TEORIJSKA I PRIMENJENA MEHANIKA 4, pp. 106—114, 1978.

SOLVING A MIXED BOUNDARY VALUE PROBLEM FOR THE BENDING OF A PLATE

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1. Introduction

Let the plate D of thickness $2h$ be loaded in D continuously with the densities of force $F_2(\zeta)$ and couple $M_x(\zeta)+i M_y(\zeta)$ on the element$d\xi\, d\eta$. If notations

$$F(\zeta)=\frac{3}{2\pi h^2}F_z(\zeta)$$

and

$$M(\zeta)=-\frac{3i}{2\pi h^2}[M_x(\zeta)+i M_y(\zeta)]$$

are introduced, then the following equations for the deflection w can be obtained [1]:

$$w=-\frac{1-\nu}{16\mu}\iint_D\{\ln(z-\zeta)(\bar z-\bar\zeta)[(z-\zeta)(\bar z-\bar\zeta)F(\zeta)+(z-\zeta)\overline{M(\zeta)}+$$

$$+(\bar z-\bar\zeta)M(\zeta)]-(z-\zeta)[\overline{M(\zeta)}-\bar\zeta F(\zeta)]-(\bar z-\bar\zeta)[M(\zeta)-\zeta F(\zeta)]\}\,d\xi\, d\eta-$$

$$(1—1)\qquad \frac{1-\nu}{16\mu}\{\bar z\varphi_0(z)+z\overline{\varphi_0(z)}+\chi_0(z)+\overline{\chi_0(z)}$$

where $\varphi_0(z)$ and $\chi_0(z)$ in D are holomorphic functions if D is a simply connected domain. In the above equation ν is Poisson's number and μ is Lame's coefficient. All the notations and introductions of functions are adapted to the paper [2]. Further, the next expression will be important

$$\frac{\partial w}{\partial\bar z}=-\frac{1-\nu}{16\mu}\iint_D\left\{zF(\zeta)+\frac{z-\zeta}{\bar z-\bar\zeta}\overline{M(\zeta)}+\ln(z-\zeta)(\bar z-\bar\zeta)[(z-\xi)F(\zeta)+\right.$$

$$(1—2)\qquad \left.+M(\zeta)]\right\}d\xi\, d\eta-\frac{1-\nu}{16\mu}[\varphi_0(z)+z\overline{\varphi_0'(z)}+\overline{\psi_0(z)}]$$

$$\psi_0(z)=\chi_0'(z).$$

If the bending is treated according to Kirchhoff's theory, we also get

$$-\frac{12}{(1-\nu)h^2}\int^{z}\left[G+i\int^{s}H\,ds\right]d\zeta=-(\varkappa+1)\iint_D[(z-\zeta)F(\zeta)+M(\zeta)]\cdot$$

$$\cdot\ln(z-\zeta)\,d\xi\,d\eta+\iint_D\left\{z\,F(\zeta)+\frac{z-\zeta}{\bar z-\bar\zeta}\overline{M(\zeta)}+\ln(z-\zeta)(\bar z-\bar\zeta)[(z-\zeta)(\zeta)+\right.$$

$$\left.+M(\zeta)]\right\}d\xi\,d\eta+[-\varkappa\varphi_0(z)+z\,\overline{\varphi_0'(z)}+\overline{\psi_0(z)}]-i\,Kz-K_1 \tag{1—3}$$

where G is the bending couple and H the generalized shear force per unit of the plate cross-section,

$$\varkappa=\frac{3+\nu}{1-\nu},$$

K is an arbitrary real and K an arbitrary complex constant. By using the notations $\tilde\varphi(z)$, $\tilde\varphi_1(z)$ and $\tilde\psi(z)$ from [3], the equations (1—2) and (1—3) can be shortened:

$$\frac{\partial w}{\partial\bar z}=-\frac{1-\nu}{16\mu}\{[\tilde\varphi(z)+z\,\overline{\tilde\varphi_1(z)}+\overline{\tilde\psi(z)}]+$$

$$+[\varphi_0(z)+z\,\overline{\varphi_0'(z)}+\overline{\psi_0(z)}]\} \tag{1—4}$$

$$-\frac{12}{(1-\nu)h^2}\int^{z}\left[G+i\int^{s}H\,ds\right]d\zeta=-(\varkappa+1)\tilde\varphi(z)+[\tilde\varphi_1(z)+z\,\overline{\tilde\varphi(z)}+$$

$$+\overline{\tilde\psi(z)}]+[-\varkappa\varphi_0(z)+z\,\overline{\varphi_0'(z)}+\overline{\psi_0(z)}]+i\,Kz+K_1. \tag{1—5}$$

2. The definition of a mixed boundary value problem for the bending of a circular plate

Let the boundary curve C of circle D be divided by $2\,n$ points into the sections

$$[a_1, b_1],\ [b_1, a_2],\ [a_2, b_2],\ \ldots,\ [a_n, b_n],\ [b_n, a_1].$$

Then we write

$$C_1=\bigcup_{k=1}^{k=n}[a_k, b_n]\quad\text{and}\quad C_2=\bigcup_{k=1}^{k=n}[b_k, a_{k+1}]$$

where $a_{n+1}=a_1$. Let the boundary value problem be expressed in this way: let the plate be clamped on the union of sections C_1 and the boundary curve C be free on the union of sections C_2 (unloaded), consequently:

$$w=0,\quad \frac{\partial w}{\partial r}=0,\quad z\in C_1 \tag{2—1}$$

$$G=0,\quad H=0,\quad z\in C_2. \tag{2-2}$$

With the above two conditions and at the given functions $\tilde{\varphi}(z)$, $\tilde{\varphi}_1(z)$ and $\tilde{\psi}(z)$ the deflection w should be determined. Mathematically the above two equations, considering (1—2) and (1—3), can be expressed in the following way:

$$[\tilde{\varphi}(z)+z\overline{\tilde{\varphi}_1(z)}+\overline{\tilde{\psi}(z)}]^+ + [\varphi_0(z)+z\overline{\varphi_0{}'(z)}+\overline{\psi_0(z)}]^+ = 0, \qquad z\in C_1 \tag{2—3}$$

$$\begin{aligned}&[-\varkappa\tilde{\varphi}(z)+z\overline{\tilde{\varphi}_1(z)}+\overline{\tilde{\psi}(z)}]^+ + [-\varkappa\varphi_0(z)+z\overline{\varphi_0{}'(z)}+\\ &+\overline{\psi_0(z)}]^+ = iKz+K_1, \qquad z\in C_2.\end{aligned} \tag{2—4}$$

The constants K and K_1 are (except on one section $[b_{k_0}, a_{k_0+1}]$ where they can be taken as equal to 0) still unknown and not arbitrary any more. So they will still have to be determined for each section separately. In addition, the fact that the condition (2—3) is not equivalent to the condition (2—1) if $n>1$, will also have to be considered.

3. The course of the solving of the boundary value problem

First let for loading of the plate in the domain D the validity of the following equations be required:

$$\iint_D F(\zeta)\,d\xi\,d\eta = 0 \tag{3—1}$$

$$\iint_D [M(\zeta)-\zeta F(\zeta)]\,d\xi\,d\eta = 0. \tag{3—2}$$

Now the functions $f_1(z)$, $f_2(z)$, $f_3(z)$ and f_4 are introduced similarly as it is done in [3]. Let us then consider the characteristics of these functions for large values of $|z|$. Consequently, the equations (2—3) and (2—4) can be written es below:

$$\begin{aligned}&\left[zf_1(z)+f_2(z)+z\bar{\varphi}_0{}'\left(\frac{1}{z}\right)+\bar{\psi}_0\left(\frac{1}{z}\right)\right]^- - \left[-z\bar{f}_1\left(\frac{1}{z}\right)-\bar{f}_1{}'\left(\frac{1}{z}\right)-\right.\\ &\left.-\bar{f}_2{}'\left(\frac{1}{z}\right)+\bar{f}_4\left(\frac{1}{z}\right)-\bar{f}_3\left(\frac{1}{z}\right)-\varphi_0(z)\right]^+ = 0, \qquad z\in C_1\end{aligned} \tag{3—3}$$

$$\begin{aligned}&\left[-\varkappa zf_1(z)-\varkappa f_2(z)+z\bar{\varphi}_0{}'\left(\frac{1}{z}\right)+\bar{\psi}_0\left(\frac{1}{z}\right)\right]^- - \left[-z\bar{f}_1\left(\frac{1}{z}\right)-\bar{f}_1\left(\frac{1}{z}\right)-\right.\\ &\left.-\bar{f}_2{}'\left(\frac{1}{z}\right)+\bar{f}_4\left(\frac{1}{z}\right)-\bar{f}_3\left(\frac{1}{z}\right)+\varkappa\varphi_0(z)\right]^+ = iK_l z+K_{1l},\\ &1\leqslant l\leqslant n,\\ &z\in[b_l, a_{l+1}].\end{aligned} \tag{3—4}$$

Now let be

$$(3\text{—}5)\qquad F_1(z) = -\varkappa z f_1(z) - \varkappa f_2(z) + z\bar{\varphi}_0'\left(\frac{1}{z}\right) + \bar{\psi}_0\left(\frac{1}{z}\right) - \bar{\alpha}_1 z, \qquad |z|>1,$$

and

$$(3\text{—}6)\qquad F_2(z) = -z\bar{f}_1\left(\frac{1}{z}\right) - \bar{f}_1'\left(\frac{1}{z}\right) - \bar{f}_2'\left(\frac{1}{z}\right) + \bar{f}_4\left(\frac{1}{z}\right) - \bar{f}_3\left(\frac{1}{z}\right) + \varkappa\varphi_0(z) - \bar{\alpha}_1 z, \qquad |z|<1$$

where α_1 has the following meaning:

$$(3\text{—}7)\qquad \varphi_0(z) = \alpha_0 + \alpha_1 z + O(z^2).$$

$F_1(z)$ is holomorphic outside D and bounded at the infinity, but $F_2(z)$ is holomorphic inside D. Out of (3—4) it follows that

$$(3\text{—}8)\qquad F_1^-(z) - F_2^+(z) = i K_l z + K_{1l}, \qquad z \in C_2.$$

If shorter notation is introduced

$$(3\text{—}9)\qquad g(z) = \begin{cases} -(i K_l z + K_{1l}), & z \in [b_l, a_{l+1}], \quad 1 \leqslant l \leqslant n \\ 0 & , \quad z \in C_1 \end{cases}$$

then (3—8) yields

$$(3\text{—}10)\qquad F(z) = \frac{1}{2\pi i} \int\limits_{C_2} \frac{g(\zeta)\, d\zeta}{\zeta - z} + F^*(z)$$

where $F^*(z)$ is a regular function everywhere in the closed plane, except perhaps on C_1.

Further there is still:

$$(3\text{—}11)\qquad \begin{aligned} F(z) &= F_2(z), \qquad |z|<1 \\ F(z) &= F_1(z), \qquad |z|>1 \end{aligned}$$

and

$$(3\text{—}12)\qquad F_0(z) = \frac{1}{2\pi i} \int\limits_{C_2} \frac{g(\zeta)\, d\zeta}{\zeta - z} = -\frac{1}{2\pi i} \sum_{l=1}^{l=n} \left[i K_l (a_{l+1} - b_l) + (iz K_l + K_{1l}) \ln \frac{a_{l+1} - z}{b_l - z} \right].$$

The following notations are further used

$$(3\text{—}13)\qquad \Phi_1(z) = z f_1(z) + f_2(z) + z\bar{\varphi}_0\left(\frac{1}{z}\right) + \bar{\psi}_0\left(\frac{1}{z}\right) - \bar{\alpha}_0 z \qquad |z|>1$$

and

$$\Phi_2(z) = -z\bar{f}_1\left(\frac{1}{z}\right) - \bar{f}_1'\left(\frac{1}{z}\right) - \bar{f}_2'\left(\frac{1}{z}\right) + \bar{f}_4\left(\frac{1}{z}\right) - \bar{f}_3\left(\frac{1}{z}\right) -$$

(3—14)

$$-\varphi_0(z) - \bar{\alpha}_1 z, \qquad |z| < 1.$$

The first function is holomorphic outside D and bounded in the infinity while the second one is holomorphic inside D. Obviously there is still

(3—15) $$\Phi_1(z) = F_1(z) + (\varkappa + 1)[zf_1(z) + f_2(z)]$$

and

$$\Phi_2(z) = -\frac{1}{\varkappa} F_2(z) + \frac{\varkappa + 1}{\varkappa}\left[-z\bar{f}_1\left(\frac{1}{z}\right) - \bar{f}_1'\left(\frac{1}{z}\right) - \bar{f}_2'\left(\frac{1}{z}\right) +\right.$$

(3—16)

$$\left. + \bar{f}_4\left(\frac{1}{z}\right) - \bar{f}_3\left(\frac{1}{z}\right) - \bar{\alpha}_1 z.\right]$$

Considering (3—12) and the last two equations, the equation (3—3) can be written:

(3—17) $$\varkappa F^{*-}(z) + F^{*+}(z) = A(z) \qquad z \in C_1$$

where the notations were used:

$$A(z) = (\varkappa + 1)\left\{-F_0(z) + \left[-z\bar{f}_1\left(\frac{1}{z}\right) - \bar{f}_1'\left(\frac{1}{z}\right) - \bar{f}_2'\left(\frac{1}{z}\right) + \bar{f}_4\left(\frac{1}{z}\right) -\right.\right.$$

(3—18)

$$\left.\left. -\bar{f}_3\left(\frac{1}{z}\right) - \bar{\alpha}_1 z\right]^+ - \varkappa[zf_1(z) + f_2(z)]^-\right\}, \qquad z \in C_1.$$

In the equation (3—17), outside C_1, $F^*(z)$ is always a regular function. From [4] we understand that is obtained in the following way:

(3—19) $$F^*(z) = \frac{\chi_0(z)}{2\pi i}\int_{C_1}\frac{A(\zeta)\,d\zeta}{\chi_0^+(\zeta)\cdot(\zeta - z)},$$

where there is

(3—20) $$\chi_0(z) = \prod_{k=1}^{k=n}(z - a_k)^{\gamma}(z - b_k)^{1-\gamma}$$

(3—21) $$\gamma = \frac{1}{2} + i\frac{\ln\varkappa}{2\pi}$$

and still due to the required regularity in $z = \infty$

(3—22) $$\int_{C_1}\frac{A(\zeta)\zeta^k}{\chi_0^+(\zeta)}d\zeta = 0, \qquad 0 \leqslant k \leqslant n - 2,$$

when $n = 1$, the above conditions are not present.

Let us see if the presented problem is solvable. If $K_1 = 0$ and $K_{11} = 0$ are taken, then the constants K_l with $2 \leqslant l \leqslant n$, $\operatorname{Re}[K_{1l}]$ and $\operatorname{Im}[K_{1l}]$ with $2 \leqslant l \leqslant n$

are determined by $3n-3$ real numbers. Together with $\mathrm{Re}[\alpha_1]$ and $\mathrm{Im}[\alpha_1]$ there are $3n-1$ real numbers, which should be determined. On the other side, by equating the coefficients of the first power of z on both sides in equations (3—22) and (3—6) we obtain $2n$ equations in the real domain for the unknown numbers. Additional $n-1$ equations result from the relations:

$$w(\tilde{\alpha}_k) - w(\tilde{\alpha}_1) = w_0(\tilde{\alpha}_k) - w_0(\tilde{\alpha}_1) \qquad 2 \leqslant k \leqslant n \tag{3—23}$$

where w means the actually calculated deflection and w_0 the one that is required for a given point. Here $\tilde{\alpha}_k$ is an arbitrary point on $[a_k, b_k]$, $2 \leqslant k \leqslant n$ and $\tilde{\alpha}_1$ an arbitrary point on $[a_1, b_1]$.

Consequently, the number of equations corresponds to the number of unknown quantities. With zero loadings the solving of the following problem is known: $\varphi_0(z)=0$, $\psi_0(z)=0$ and $K_1=K_{1l}=0$, $1 \leqslant l \leqslant n$, thus also $\alpha_1=0$. As at that time the equation system is homogeneous, this means that it is a Cramer's system. Arranging the equation system according to the usual manner, we can easily see that the coefficients with the unkown quantities are independent from loading. It means that the equation system for determination of unknown quantities is at an arbitrary loading uniquely solvable.

Finally (3—6) yields

$$\begin{aligned} \varkappa\varphi_0(z) = \bar{\alpha}_1 z + z\bar{f}_1\left(\frac{1}{z}\right) + \bar{f}_1'\left(\frac{1}{z}\right) + \bar{f}_2'\left(\frac{1}{z}\right) - \bar{f}_4\left(\frac{1}{z}\right) + \\ + \bar{f}_3\left(\frac{1}{z}\right) + F_0(z) + F^*(z) \end{aligned} \tag{3—24}$$

and (3—5) yields

$$\psi_0(z) = \varkappa\left[\frac{1}{z}\bar{f}_1\left(\frac{1}{z}\right) + \bar{f}_2\left(\frac{1}{z}\right)\right] - \frac{\varphi_0'(z) - \alpha_1}{z} + \bar{F}_0\left(\frac{1}{z}\right) + \bar{F}^*\left(\frac{1}{z}\right). \tag{3—24'}$$

In the case $n=1$ there are no conditions for form (3—22) and (3—23). The only constants that have to be determined are $\mathrm{Re}[\alpha_1]$ and $\mathrm{Im}[\alpha_1]$. If now $F_0(z)=0$ is considered, α_1 can be calculated out of the equation:

$$F^{*\prime}(0) = \varkappa\alpha_1 - \bar{\alpha}_1 + \iint_D [\bar{\zeta} M(\zeta) + \zeta \overline{M(\zeta)} - \zeta\bar{\zeta} F(\zeta)]\, d\xi\, d\eta. \tag{3—25}$$

However, α_1 is contained also on the left side of the equation. The equation has a unique solution.

The unfulfilment of the equations (3—1) and (3—2) has no essential meaning. At arbitrary values of the left sides of these equations the problem has a unique solution.

4. Some remarks on solving the boundary-value problem considering the improved theory of the bending of a plate

In this case, when formulating the same boundary value problem, the equation (2—4) can be taken while the equation (2—3) changes. In the first bracket (2—3) on the left only the boundary values of holomorphic functions prove to appear while in the second bracket the member $\alpha\overline{\varphi_0''(z)}$ where α is a given constant has to be joined additionally. Treating the case analogously with the preceding one, we come to the same equation (3—8) and to the changed form of equation (3—17) as below:

$$\varkappa F^{*-}(z) + [F^*(z) + \alpha \overline{F^{*''}(z)}]^+ = A_0(z), \qquad z \in C_1 \tag{4—1}$$

where the boundary values of (known) holomorphic functions only and constants K_l, K_{1l}, $1 \leqslant l \leqslant n$, as well as α_1 appear in $A_0(z)$. This equation is in comparison to the equation (3—17) far more complicated, however, it can be solved by means used for solving singular integral equations with Cauchy's kernel in a generalized form [4], [5]. At the required regularity of function $F^*(z)$ in $z=\infty$ the conditions analoguous to those in (3—22) and (3—23) have still to be fulfilled. Fulfilling these conditions it is possible to determine uniquely the constants appearing in $A_0(z)$. For the expression of $\varphi_0(z)$ and $\psi_0(z)$ the equations (3—24) and (3—25) are again obtained.

The course of the solving of equation is not given here in detail, because the general function form $A_0(z)$ it does not offer an elementary expression for the solution $F^*(z)$ if C_1 is not the entire circle C. From (4—1) it can be concluded that the boundary value problems, considering the improved theory of bending, are mathematically far more difficult than when the same problems are solved by Kirchhoff's theory. It is therefore more advisable in practice to content oneself with some approximate solution of the equation (4—1) than to follow the general method of N. P. Vekua. This can be done if we write the equation (4—1) in the form as below:

$$\varkappa F^{*-}(z) + F^{*+}(z) = A_0(z) - \alpha \overline{F^{*''}(z)^+}, \qquad z \in C_1.$$

Considering (3—19) it follows:

$$F^*(z) = \frac{\chi_0(z)}{2\pi i} \int_{C_1} \frac{A_0(\zeta) - \alpha \overline{F^{*''}(\zeta)^+}}{\chi_0^+(\zeta)\cdot(\zeta - z)}\, d\zeta. \tag{4—2}$$

At low values of α an approximation will be obtained by substituting the expression $F^{*''}(\zeta)^+$ for the expression $F_0^{*''}(\zeta)^+$ if $F_0^*(z)$ means the solution of equation (4—1) at $\alpha = 0$.

Thus an elementarily expressed approximation is reached:

$$F^*(z) \approx \frac{\chi_0(z)}{2\pi i} \int_{C_1} \frac{A_0(\zeta) - \alpha\, \overline{F_0^{*''}(\zeta)^+}}{\chi_0^+(\zeta)\cdot(\zeta - z)}\, d\zeta. \tag{4—3}$$

5. Comparison of the solved boundary value problem with the analogous boundary value problem of a plate, unloaded in D

For a plate that is in D unloaded, that is, if $F(\zeta) = M(\zeta) = 0$, $\zeta \in D$ we have

$$\tilde{\varphi}(z) = \tilde{\varphi}_1(z) = \tilde{\psi}(z) = 0 \tag{5—1}$$

$$f_1(z) = f_2(z) = f_3(z) = f_4(z) = 0 \tag{5—2}$$

and the boundary value conditions are not homogeneous any more:

$$\frac{16\mu}{1-\nu}\cdot\frac{\partial w}{\partial \bar{z}}=h_1(z), \qquad z\in C_1 \tag{5—3}$$

$$\frac{12}{(1-\nu)h^2}\int^{z}\left[G+i\int^{s}H\,ds\right]d\zeta=h_2(z)+i\,K_l\,z+K_{1\,l}, \qquad z\in C_2 \tag{5—4}$$

because $h_1(z)$ and $h_2(z)$ are not equal to zero. In this case, on the left sides of these conditions only the boundary values of the functions $\varphi_0(z)$, $\varphi_0{}'(z)$ and $\psi_0(z)$ are to be found. Since the right sides as well can be treated as a linear combination of the boundary values of analytical functions, for example:

$$h_1(z)=g_1^{+}(z)-g_1^{-}(z),\ g_1(z)=\frac{1}{2\pi i}\int_{C_1}\frac{h_2(\zeta)\,d\zeta}{\zeta-z} \tag{5—5}$$

and similarly for $h_2(z)$, then it can be seen that it is possible to transform the problem (5—3)—(5—4) into the same form of solving as the afore-treated problem. The equations (3—8) and (3—17) are again obtained and with the improved theory too (4—1). In the same way also the treating of the boundary value problem for a plate, loaded in D with unhomogeneous boundary value conditions also takes exactly the same course, which can be easily understood. Consequently, it can be concluded that, in solving the boundary value problem by the discussed method, a load in D does not represent any additional difficulty in mathematical sense.

REFERENCES

[1] Савин, Г. Н., Прусов, Й. А.: *Об одной решении основных задач изгиба изотропних плит для некоторих областей* РМ, т 5, в 6, п 66—73, 1965

[2] Muršič, M.: *Koncentracija napetosti na robovih lukelj v zmerno debelih elastičnih ploščah*, Ljubljana 1965

[3] Krušič, B.: *A contribution to the solving of the fundamental boundary-value problems of bending of moderate thick plates*, TPM

[4] Мусхелишвили, Н. J.: *Сингульарные интегральные уравнения*, Наука, Москва 1965

[5] Векуа, Н. П.: *Сисшемы сингуларных интегральных уравнений.* Наука, Москва 1970

LÖSUNG EINER GEMISCHTEN RANDWERTAUFGABE DER PLATTENBIEGUNG

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Zusammenfassung

In diesem Aufsatz wird eine Methode zur Lösung einer gemischten Randwertaufgabe der Plattenbiegung behandelt. Die Platte is stetig mit Kraft und Moment belastet. Es wird festgestellt, dass die Behandlung dieser Randwertaufgabe im mathematischen Sinne zu wesentlichen Schwierigkeiten führt, wenn anstatt der Kirchhoffs Theorie die verbesserte Theorie gebraucht wird.

REŠITEV NEKEGA MEŠANEGA ROBNEGA PROBLEMA UPOGIBA PLOŠČE

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Povzetek

V tem članku je obravnavana neka metoda reševanja mešanega robnega problema upogiba tanke plošče, ki je obremenjena zvezno s silo in momentom. Pokazano je, da vodi obravnava takega robnega problema do bistvenih komplikacij v matematičnem smislu, če uporabimo namesto Kirchhoffove teorije posplošeno teorijo upogiba.

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