" solutions:

$$\frac{\partial^3 \Phi^{(1/2)}}{\partial \eta^3}+\Phi^{(1/2)}\frac{\partial^2 \Phi^{(1/2)}}{\partial \eta^2}+\beta(x)\left[1-\left(\frac{\partial \Phi^{(1/2)}}{\partial \eta}\right)^2\right]=0. \tag{3.4}$$

The notation (1) is referred to the one-parameter solution, and (1/2) to the one-parameter localized solution, that we will call half-parameter solution.

Thereby following boundary conditions emerge from (2.16):

$$\begin{aligned}&\eta=0: && \Phi^{(1)}=\frac{\partial \Phi^{(1)}}{\partial \eta}=0;\\&\eta\to\infty: && \frac{\partial \Phi^{(1)}}{\partial \eta}\to 1;\\&f_1=f_2=\cdots=0: && \Phi^{(1)}=\Phi_0,\end{aligned} \tag{3.5}$$

which are valid for both equations (3.3) and (3.4).

It is to be mentioned here that the equation (3.4) was first derived and numerically solved by Hartree.

Hence, it can be concluded that (3.2) can be regarded as a generalized Falkner-Skan equation. On the other hand, however, the presented way of universalization and the values $a_0 = b_0 = 2$ offer obviously a possibility for a new interpretation of the equation (3 4). It can be, namely, considered as an universal equation of the boundary-layer in the one-parameter localized approximation. Therefore, the comparison of its solution, that is of the known results by Hartree, with the solution of the full one-parameter equation (3.3), would give an idea about the alterations caused by the "localization" in regard to f_1.

For this purpose the numerical integration of the equation (3.3) with the corresponding boundary conditions (3.5) was performed on a computer. The method by Simuni & Terentiew ([7]), previously developed for the numerical integration of the one-parameter universal solution of Loitsianskii, was used. In the meantime the method was successfully adapted for equations of similar type, obtained during our investigations, showing a satisfactory, stability, that was noticed earlier.

We cite here the calculated results, some of which have been shown in a previous paper (Saljnikov [9]), due to the more complete and more systematic review of various solutions, obtained by related methods. Namely, we compare them with the one-parameter-localized solution, i. e. with the half-parameter solution by Hartree with the one-parameter solution by Loitsianskii, as well as with our recent results, that will be discussed in the next section.

For a better clearness, we introduce the notations, that will be persistently used in the table T-1 of universal values of characterised quantities and in all diagrams. The quantities, e. g. the curves noted by:

$L(1)$ — are referred to the one-parameter solution by Loitsianskii ([3]);

$H(1)$ — are referred to the one-parameter solution of the equation (3.3), (Saljnikov [9]);

$H(1/2)$ — are referred to the Hartree's solution of the Falkner-Skan equation (3.4);

$S(1)$ — are referred to our one-parameter solution ($a_0 = 0.4408$; $b_0 = 5.714$);

$S(1/2)$ — are referred to our one-parameter localized, e. g. half-parameter solution ($a_0 = 0.4408$. $b_0 = 5.714$);

T — are referred to the exact solution by Terrill ([13]).

It is to be mentioned that the solutions noted with $S(1)$ and $S(1/2)$, that actually represent basic results of our investigations, will be considered in continuation.

The universal characteristic functions ζ, H^+ and F are given in the table T-1 and in the diagram in Fig. 1. It can be noticed that, they almost coincide for all three one-parameter solutions ($L(1)$; $H(1)$; $S(1)$), as well as for two half-parameter solutions, with the exception of the neighbourhood of the separation point and the stagnation point, where the corresponding results are still close enough. Such results are the consequence of the similarity of the corresponding universal equations. The equations of Loitsianskii and (4.3), i. b. (3.3) belong, namely, to the same type of differential equations and can be reduced from one form to another by suitable transformations. Therefore, from the behaviour of the functions ζ, H^+ and F very little can be concluded about the accuracy and the convergence, that are reached in the practical use of these methods. Something more can be said about that, however, when one applies the various universal solutions from the table T-1 to a concrete problem of the boundary layer.

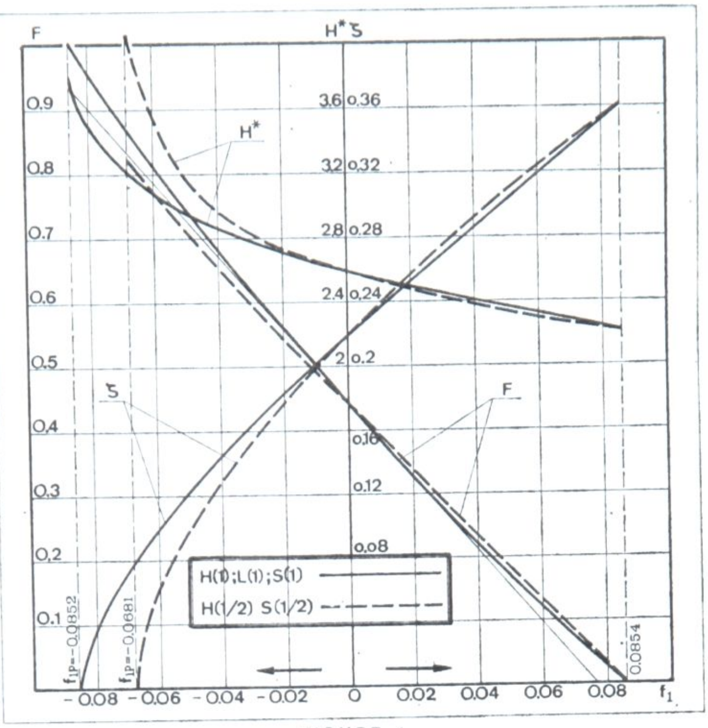


FIGURE 1

For this purpose the potential incompressible flow about a circular cylinder will be used. For this case, namely, Terrill ([13]) obtained a solution by the finite difference method, carrying out the direct numerical integration of

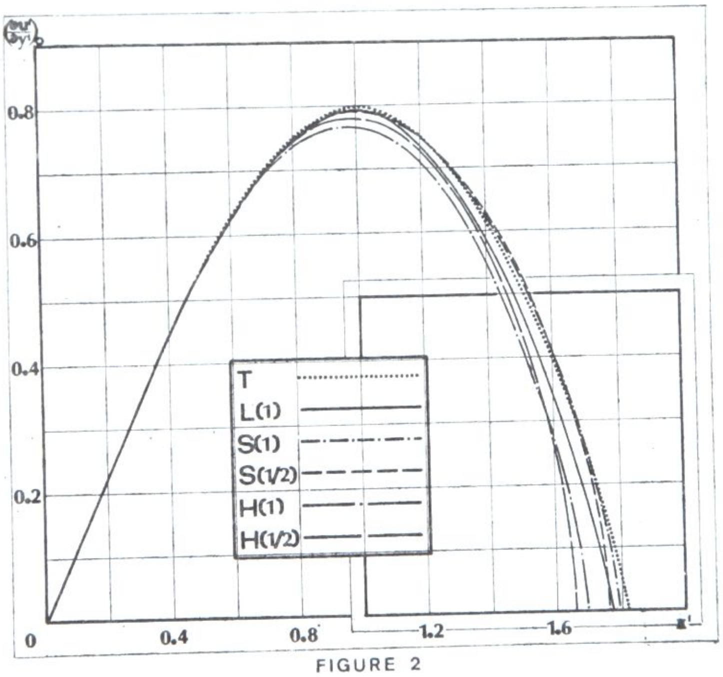


FIGURE 2

the corresponding boundary layer equations. Considering this solution as an exact one, Loitsianskii ([3]) compared his results with it. It seems to us therefore that it will be suitable to test various methods just on that example.

The corresponding velocity

(3.6) $$U(x) = 2\,U_\infty \sin(x/R)$$

by means of:

(3.7) $$U_0 = 2\,U_\infty; \qquad x/R = x'; \qquad U^* = U/U_0,$$

reduces to the nondimensional form:

(3.8) $$U^* = \sin x'.$$

For this flow, the nondimensional quantity $(\partial u'/\partial y')_0$ characterizing the skin friction, and three first form parameters-f_1, f_2, f_3 have been calculated.

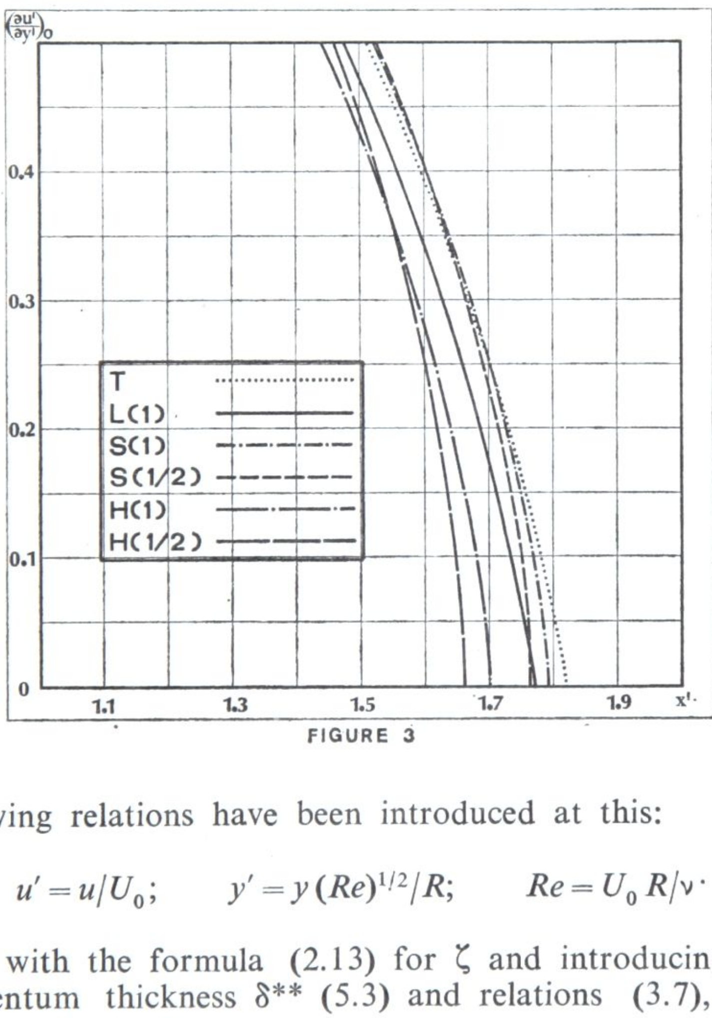


FIGURE 3

The following relations have been introduced at this:

(3.9) $$u' = u/U_0; \qquad y' = y\,(Re)^{1/2}/R; \qquad Re = U_0\,R/\nu\cdot$$

Beginning with the formula (2.13) for ζ and introducing the transforms (2.3), the momentum thickness δ^{**} (5.3) and relations (3.7), (3.9). a general formula has been obtained:

(3.10) $$\left(\frac{\partial u'}{\partial y'}\right)_{y'=0} = \frac{(U^*)^{1+\frac{b_0}{2}}}{\left(a_0 \int\limits_0^{x'} (U^*)^{b_0-1}\, dx'\right)^{1/2}}\, \Phi''(0).$$

$\Phi''(0)$ has been introduced instead of $(\partial^2 \Phi/\partial \eta^2)_{\eta=0}$ in (3.10) and that will be used in this paper in what follows.

Entering the velocity distribution U^* (3.8) in the (3.10), the last expression becomes suitable for the chosen example, whereby the calculation has been carried out for the case considered ($a_0 = b_0 = 2$). The values $\Phi''(0)$, referring to the corresponding universal solutions denoted by $H(1)$ and $H(1/2)$, are determined from the table T-1 by the procedure, to which a special attention in the Section 5 will be given.

The obtained results are shown in Fig. 2, whereby the neighbourhood of the separation point is given in a larger scale in Fig. 3. For the comparison, the corresponding results by Loitsianskii — $L(1)$ and the exact solution by Terrill — T are plotted in the same diagrams.

The calculation of the parameters f_1, f_2, f_3 has been made by means of (2.7), (2.4) and (2.5). The first parameter has been determined, by means of (2.7) (for $a_0 = b_0 = 2$) from the table T-1, by the procedure, already mentioned, about which will be said more in the Section 5. After that, by a combination of the formulae (2.4), (2.5), the expression for:

$$f_k = U^{k-1} \frac{d^k U}{dx^k} \left(\frac{f_1}{dU/dx}\right)^k; \qquad k = 1,\ 2, \ldots \infty, \tag{3.11}$$

has been obtained, from which (for $k = 2,\ 3$) follow:

$$f_2 = U \frac{d^2 U/dx^2}{(dU/dx)^2} (f_1)^2; \qquad f_3 = U^2 \frac{d^3 U/dx^3}{(dU/dx)^3} (f_1)^3. \tag{3.12}$$

Introducing the adopted expression for $U(x)$ (3.6), the final formulae for f_2 and f_3 has been obtained from (3.12). Thereby, the previously calculated values of the parameter f_1 have been used.

The obtained results are shown in Fig. 4, whereby the corresponding curves by Loitsianskii are plotted for comparison.

We next consider the distribution $(\partial u'/\partial y')_{y'=0}$ denoted by $H(1)$ and $H(1/2)$ on the diagrams in Fig. 2 and 3. It can be concluded that the full one-parameter equation (3.3) of "similar" solutions gives more accurate results than the Falkner-Skan equation (3.4), what should be expected due to the performed localization.

On the other hand, however, from the comparison of the curves $H(1)$, $L(1)$ and T it follows that in spite of the certain improvement of accuracy regarding to Hartree's half-parameter results $H(1/2)$, one-parameter "similar" solutions $H(1)$ fall in regard to the exact values by Terrill — T, behind the corresponding results due to Loitsianskii — $L(1)$. From further considerations it will be seen, that the reason for that is the unsatisfactory "strong" first parameter of the adopted set.

The remaining requirements, however set in the introductory considerations, have been fulfilled already in the first attempt ($a_0 = b_0 = 2$). Namely, the distribution of f_1, f_2, f_3 (see the diagram in Fig. 4) shows that the convergence of $H(1)$ is enough fast. The character of the alterations of the corresponding parameters similar as in case by Loitsianskii — $L(1)$ speaks in favour of that. This conclusion is even uninfluenced by the fact, that all three parameters fail near the stagnation point. They tend there, to infinity, and as a consequence, as seen from (3.12), an unsuitable behaviour of the first parameter f_1 in this region.

It is to be emphasized that the property of the method, that represents the main goal in these investigations, concerning the superfluousness of the

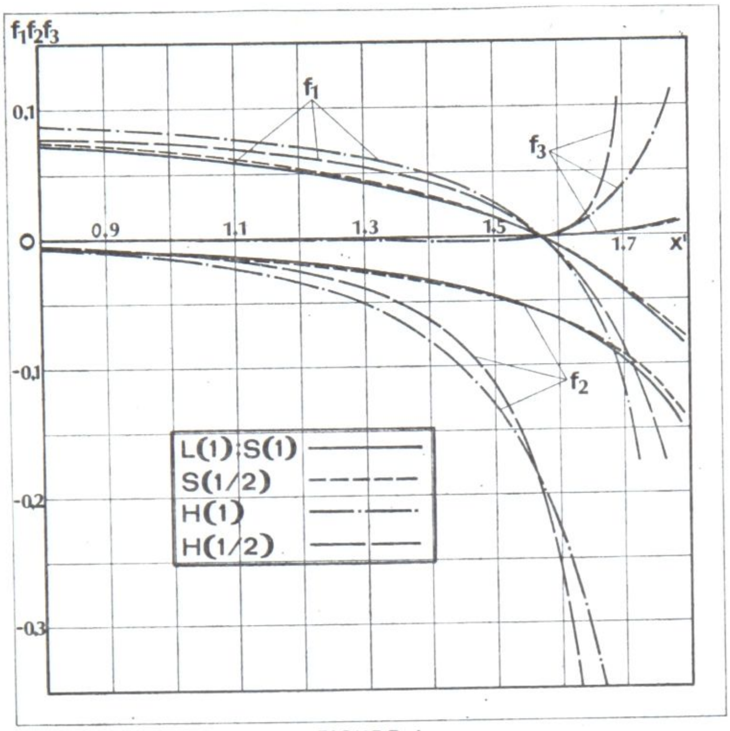


FIGURE 4

integration of the momentum equation, has been already provided in this attempt ($a_0 = b_0 = 2$). The additional integration has been actually avoided, because the first parameter (2.7) itself, by means of which the universal solutions from the table T-1 are used for the calculation of the boundary layer, as it will be seen in the Section 5, can be considered as a result of the formal integration of the momentum equation (2.12) in the form:

$$\frac{df_1}{dx} = \frac{U'}{U} F + \frac{U''}{U'} f_1,$$

with the following characteristic function:

$$F = a_0 B^2 - b_0 f_1 + \frac{2\,UB'}{U'\,B} f_1 . \tag{3.13}$$

It obviously follows from these considerations that further investigations should be directed to a strengthening of the first parameter, what would fulfil the last remaining requirement.

4. The determination of optimal values of the constants a_0 and b_0

The behaviour of the function F (3.13) has been considered for this purpose. It reduces by means of (2.14) to the following universal form:

$$F = a_0 B^2 - b_0 f_1 + \frac{2}{B} \sum_{k=1}^{\infty} \theta_k \frac{\partial B}{\partial f_k}, \tag{4.1}$$

namely, the behaviour of its one-parameter approximation ($f_1 \neq 0$; $f_2 = f_3 = \cdots = 0$; $\theta_1 = F^{(1)} f_1$) is

$$F(f_1) = F^{(1)} = \frac{a_0 B^2 - b_0 f_1}{1 - \frac{2}{B} f_1 \frac{\partial B}{\partial f_1}}, \tag{4.2}$$

that shall be used in the following considerations.

Integrating numerically the universal equation (2.15) in one-parameter approximation:

$$\frac{\partial^3 \Phi^{(1)}}{\partial \eta^3} + \frac{1}{2B^2}\left[a_0 B^2 + (2-b) f_1\right] \Phi^{(1)} \frac{\partial^2 \Phi^{(1)}}{\partial \eta^2} + \frac{f_1}{B^2}\left[1 - \left(\frac{\partial \Phi^{(1)}}{\partial \eta}\right)^2\right] = \frac{F^{(1)} f_1}{B^2}\left(\frac{\partial \Phi^{(1)}}{\partial \eta} \frac{\partial^2 \Phi^{(1)}}{\partial \eta \, \partial f_1} - \frac{\partial \Phi^{(1)}}{\partial f_1} \frac{\partial^2 \Phi^{(1)}}{\partial \eta^2}\right), \tag{4.3}$$

with the corresponding boundary conditions (2.16)

$$\begin{aligned} \eta = 0: &\quad \Phi^{(1)} = \frac{\partial \Phi^{(1)}}{\partial \eta} = 0; \\ \eta \to \infty: &\quad \frac{\partial \Phi^{(1)}}{\partial \eta} \to 1; \\ f_1 = f_2 = \cdots = 0: &\quad \Phi^{(1)} = \Phi_0, \end{aligned} \tag{4.4}$$

whereby the already mentioned method by Simuni & Terentiew [7] has been used, it has been concluded that $F^{(1)}$ simultaneously calculated by means of (4.2), changes only slightly with a_0 and b_0. It follows from that fact, already noted and explained in the previous Section, that $F^{(1)}$ is almost independent of a_0 and b_0. The behaviour of the parameters f_k and consequently the accuracy given by the universal solution in the neighbourhood of the separation point depend on them, however. It is to be noticed here again, that the solution with $a_0 = b_0 = 2$, although methodologically interesting because it represents a generalization of the "similar" solutions, did not give expected results, as we have seen in the previous Section.

That is why the optimal values of a_0 and b_0, by means of which the universal solution of the system (4.3), (4.4) would give the furthest possible approach to the exact results, are to be found. We succeeded to find them combining two facts. The first one is, that we can consider the distribution of function $F^{(1)}$ independent on a_0 and b_0. In other words, $F^{(1)}$ can be determined for any pair of a_0 and b_0 and be considered as known in what follows. The other fact, however, results from the way in which the numerical integration of (4.3), (4.4) is performed by the cited finite difference method. One starts, from the value $f_1 = 0$, defining the Blasius problem of a flat plate. The corresponding solution Φ_0 is determined in that point, that in accordance with the last boundary condition (4.4) represents the initial value for further integration. Thereby, the integration is performed, as shown by arrows on the diagram in Fig. 1, either in the direction of successive growth of positive values of the parameter f_1 ($f_1 > 0$), i. e. toward the stagnation point, or in the direction of successive growth of its negative values ($f_1 < 0$), i. e. toward the separati on point.

With in regard to this method of integration, it is appropriate to choose those values of a_0 and b_0, by which the position of the tangent (see the diagram on the Fig. 1) on the curve $F(f_1)$ in the point $f_1=0$ is defined. Such a conclusion is led by the fact, that Loitsianskii [3] developed the procedure of the before mentioned additional integration of the momentum equation, starting with $F^{(1)}$ in the form:

(4.5) $$F(f_1)=a-bf_1+\varepsilon(f_1),$$

where $\varepsilon(f_1)$ expresses the deviation of the function $F^{(1)}$ from the linear one, determined by the tangent in the point $f_1=0$. It is to be mentioned that the corresponding values of the constants a and b have been determined in the same paper from a series in terms of f_k for the function F. It follows, namely from this series taken in one-parameter approximation in the point $f_1=0$ that:

(4.6) $$\mathrm{a}=0.4408; \qquad \mathrm{b}=5.714.$$

As expected, we are getting the same results using the table T-1 also, when calculating for the cases denoted by $L(1)$ and $H(1)$ the corresponding quantities $F(0)$ and $(dF^{(1)}/df_1)_0$, whose numerical values are therefore:

(4.7) $$F(0)=0.4408; \qquad (dF^{(1)}/df_1)_0=-5.714.$$

Since on the other hand, from (4.2) we have:

(4.8) $$F(0)=a_0\,B(0)^2; \qquad (dF^{(1)}/df_1)_0=4\,a_0\,B(0)\,(dB/df_1)_0-b_0,$$

we obtain by the comparison of (4.7) and (4.8):

(4.9) $$a_0=0.4408/B(0)^2$$

(4.10) $$b_0=5.714+4\,a_0\,B(0)\,(dB/df_1)_0.$$

Making use of the relations (4.9), (4.10) it can be concluded that by the choice of a_0 and b_0 a normalization of $B(0)$ and $(dB/df_1)_0$, defining the form and the position of the curve $B(f_1)$ in the initial point of integration $f_1=0$, is made at the same time.

The former investigations, performed for various values of a_0 and b_0, showed that the optimal results, in the sense of the requirements, set earlier, are reached, as it will be seen in the next Section, with:

(4.11) $$a_0=0.4408; \qquad b_0=5.714.$$

It is to be mentioned that the testing of various solutions has been carried out on the example of the flow around a circular cylinder, adopted earlier for this purpose, in the way shown in the Section 3.

Therefore, choosing optimal values of the constants a_0; b_0, we stopped at the values (4.11), that obviously coincide with the constants a; b (4.6) and that we adopt for further work.

In this connection it is to be reminded that in the Section 3 of this paper, in order to generalize the equation of "similar" solutions, $a_0=2$ has been adopted. The value $B(0)=0.4694$ follows then from (4.9), that can be easily noticed on the curve $B(f_1)$ denoted with $H(1)$ in Fig. 6. Since here, however, $a_0=0.4408$ has been adopted, the quantity $B(0)$ is normalized as $B(0)=1$, what can be immediately verified on the curve $B(f_1)$ denoted with $S(1)$ in Fig, 7.

	f_1	—0.0852	—0.0800	—0.0700	—0.0681	—0.0600	—0.0500	—0.0400	—0.0300	—0.0200	—0.0100
ζ	L (1)	0.0000	0.0397	0.0746		0.1015	0.1249	0.1462	0.1662	0.1851	0.2034
	H (1)	0.0000	0.0385	0.0737		0.1007	0.1242	0.1456	0.1656	0.1846	0.2028
	H (1/2)				0.0000	0.0645	0.1014	0.1306	0.1561	0.1791	0.2005
	S (1)	0.0000	0.0383	0.0737		0.1007	0.1242	0.1456	0.1657	0.1847	0.2029
	S (1/2)				0.0000	0.0646	0.1015	0.1307	0.1562	0.1792	0.2005
H^*	L (1)	3.8150	3.4410	3.2051		3.0575	2.9458	2.8538	2.7754	2.7063	2.6441
	H (1)	3.7477	3.4543	3.2134		3.0638	2.9510	2.8587	2.7798	2.7105	2.6484
	H (1/2)				4.0300	3.3597	3.1140	2.9560	2.8373	2.7416	2.6612
	S (1)	3.7785	3.4562	3.2138		3.0639	2.9508	2.8583	2.7794	2.7101	2.6479
	S (1/2)				4.0300	3.3586	3.1133	2.9555	2.8367	2.7410	2.6607
F	L (1)	0.9909	0.9500	0.8779		0.8099	0.7444	0.6807	0.6189	0.5585	0.4997
	H (1)	1.0012	0.9575	0.8800		0.8108	0.7447	0.6807	0.6186	0.5578	0.4985
	H (1/2)				0.8210	0.7723	0.7142	0.6576	0.6024	0.5428	0.4942
	S (1)	1.0116	0.9659	0.8847		0.8124	0.7452	0.6809	0.6185	0.5579	0.4988
	S (1/2)				0.8210	0.7722	0.7143	0.6578	0.6026	0.5480	0.4942
A	H (1)	2.0289	1.8555	1.7013		1.5981	1.5158	1.4450	1.3816	1.3233	1.2686
	H (1/2)				2.3590	1.9183	1.7254	1.5885	1.4774	1.3814	1.2954
	S (1)	3.8279	3,5026	3.2457		3.0865	2.9663	2.8684	2.7853	2.7130	2.6490
	S (1/2)				3.9163	3.3147	3.0696	2.9164	2.8052	2.7189	2.6493
B	H (1)	0.5411	5.5371	0.5294		0.5216	0.5137	0.5055	0.4970	0.4882	0.4790
	H (1/2)				0.5852	0.5710	0.5541	0.5374	0.5207	0.5039	0.4868
	S (1)	1.0156	1.0134	1.0100		1.0074	1.0052	1.0035	1.0021	1.0011	1.0004
	S (1/2)				0.9902	0.9869	0.9860	0.9868	0.9889	0.9919	0.9957
$\Phi''(0)$	H (1)	0.0110	0.0717	0.1393		0.1931	0.2418	0.2881	0.3333	0.3782	0.4234
	H (1/2)				0.0000	0.1130	0.1830	0.2431	0.2998	0.3555	0.4118
	S (1)	0.0029	0.0378	0.0730		0.1000	0.1236	0.1451	0.1653	0.1845	0.2028
	S (1/2)				0.0000	0.0655	0.1029	0.1325	0.1579	0.1807	0.2014
f_1/B^2	H (1)	—0.2903	—0.2773	—0.2497		—0.2205	—0.1895	—0.1565	—0.1215	—0.0839	—0.0436
	H (1/2)				—0.1988	—0.1840	—0.1629	—0.1385	—0.1106	—0.0788	—0.0422
	S (1)	—0.0827	—0.0779	—0.0686		—0.0591	—0.0495	—0.0397	—0.0299	—0.0199	—0.0100
	S (1/2)				—0.0694	—0.0616	—0.0514	—0.0411	—0.0307	—0.0203	—0.0101

T-1

0.0000	0.0100	0.0200	0.0300	0.0400	0.0500	0.0600	0.0700	0.0800	0.0854
0.2204	0.2375	0.2542	0.2706	0.2868	0.3028	0.3188	0.3348	0.3510	0.3601
0.2204	0.2376	0.2543	0.2707	0.2869	0.3038	0.3189	0.3349	0.3512	0.3594
0.2204	0.2393	0.2574	0.2747	0.2913	0.3073	0.3228	0.3378	0.3524	0.3592
0.2205	0.2376	0.2543	0.2707	0.2869	0.3030	0.3189	0.3349	0.3511	0.3600
0.2205	0.2394	0.2574	0.2747	0.2913	0.3073	0.3228	0.3378	0.3523	0.3600
2.5919	2.5384	2.4903	2.4449	2.4014	2.3599	2.3196	2.2802	2.2403	2.1730
2.5919	2.5397	2.4910	2.4452	2.4017	2.3580	2.3197	2.2801	2.2401	2.2084
2.5919	2.5308	2.4763	2.4270	2.3821	2.3409	2.3029	2.2676	2.2347	2.2060
2.5913	2.5391	2.4904	2.4446	2.4011	2.3594	2.3190	2.2794	2.2397	2.2154
2.5913	2.5303	2.4757	2.4265	2.3817	2.3405	2.3026	2.2674	2.2347	2.2154
0.4408	0,3847	0.3293	0.2750	0.2219	0.1701	0.1197	0.0708	0.0239	0.0000
0.4406	0.3841	0.3289	0.2745	0.2214	0.1671	0.1196	0.0707	0.0233	0.0000
0.4408	0.3879	0.3357	0.2838	0.2320	0.1805	0.1292	0.0781	0.0272	0.0000
0.4409	0.3844	0.3290	0.2748	0.2218	0.1699	0.1194	0.0704	0.0233	0.0000
0.4410	0.3881	0.3357	0.2838	0.2320	0.1805	0.1292	0.0781	0.0270	0.0000
1.2166	1.1661	1.1167	1.0673	1.0171	0.9623	0.9090	0,8456	0.7607	0.6540
1.2166	1.1428	1.0729	1.0057	0.9403	0.8762	0.8124	0.7484	0.6831	0.6421
2.5916	2.5401	2.4935	2.4513	2.4134	2.3800	2.3516	2.3302	2.3212	2.3261
2.5917	2.5431	2.5015	2.4655	2.4341	2.4064	2.3817	2.3596	2.3396	2.3261
0.4694	0.4591	0.4483	0.4365	0.4235	0.4081	0.3919	0.3709	0.3396	0.2946
0.4694	0.4516	0.4333	0.4144	0.3947	0.3743	0.3528	0.3300	0.3057	0.2917
1.0001	1.0004	1.0012	1.0027	1.0051	1.0087	1.0140	1.0222	1.0364	1.0500
1.0001	1.0051	1.0104	1.0161	1.0220	1.0281	1.0344	1.0407	1.0469	1.0500
0.4696	0.5174	0.5673	0.6203	0.6776	0.7444	0.8139	0.9031	1.0341	1.2199
0.4696	0.5300	0.5940	0.6628	0.7379	0.8210	0.9150	1.0237	1.1529	1.2314
0.2205	0.2374	0.2537	0.2693	0.2841	0.2978	0.3102	0.3207	0.3268	0.3428
0.2205	0.2382	0.2548	0.2703	0.2850	0.2989	0.3121	0.3246	0.3365	0.3428
0.0000	0.0474	0.0995	0.1575	0.2230	0.3002	0.3907	0.5089	0.6937	0.9839
0.0000	0.0490	0.1065	0.1747	0.2567	0.3569	0.4821	0.6427	0.8561	1.0036
0.0000	0.0099	0.0199	0.0298	0.0395	0.0491	0.0583	0.0669	0.0744	0.0774
0.0000	0.0099	0.0196	0.0291	0.0383	0.0473	0.0561	0.0646	0.0730	0.0774

Choosing the constant $b_0=2$ (in the Section 3), it follows from (4.10): $(dB/df_1)_0=-0.9890$, what can be easily verified on the curve $B(f_1)$ denoted with $H(1)$ in Fig. 6 and in the table T-1. In this Section, however, with regard to the adopted value $b_0=5.7140$, the slope of the curve $B(f_1)$ has been normalized by means of (4.10) to be: $(dB/df_1)_0=0$, what can be also verified on the curve $S(1)$ in Fig. 7.

With the adopted values $a_0=0.4408$ and $b_0=5.7140$ the numerical integration of the system (4.3), (4.4) has been performed, whereby the obtained results, denoted by $S(1)$, are given in the table T-1 and in the form of a diagram in Fig. 1 and 7. The same system has been in addition integrated in the one-parameter localized approximation ($\partial/\partial f_1=0$). The corresponding results denoted by $S(1/2)$ are given in the table T-1 and shown in Figs. 1 and 7. It can be noticed that the characteristic functions ζ, H^+ and F, denoted by $S(1)$ on the diagram in Fig. 1, coincide with the corresponding curves $L(1)$, $H(1)$, as expected. It is also valid for the curves denoted by $S(1/2)$, that coincide with the corresponding curves $H(1/2)$ on the same diagram.

5. The efficiency of the obtained universal solution and its practical application

Before we show the procedure for practical use of the universal solution, obtained in the preceding Section, we will consider its quality from the point of view of the requirements, set in the introduction. For this purpose the flow around a circular cylinder, as in the Section 3, will be used.

Since the behaviour of the first parameter f_1, as seen from the previous considerations, is of the primary importance for obtaining better results, we will consider first its distribution along the contour, especially near the separation point.

The calculation of the parameter f_1 has been carried out making use of the relation:

$$\frac{f_1}{B^2}=\frac{a_0\,U'}{U^{b_0}}\int_0^x U^{b_0-1}\,dx \qquad \left(\text{for } \begin{matrix} a_0=0.4408 \\ b_0=5.7140 \end{matrix}\right), \tag{5.1}$$

that follows from (2.7). It is to be mentioned that (5.1) plays a leading role in making a connection between the universal solutions, illustrated by the table T-1, and the special data of every concrete flow, expressed through the free stream velocity $U(x)$.

In the case considered, after substituting (3.6) into (5.1), the left-hand side of (5.1), i. e. $(f_1/B^2)_0$, for $x=x_0$, is calculated. The corresponding value of $(f_1)_0$ is then determined from T-1. From the obtained distributions, denoted by $S(1)$ and $S(1/2)$ in Fig. 4, can be concluded that in comparison with the previous results, denoted by $H(1)$ and $H(1/2)$, the quality of the first parameter f_1 has been considerably improved. That can be seen from its behaviour near the separation point ($x_p'=1.823$), that, as known, represents a singular point of the boundary layer equations. The absolute values of the ordinates of the curves $S(1)$ and $S(1/2)$ increase much slower in approaching the separation point, than the curves $H(1)$ and $H(1/2)$, that tend to infinity.

Since the curves $S(1)$ and $L(1)$ coincide on the same diagram (see Fig. 4), it is to be expecjed that the results obtained in this paper in one-parameter approximation will be at least as good as the results by Loitsianskii. In order to check it the nondimesional skin friction $(\partial u'/\partial y')\,y'=0$ has been calculated. Thereby, (3.10) with (3.8) and $a_0=0.4408$; $b_0=5.714$ have been used. The corresponding values of $\Phi''(0)$ have been then determined from T-1 from the solutions denoted by $S(1)$ and $S(1/2)$. The calculated distributions of $(\partial u'/\partial y')_0$ denoted by $S(1)$ and $S(1/2)$ have been plotted on the diagram in Fig. 2, whereby the neighbourhood of the separation point has been shown in a larger scale on the diagram in Fig. 3. It can be seen that the position of $S(1)$ and $S(1/2)$ with regard to the exact results by Terrill — T, is much more convenient than the position of $H(1)$ and $H(1/2)$. It is to be emphasized, however, that a considerable improvement can be noticed in comparison to the curve $L(1)$ also, from which not only the results $S(1)$, obtained in the one-parameter approximation, are more accurate now, but also the results $S(1/2)$ calculated by means of the one-parameter localized solution, excepting an immediate neighbourhood of the separation point. The position of the separation point, defined as known by the intersection of $(\partial u'/\partial y')_{y'=0}=f(x')$ with x'-axis and denoted by P, is determined in the case of $L(1)$ by: $x_p'=1.770$ and in case $S(1)$ by: $x_p'=1.789$. It is to be reminded here that Terrill obtained $x_p'=1.823$, that can be considered as an exact value.

All that speaks for the benefit of the fact that by the choice of optimal values of the constants: $a_0=0.4408$; $b_0=5.714$ the first parameter f_1 has become considerably "stronger" and that the results calculated on the basis of the corresponding universal solution in the one-parameter approximation $S(1)$, in view of practical applications, can be considered as almost exact.

After such a conclusion, it is reasonable to expect due to (3.12) that the distributions of the following two parameters f_2 and f_3 will be also improved. Considering the curves $f_2(x')$ and $f_3(x')$, calculated by means of (3.12) and denoted by $S(1)$ and $S(1/2)$ on the diagram in Fig. 4 and from their comparison with the corresponding curves denoted by $H(1)$ and $H(1/2)$, follows that our expectations have been true. Namely, the absolute values of the ordinates of the curves $S(1)$ and $S(1/2)$ increase much slower by approaching to the separation point ($x_p'=1.823$), than the ordinates of the curves $H(1)$ and $H(1/2)$, that tend to infinity. Thereby while a slight discrepancy in the curve $S(1/2)$ can be noticed comparing with $L(1)$, the curves $S(1)$ and $L(1)$ fully coincide. Therefore, the parameter f_2 changes absolutely at both curves in the limits $0\div0.15$, and the parameter f_3 remains along the whole contour very close to zero, excepting the neighbourhood of the separation point, where gradually increases to the value given by 0.0125. Such a behaviour of f_2 and f_3 provides the existence of the so named physical convergence to the solution of Loitsianskii [3], which is fast enough, as seen from his two-parameter results. That is why the same property of the solution of the system (2.15), (2.16) should be conjectured.

The results by Mirgaux [4], who determined the solution of this system in the two-parameter approximation and applied it to the flow around a circular cylinder, speak for the benefit of that. Namely, the distribution of $(\partial u'/\partial y')_{y'=0}$, in spite of some insignificant simplifications introduced by the numerical integration, almost does not differ from the exact results by Terrill along the whole contour of the cylinder, excepting the immediate neighbourhood of the separation point. Thereby, the position of this point is determined by $x_p' = 1.815$. Its value, obtained by the two-parameter solution by Loitsianskii, is however: $x_p' = 1.833$. The discrepancy of the Mirgaux's result ($x' = 1.815$) from the Terrill's one ($x_p' = 1.823$) is: $x_p' = -0.008$. In the case of the Loitsianskii's result ($x_p' = 1.833$) we have: $x_p' = 0.0.10$.

Since the former results in this Section are based on the flow around a circular cylinder, an interesting and at the same time a natural question arises concerning the quality of the obtained universal solution in case of its application to the contours that are relatively slender than a circular one and that would be more close to the real airfoils.

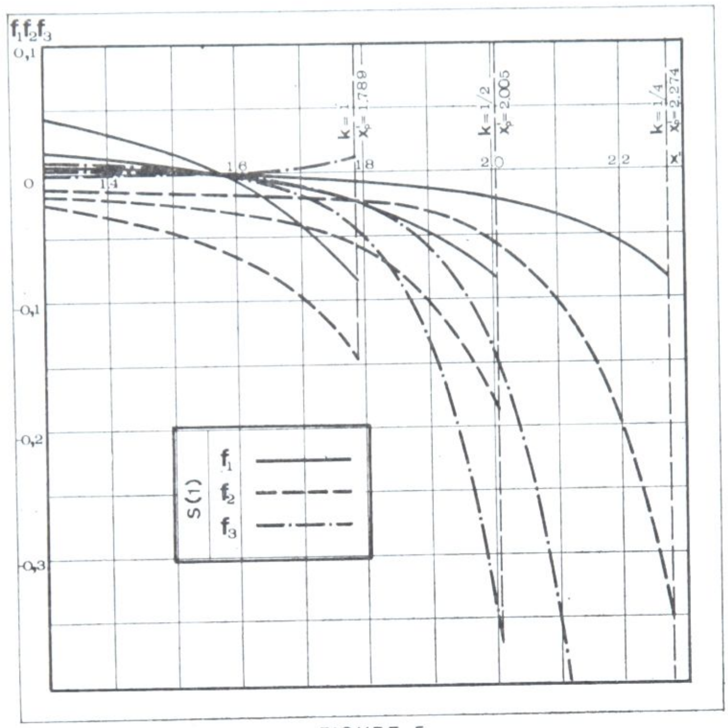


FIGURE 5

As an answer to this question the results of Ivanović [2], can be used who calculated the boundary layer on elliptic contours for various ratios of their halfaxis $k = b/a$. From the distributions of $(\partial u'/\partial y')_{y'=0} = f(x')$, calculated by Ivanović follows, that the character of the mutual position of the curves, corresponding to the individual universal solutions in the table T-1, is not changed considerably with the decrease of k, comparing with the flow around the circular cylinder (see Fig. 2 and 3). In other words, the order of the curves in the neighbourhood of the separation point remains unaltered independently of k, that is: $H(1)$, $L(1)$, $S(1)$ in the direction of x'. A change in the position of an exact solution, that could be in principle found by the procedure similar to that by Terrill, should not be expected also. That is why the order of individual universal solutions in the one-parameter approximation can be supposed with the great certaintly to be the same as in the case of flow around a circular cylinder.

The distributions of the first parameter f_1, calculated by Ivanović for various values of k and shown on the diagram in Fig. 5 speak for the benefit of that. It can be seen, from their behaviour that not only the character of the absolute value of f_1 is similar for various k, but the values of f_1 in the separation point, determined in the one-parameter approximation and denoted by P, do not differ (for $k=1$; $1/2$; $1/4$: $f_{1p}=-0.085$). It points out that the "strength" of the parameter f_1 is independent from the slenderness of the profile and confirms at the same time the assumption concerning the steadiness of the quality of the corresponding one-parameter solution.

This conclusion does not hold for the parameters f_2 and f_3 any more. From the same diagram (see Fig. 5) follows, that the absolute values of f_2 and f_3, in spite of the similar character of their distributions, increase with the decrease of k. For example for $k=1$: $f_{2p}=-0.150$ for $k=1/2$; $f_{2p}=-0.190$; for $k=1/4$: $f_{2p}=-0.350$. It points out to a certain deterioration of the physical convergence, that takes place with the decreasing of the thickness of the contour, that is with the increase of its slenderness. This phenomenon should be expected with regard to the known similar behaviour of the series solutions of the boundary layer theory. It does not become evident, however, until the universal solution in the two-parameter approximation is applied, that due to the high quality one-parameter approximation is almost unnecessary from the practical point of view.

We will present now ₁he procedure for the practical use of the universal solutions from the table T-1. Thereby the solutions $H(1/2)$, $H(1)$, $S(1/2)$, $S(1)$ are meant, for which the additional integration of the momentum equation in contrast to the solution $L(1)$ is not necessary.

Starting with the formulae for:

— the displacement thickness δ^* (2.13)
— the momentum thickness δ^{**} (2.6) and
— the skin friction

(5.2) $$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

where μ is coefficient of dynamic viscosity, one obtains after employing the transforms (2.3):

$$\delta^*=\frac{\left(a_0\,\nu\int\limits_0^x U^{b_0-1}\,dx\right)^{1/2}}{U^{-b_0/2}}A \quad\text{with}\quad A=\int\limits_0^\infty\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta;$$

(5.3) $$\delta^{**}=\frac{\left(a_0\,\nu\int\limits_0^x U^{b_0-1}\,dx\right)^{1/2}}{U^{b_0/2}}B \quad\text{with}\quad B=\int\limits_0^\infty\frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta;$$

$$\tau_w=\frac{\mu\,U^{1+\frac{b_0}{2}}}{\left(a_0\,\nu\int\limits_0^x U^{b-1}\,dx\right)^{1/2}}\Phi''(0),$$

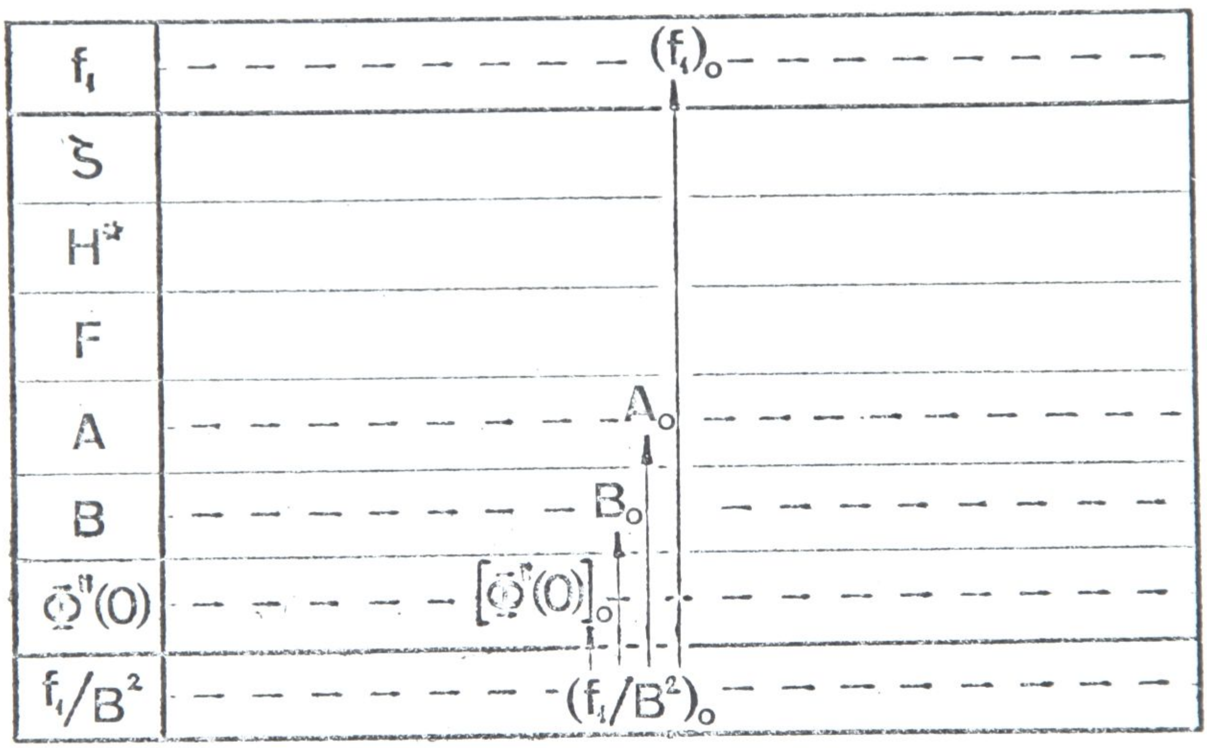
By the calculations of the usual characteristic quantities of the boundary layer the free stream velocity $U(x)$ is introduced into (5.3), while the values of the constants a_0; b_0 are fitted with the corresponding universal solution. It is to be reminded here that $a_0=b_0=2$ have been adopted for the solutions denoted with $H(1/2)$ and $H(1)$, and $a_0=0.4408$, $b_0=5.714$, for $S(1/2)$ and $S(1)$.

The corresponding values of A, B and $\Phi''(0)$ are determined from the table T-1 by means of (5.1), after the constants a_0; b_0 are fitted in it also. Into the right-hand side of (5.1) the free stream velocity $U(x)$ is introduced first in order to calculate the numerical value of (5.1) for given $x=x_0$. The left-hand side of (5.1) $(f_1/B^2)_0$ is determined in this way also, by means of which the corresponding values $(f_1)_0$, A_0, B_0 and $[\Phi''(0)]_0$ are easily determined from the table T-1 (see the following principal scheme.

The diagrams in Fig. 6 and 7 can be used as a kind of nomograms for $(f_1)_0$, A_0, B_0 and $[\Phi''(0)]_0$ on the basis of $(f_1/B^2)_0$ in case when a special accuracy is not necessary, in order to obtain approximate values fast. This procedure is shown on both diagrams (see Fig. 6 and 7) in a schematic way.

Namely, beginning with the ordinate, corresponding to the given value of $(f_1/B^2)_0$, a straight line parallel to the f_1 — axis is drawn first (follow the direction denoted by arrows on the diagrams) to the intersection with the distributions of f_1/B^2 denoted with $H(1)$ and $H(1/2)$ (see Fig. 6), namely with the corresponding distributions denoted with $S(1)$ and $S(1/2)$ (see Fig. 7). The straight lines parallel with the ordinate axis are drawn afterwards from these points. They intersect the corresponding distributions B, $\Phi''(0)$ and A and determine the desired values of $(f_1)_0$.

T-1



Drawing lines parallel with f_1 — axis from the intersections with the mentioned distributions, the remaining quantities A_0, B_0 and $[\Phi''(0)]_0$ are determined for the universal solutions $H(1)$ and $H(1/2)$ in Fig. 6 and for $S(1)$ and $S(1/2)$ in Fig. 7.

After substituting these quantities into (5.3), no matter how they are obtained — by using the table T-1, or graphically from the diagrams in Fig. 6 and 7, the desired quantities $\delta^*(x_0)$, $\delta^{**}(x_0)$ and $\tau_w(x_0)$ are finaly easily calculated.

6. Conclusions

From the previous Sections follows that all demands, stated at the beginning of this paper to the parametric method of Loitsianskii's type for the solution of various problems of boundary layer theory have been fulfilled in a satisfactory way. Originating from their basic ideas, but introducing more appropriate variables, a slightly different procedure has been developed, that makes prossible to calculate the characteristic quantities of the boundary layer

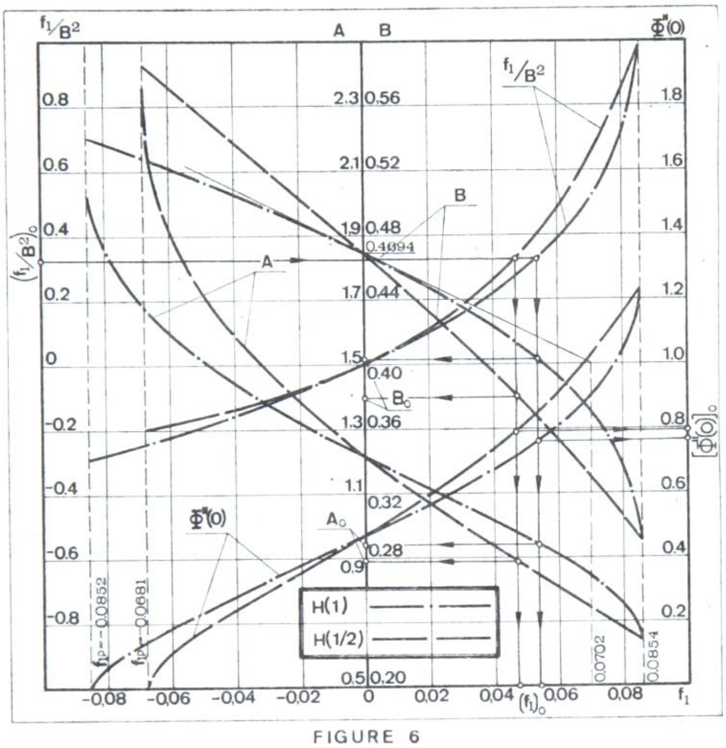


FIGURE 6

very fast, efficiently and accuratly enough, using thereby exclusively the table T-1 and the ready formulae (5.3). Using the diagrams in Fig. 7 in the role of nomograms, makes these caculations more efficient and simple, when less precision is expected from the results, as shown in the preceding Section. All that makes that the method by Loitsianskii [3], improved in this paper, becomes in comparison with other known analytical procedures more accessible and acceptable for the application in engineering practice.

Namely, by the procedure used by Terrill [13] for the flow around a circular cylinder, the boundary layer for any other flow can be calculated very accuratly without any doubt. This way, however, regarding the numerical integration in every special case is neither economical, nor simple at frequent applications, in contrast, with the procedure proposed here. This fact becomes especially apparent when characteristics of the boundary layer in a section determined by $x = x_0$ are to be calculated accuratly, whereby the prehistory of the boundary layer is not especially interesting.

It is to be mentioned here that the proposed procedure can be successfully used at an experimentally given free stream velocity also, what is of particular interest for engineering practice. In this case one should form an interpolation polynomial on the basis of experimental data and apply the procedure presented here for the characteristic quantities of the boundary layer.

The results obtained on the basis of the one-parameter localized solution $S(1/2)$ are especially characteristic for the accuracy of the procedure. It can be seen, in Fig. 2 and 3 that the results, obtained by means of this solution approach the exact values T on a large part of the skin friction distribution, differing thereby insignificantly from the one-parameter curve $S(1)$. On this

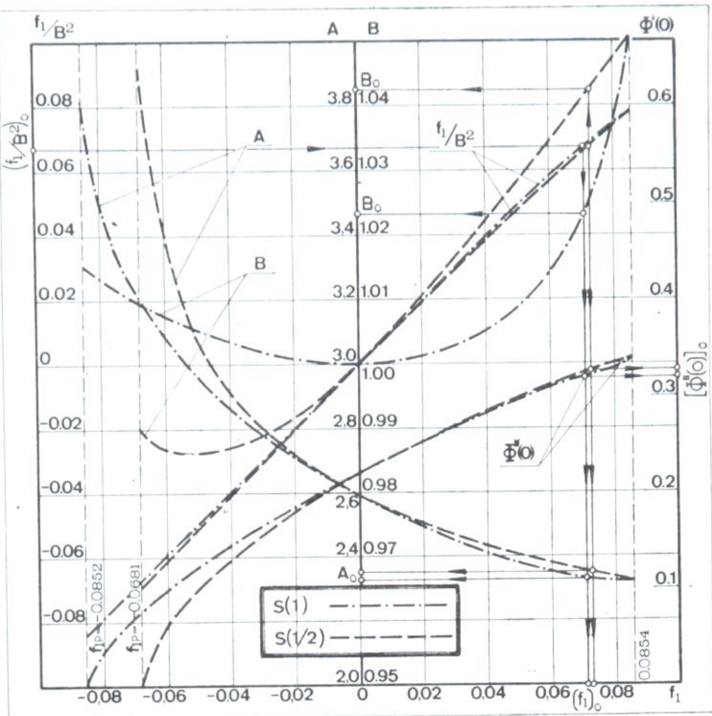


FIGURE 7

part the results $S(1/2)$ are better not only from the corresponding distributions $H(1/2)$ and $H(1)$, what should be expected regarding the analysis carried out, but from the results denoted by $L(1)$, what could be surprising at the first sight. Of course the immediate neighbourhood of the separation point, where the curve $L(1)$ approaches a little better to T, than $S(1/2)$, is to be excluded here.

The reason for the improvement of one-parameter results in comparison with the corresponding results due to the method by Loitsianskii [3] should be probably sought in the basic difference between both procedures, expressed through the distinct ways of integration of the impulse equation (2.12). In our case, the additional integration of the equation (2.12) has been fully avoided by the appropriate choice of the transformations (2.3), because the form parameter (2.7) itself can be regarded as the result of a formal integration of the impulse equation, as mentioned before. The calculation of the characteristic function F in the course of numerical integration (by means of (4.2) in the case of the one-parameter approximation, for example) has been reached in this way. The determination of the function $F^{(1)}$ performs, therefore, automatically, parallel to the solution of the system of universal differential equations. In contrast with that, the integration of the impluse equation performs additionally, namely, for every special case of flow in the method by Loitsianskii [3]. Thereby, the function $F^{(1)}$ is supposed in the form of (4.5), where the correction $\varepsilon(f_1)$ represents the deviation of the real distribution from the linear one. It is to be noted that also the corresponding procedure proposed for the integration of the impulse equation, although correct, did not appear as enough suitable and reliable for practical applications.

A particular attention to the "strengtheness" of the first parameter f_1 has been paid in this paper. The choice of optimal values of a_0 and b_0 — the task set at the beginnining of this paper — has been solved successfully. However, although the results obtained by means of $a_0 = 0.4408$ and $b_0 = 5.7140$ are extraordinary good, the procedure concerning the choice of them should be subjected to a more rigorous mathematical analysis.*)

The same can be said for the convergence of the solution proposed, for which unfortunately an exact mathematical procedure has not been developed. In absence of such a possibility, we carried out here a consideration of so named physical convergence on the basis of the behaviour of first three parameters of the set f_k (f_1, f_2, f_3), approaching the separation point. A conclusion could be drawn that the physical convergence is with a very suitable speed of approaching to the exact solution exists. The convergence is getting worse with the slenderness of the contour, as expected, but this is not of special importance from the practical point of view, because the one-parameter solution is qualitatively very stable.

It should be emphasized at the end that the main goal of this paper was of methodological character — to affirm and possibly improve the modern parameter method by Loitsianskii [3] on the example of simple two-dimensional steady boundary layer of incompressible flow. As mentioned before, this method has been already successfully extended to the more complex models of the boundary layer, as for example for compressible flows, for power law nonnewtonian flows and MHD flows. It can be interesting, however, for other problems that reduce to nonlinear differential equations of parabolic type also, as for example for the problem of nonlinear conduction of heat.

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BEITRAG ZU DEN UNIVERSELLEN LÖSUNGEN VON GRENZSCHICHTTHEORIE

Viktor N. Saljnikov

Zusammenfassung

Durch Einführung zweckmäßiger Transformationen und der seitens Loitsianskii [3] definierten Formparametermenge in dem Ausgangssystem von Differentialgleichungen der laminaren ebenen und stationären Grenzschicht hat man die universelle Gleichung betrachtetes Problems gewonnen. Die entsprechende auf der Rechnenanlage vermittels numerischer Integration berechnete und tabulierte Lösung, nämlich, kann man bei der Betrachtung beliebiges speziellen Strömungfalles inkompressibler Flüssigkeit leicht benützen. Die bei der Anwendung der ursprünglichen Methode von Loitsianskii unvermeidliche nachträgliche Integration der Impulsgrenzschichtgleichung, in diesem Falle überflüssig wird, was die Benützung dieses Verfahrens im Ingenieurpraxis wesentlich vereinfacht. Aus der Betrachtung der Kreiszylinderumströmung folgt, daß schon in der einparametrigen Näherung erhaltene Werte sehr nah zu exakten liegen. Man erblickt dabei, im Vergleich zur entsprechenden einparametrigen Lösung

von Loitsianskii die gewisse Besserung erhaltener Resultate. Zu diesem Schluß kam auch Ivanović [2] bei der Betrachtung der Zylinderumströmung elliptisches Querschnitts für verschiedene Verhältnisse der Halbachsen. Die durch Mirgaux [4] gewonnene Lösung derselben universellen Gleichung in zweiparametriger Näherung hat die weitere Resultatenbesserung ermöglicht. Damit ist die befriedigende Konvergenzgeschwindigkeit, des in dieser Arbeit entwickelten Verfahrens nachgewiesen. Man soll noch bemerken, daß es schon auf die komplizierteren physikalischen Grenzschichtmodelle mit Erfolg übertragen ist. Nämlich, auf die: kompressible Strömung (Saljnikov / Boričić [10]), Strömungen von nichtnewtonschen Potenzgesetzflüssigkeiten (Saljnikov / Đukić [11]) und magnetohydrodynamischen Strömungen (Saljnikov / Boričić [12]).

ПРИЛОГ УНИВЕРЗАЛНИМ РЕШЕЊИМА ТЕОРИЈЕ ГРАНИЧНОГ СЛОЈА

Викшор Н. Саљников

Резиме

Увођењем сврсисходних трансформација и скупа параметара Лојцјанског [3] у почетни систем диференцијалних једначина ламинарног раванског стационарног граничног слоја добијена је универзална једначина. Њено решење, наиме, срачунато нумеричком интеграцијом на електронском рачунару и сређено у таблицама, може се лако користити при решавању ма ког специјалног случаја струјања нестишљивог флуида. Допунска интеграција импулсне једначине граничног слоја неопходна код примене првобитне методе Лојцјанског [3] у овом случају постаје сувишна, што битно упрошћава коришћење овог поступка у инжењерској пракси. Из разматрања опструјавања цилиндра кружног пресека следује, да се већ у једнопараметарском приближењу постижу вредности блиске тачним. При томе се у поређењу са одговарајућим једнопараметарским решењем Лојцјанског уочава извесно побољшање добијених резултата. До овог закључка је дошао и Ивановић [2] при разматрању опструјавања цилиндара пресека елипсе за различите односе њених полуоса. Решење исте универзалне једначине у двопараметарском приближењу добијено од страна Миргоа [4], омогућило је даље побољшање резултата. Тиме је потврђена задовољавајућа брзина конвергенције, коју обезбеђује поступак развијен у овом раду. Треба још напоменути, да је он већ са успехом проширен и на сложеније физичке моделе граничног слоја. Наиме, на струјања стишљивог флуида (Саљников и Боричић [10]), струјања нењутновских степених течности (Саљников и Ђукић [11]) и магнетохидродинамичка струјања (Саљников и Боричић [12]).