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FIELD EQUATIONS FOR ISOTROPIC MICROPOLAR ELASTIC DIELECTRICS

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1. Introduction

A nonlinear theory of micropolar elastic dielectric is presented in [1]. Differential equations of motion and the balance equation for specific internal energy of deformation and polarization are obtained in the form:

$$\rho \dot{v}^i = t^{ij}_{\ ,j} + F^i + \rho f^i, \tag{1.1}$$

$$\rho \Gamma^{ij} = t^{[ij]} + m^{ijk}_{\ \ ,k} + \rho l^{ij} + r^{[i} p^{j]} + r^{[i|k|} p^{j]}_{,k}, \tag{1.2}$$

$$\rho \Gamma^i = r^{ik}_{\ ,k} - r^i + \varepsilon^i, \tag{1.3}$$

$$\rho \dot{w} = t^{ij} (v_{i,j} - \nu_{ij}) + m^{ijk} \nu_{ij,k} + r^i (\dot{p}_i - \nu_{ij} p^j) + r^{ik} (\dot{p}_{i,k} - \nu_{ij} p^j_{,k}). \tag{1.4}$$

The quantities in these equations are

t^{ij} — the nonsymmetric stress tensor,

F^i — the electromagnetic force per unit volume, which is determined by the expression

$$F^i = s^{ij}_{\ ,j} - \frac{1}{c} \dot{g}^i, \tag{1.5}$$

where s^{ij} is the electromagnetic stress tensor, c — the velocity of light and g^i — the electromagnetic impulse,

f^i — the mechanical body force per unit mass,

m^{ijk} — the couple stress tensor,

l^{ij} — the mechanical body couple per unit mass,

Γ^{ij} — the mechanical inertial spin, determined by

$$\Gamma^{ij} = I^{KL} \ddot{\chi}^{[i}_{\cdot K} \chi^{j]}_{\cdot L}, \tag{1.6}$$

where $I^{KL} = I^{LK}$ are microinertia coefficients and $\chi^k_{\cdot K}$ the orthogonal tensor determining the independent rigid rotations of dielectric's particles,

r^i — the local electrical vector of the field,

r^{ij} — the local electrical tensor of the field,

ε^i — the dynamical electric field, determined by the expression

$$\varepsilon^i = E^i + \frac{1}{c} \varepsilon^{ijk} v_j B_k, \tag{1.7}$$

where E^i is the electric field at rest, B_k the magnetic induction and ε^{ijk} the Ricci's permutation tensor,

Γ^i — the inertial spin of polarization, given by the expression

$$\Gamma^i = \nu \ddot{p}^i, \tag{1.8}$$

where ν is the polarization inertial coefficient and p^i the polarization vector, and

ν_{ij} — the giration skew-symmetric tensor, determined by the expression

$$\nu_{ij} = -\nu_{ji} = \dot{\chi}_{iK} \chi^{K}_{\cdot j}, \qquad (\chi_{iK} \chi^{K}_{\cdot j} = g_{ij}). \tag{1.9}$$

The boundary conditions are of the form

$$T^i = \left(t^{ij} - \left[\left| s^{ij} + \frac{v^j}{c} g^i \right| \right] \right) n_j, \qquad M^{ij} = m^{ijk} n_k, \qquad 0 = r^{ij} n_j, \tag{1.10}$$

where T^i is the mechanical surface force per unit area, M^{ij} the mechanical surface couple per unit area and n_j the outward unit normal vector of the dielectric boundary surface.

Using the specific internal energy of deformation and polarization, as a function of the form

$$w = w\left(x^k_{;K},\ \chi^k_{\cdot K},\ \chi^k_{\cdot L;K},\ p^k,\ p^k_{;K}\right), \tag{1.11}$$

in the paper [1] the nonlinear constitutive equations are obtained in the form

$$
\begin{aligned}
t^{ij} &= \rho g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K}, \\
m^{ijk} &= \rho \frac{\partial w}{\partial \chi^l_{\cdot L;K}} g^{l[i} \chi^{j]}_{\cdot L} x^k_{;K}, \\
r^i &= \rho g^{il} \frac{\partial w}{\partial p^l}, \\
r^{ij} &= \rho g^{il} \frac{\partial w}{\partial p^l_{;K}} x^j_{;K},
\end{aligned} \tag{1.12}
$$

and the condition of objectivity in the form

$$
\begin{aligned}
&\left(g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K} + g^{il} \frac{\partial w}{\partial \chi^l_{\cdot K}} \chi^j_{\cdot K} + g^{il} \frac{\partial w}{\partial \chi^l_{\cdot L;K}} \chi^j_{\cdot L;K} \right. \\
&\quad \left. + g^{il} \frac{\partial w}{\partial p^l} p^j + g^{il} \frac{\partial w}{\partial p^l_{;K}} p^j_{;K} \right)_{[ij]} = 0.
\end{aligned} \tag{1.13}
$$

2. Constitutive equations

For an isotropic micropolar dielectric we can introduce the following spatial tensors:

$$\varepsilon_{ij} = g_{ij} - \chi_{iK} X^K_{;j}, \tag{2.1}$$

$$k_{ijk} = G^{KL} \chi_{iK;M} X^M_{;k} \chi_{jL} = -k_{jik}, \tag{2.2}$$

(2.3) $$\pi_i = g_{il} p^l = p_i,$$

(2.4) $$\pi_{ij} = p_{i;K} X^K_{;j} = p_{i,j},$$

so that the specific internal energy of deformation and polarization can be considered as a function of the form

(2.5) $$w = w(\varepsilon_{ij}, k_{ijk}, \pi_i, \pi_{ij}).$$

Using the functional form (2.5) and the expressions (2.1) — (2.4), the nonlinear constitutive equations (1.12) become

(2.6) $$t^{ij} = \rho\left(\frac{\partial w}{\partial \varepsilon_{ij}} - \frac{\partial w}{\partial \varepsilon_{kj}} \varepsilon_k^{\cdot i} - \frac{\partial w}{\partial k_{klj}} k_{kl}^{\cdot\cdot i} - \frac{\partial w}{\partial \pi_{kj}} \pi_k^{\cdot i}\right),$$

(2.7) $$m^{ijk} = \rho \frac{\partial w}{\partial k_{ijk}},$$

(2.8) $$r^i = \rho \frac{\partial w}{\partial \pi_i},$$

(2.9) $$r^{ij} = \rho \frac{\partial w}{\partial \pi_{ij}}.$$

These equations, however, are not form-invariant with respect to superposed rigid motions. Therefore, the condition of objectivity must be satisfied:

(2.10) $$\left(\frac{\partial w}{\partial \varepsilon_{ik}} \varepsilon_k^{\cdot j} - \frac{\partial w}{\partial \varepsilon_{kj}} \varepsilon_k^{\cdot i} - \frac{\partial w}{\partial k_{klj}} k_{kl}^{\cdot\cdot i} + 2\frac{\partial w}{\partial k_{ikl}} k^j_{\cdot kl} + \right.$$
$$\left. + \frac{\partial w}{\partial \pi_i} \pi^j + \frac{\partial w}{\partial \pi_{ik}} \pi^j_{\cdot k} - \frac{\partial w}{\partial \pi_{kj}} \pi_k^{\cdot i}\right)_{[ij]} = 0,$$

which is obtained from (1.13), using the expressions (2.1)—(2.4).

The spatial deformation gradients $X^K_{;k}$ are correlated with the gradients of displacement vector $u^K_{;k}$ by the relation

(2.11) $$X^K_{;k} = g^K_k - u^K_{;k},$$

where g^K_k are the coordinates of Euclidean shifters. If the gradients of microdisplacement vector $\varphi^K_{\cdot k}$ are introduced in a similar manner,

(2.12) $$\chi^K_{\cdot k} = g^K_k - \varphi^K_{\cdot k},$$

then, from the relation

(2.13) $$G_{KL} \chi^K_{\cdot k} \chi^L_{\cdot l} = g_{kl},$$

we get

(2.14) $$\varphi_{kl} + \varphi_{lk} - \varphi_{mk} \varphi^m_{\cdot l} = 0, \qquad (\varphi_{kl} = \varphi_{Kl} g^K_k).$$

From this it follows that in the linear theory

$$\varphi_{kl}+\varphi_{lk}=0, \tag{2.15}$$

i.e. that φ_{kl} is a skew-symmetric tensor.

Spatial deformation tensors ε_{ij} and k_{ijk}, using (2.1), (2.2), (2.11), (2.12) and (2.14), can be expressed in the form

$$\begin{aligned} \varepsilon_{ij} &= u_{i,j}+\varphi_{ji}-\varphi_{mi}\,\overset{m}{u}_{,j}=u_{i,j}-\varphi_{ij}-(u_{m,j}-\varphi_{mj})\,\overset{m}{\varphi}_{\cdot i}, \\ k_{ijk} &= \varphi_{ij,k}-\varphi_{mi}\,\overset{m}{\varphi}_{\cdot j,k}=-\varphi_{ji,k}+\varphi_{mj}\,\overset{m}{\varphi}_{\cdot i,k}. \end{aligned} \tag{2.16}$$

In the linear theory these tensors are

$$\begin{aligned} \varepsilon_{ij} &= u_{i,j}+\varphi_{ji}=u_{i,j}-\varphi_{ij}, \\ k_{ijk} &= \varphi_{ij,k}=-\varphi_{ji,k}. \end{aligned} \tag{2.17}$$

3. Linear constitutive equations

In the linear theory we omit the nonlinear terms in the equations (2.6)—(2.9), so that the linear constitutive equations are of the form

$$\begin{aligned} t^{ij} &= \rho\frac{\partial w}{\partial \varepsilon_{ij}}, \\ m^{ijk} &= \rho\frac{\partial w}{\partial k_{ijk}}, \\ r^{i} &= \rho\frac{\partial w}{\partial \pi_{i}}, \\ r^{ij} &= \rho\frac{\partial w}{\partial \pi_{ij}}, \end{aligned} \tag{3.1}$$

where

$$w=w(\varepsilon_{ij},\ k_{ijk},\ \pi_i,\ \pi_{ij}), \tag{3.2}$$

and the deformation tensors ε_{ij} and k_{ijk} are of the form (2.17).

The specific internal energy of deformation and polarization (3.2) is an isotropic function of its arguments and can be expressed in a polynomial form. In the linear theory, if we suppose that the initial stresses do not exist, the specific internal energy is a quadratic polynomial of the form

$$\begin{aligned} \rho w &= \frac{1}{2}A^{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}B^{ijklmn}k_{ijk}k_{lmn}+\frac{1}{2}C^{ij}\pi_i\pi_j+ \\ &+\frac{1}{2}D^{ijkl}\pi_{ij}\pi_{kl}+E^{ijkl}\varepsilon_{ij}\pi_{kl}+F^{ijkl}k_{ijk}\pi_l, \end{aligned} \tag{3.3}$$

where

$$A^{ijkl}=a_1 g^{ij} g^{kl}+a_2 g^{ik} g^{jl}+a_3 g^{il} g^{jk},$$

(3.4)
$$\begin{aligned} B^{ijklmn}&=b_1 (g^{jk} g^{il} g^{mn}+g^{ki} g^{jm} g^{nl}-g^{jk} g^{im} g^{nl}-g^{ki} g^{jl} g^{mn})+\\ &+b_2 (g^{il} g^{jm} g^{kn}-g^{jl} g^{kn} g^{im})+\\ &+b_3 (g^{jl} g^{km} g^{in}+g^{kl} g^{im} g^{jn}-g^{kl} g^{in} g^{jm}-g^{il} g^{jn} g^{km}),\end{aligned}$$

$$C^{ij}=c_1 g^{ij},$$

$$D^{ijkl}=d_1 g^{ij} g^{kl}+d_2 g^{ik} g^{jl}+d_3 g^{il} g^{jk},$$

$$E^{ijkl}=e_1 g^{ij} g^{kl}+e_2 g^{ik} g^{jl}+e_3 g^{il} g^{jk},$$

$$F^{ijkl}=f(g^{ik} g^{jl}-g^{jk} g^{il})$$

are isotropic tensors, a_α and b_α ($\alpha=1, 2, 3$) elastic material constants, c_1 and d_α ($\alpha=1, 2, 3$) dielectric material constants and e_α ($\alpha=1, 2, 3$) and f coupled material constants.

There is no difficulty to show, using (3.4), that the specific internal energy of deformation and polarization, presented in the form (3.3), satisfies the condition of objectivity (2.10).

Making use of (3.3) and (3.4), from (3.1) we obtain the linear constitutive equations in the form

(3.5)
$$t^{ij}=(a_1 \varepsilon_I+e_1 \pi_I) g^{ij}+a_2 \varepsilon^{ij}+a_3 \varepsilon^{ji}+e_2 \pi^{ij}+e_3 \pi^{ji},$$

(3.6)
$$\begin{aligned} m^{ijk}&=2 b_1 (k^{il}_{\cdot\cdot l} g^{jk}-k^{jl}_{\cdot\cdot l} g^{ik})+2 b_2 k^{ijk}+\\ &+2 b_3 (k^{kij}-k^{kji})+f(g^{ik} \pi^j-g^{jk} \pi^i),\end{aligned}$$

(3.7)
$$r^i=c_1 \pi^i+2fk^{il}_{\cdot\cdot l},$$

(3.8)
$$r^{ij}=(d_1 \pi_I+e_1 \varepsilon_I) g^{ij}+d_2 \pi^{ij}+d_3 \pi^{ji}+e_2 \varepsilon^{ij}+e_3 \varepsilon^{ji},$$

where $\varepsilon_I=\varepsilon^{k}_{\cdot k}=u^{k}_{,k}=\operatorname{div}\vec{u}$ and $\pi_I=\pi^{k}_{\cdot k}=p^{k}_{,k}=\operatorname{div}\vec{p}$ are the first invariants of the tensors ε_{ij} and π_{ij}. In the linear constitutive equations the total number of material constants is 14.

The linear constitutive equations, using (2.3), (2.4), (2.17), and relations

(3.9)
$$\begin{aligned} &\varepsilon^{ij}=\varepsilon^{(ij)}+\varepsilon^{[ij]}=e^{ij}+\omega^{ij}-\varphi^{ij}=e^{ij}+\varepsilon^{ijk}(\omega_k-\varphi_k),\\ &e^{ij}=\frac{1}{2}(u^{i,j}+u^{j,i}),\\ &\omega^{ij}=\frac{1}{2}(u^{i,j}-u^{j,i})=\varepsilon^{ijk}\omega_k, \qquad \left(\vec{\omega}=-\frac{1}{2}\operatorname{rot}\vec{u}\right),\\ &\varphi^{ij}=\varepsilon^{ijk}\varphi_k,\end{aligned}$$

where $\vec{\varphi}=\{\varphi_k\}$ is the vectorial representation of the angle of independent rotation of the dielectric's particles, can be written in the equivalent form

$$t^{ij}=\left(\lambda u^{k}_{,k}+e_1 p^{k}_{,k}\right) g^{ij}+2\mu e^{ij}+k\varepsilon^{ijk}(\omega_k-\varphi_k)+e_2 p^{i,j}+e_3 p^{j,i}, \tag{3.10}$$

$$m^{ij}=\nu_1 \varphi^{k}_{,k} g^{ij}+\nu_2 \varphi^{i,j}+\nu_3 \varphi^{j,i}+f\varepsilon^{ijk}\pi_k, \tag{3.11}$$

$$r^{i}=c_1 p^{i}+2f\varepsilon^{ijk}\varphi_{k,j}, \qquad (\vec{r}=c_1\vec{p}+2f\operatorname{rot}\vec{\varphi}), \tag{3.12}$$

$$\begin{aligned} r^{ij}=&\left(d_1 p^{k}_{\cdot k}+e_1 u^{k}_{,k}\right) g^{ij}+d_2 p^{i,j}+d_3 p^{j,i}+\\ &+(e_2+e_3)e^{ij}+(e_2-e_3)\varepsilon^{ijk}(\omega_k-\varphi_k), \end{aligned} \tag{3.13}$$

where λ and μ are classical Lame's constants:

$$\lambda=a_1, \qquad \mu=\frac{1}{2}(a_2+a_3),$$

and

$$k=a_2-a_3, \qquad \nu_1=2b_3, \qquad \nu_2=2b_1+2b_2-2b_3, \qquad \nu_3=-2b_1,$$

$$m^{ij}=\frac{1}{2}\varepsilon^{ikl} m_{kl}^{\cdot\cdot j}.$$

4. Partial differential equations of motion

The first Cauchy's law of motion (1.1) in the linearized form is

$$\rho_0 \ddot{u}^{i}=t^{ij}_{,j}+F^{i}+\rho f^{i}, \tag{4.1}$$

where use has been made of the law of conservation of mass in the linearized form

$$\rho=\rho_0\left(1-u^{k}_{,k}\right). \tag{4.2}$$

Using the linear constitutive equation (3.10), the system (4.1) can be expressed in the form

$$\begin{aligned} &\left(\mu+\frac{k}{2}\right)\nabla^2 u^{i}+e_2\nabla^2 p^{i}+\left(\lambda+\mu-\frac{k}{2}\right)u^{k\cdot i}_{,k}+\\ &+(e_1+e_3)p^{k\cdot i}_{,k}-k\varepsilon^{ijk}\varphi_{k,j}+F^{i}+\rho f^{i}-\rho_0\ddot{u}^{i}=0, \end{aligned} \tag{4.3}$$

or, in the vectorial form,

$$\begin{aligned} &\left(\mu+\frac{k}{2}\right)\nabla^2\vec{u}+e_2\nabla^2\vec{p}+\left(\lambda+\mu-\frac{k}{2}\right)\operatorname{grad}(\operatorname{div}\vec{u})+\\ &+(e_1+e_3)\operatorname{grad}(\operatorname{div}\vec{p})-k\operatorname{rot}\vec{\varphi}+\vec{F}+\rho\vec{f}-\rho_0\ddot{\vec{u}}=0. \end{aligned} \tag{4.4}$$

If we suppose material points of the dielectric to be infinitesimal spheres, we have

$$I^{KL} = I G^{KL}, \qquad (I = \text{const.}).$$

Using this relation and

$$\chi^{i}_{\cdot K} = g^{i}_{K} + \varphi^{i}_{\cdot K},$$

for the inertial spin (1.6) we obtain in the first approximation

$$\Gamma^{ij} = I \ddot{\varphi}^{ij} = I \varepsilon^{ijk} \ddot{\varphi}_k, \tag{4.5}$$

so that the second Cauchy's law of motion (1.2) in the linearized form becomes

$$\rho_0 I \varepsilon^{ijk} \ddot{\varphi}_k = t^{[ij]} + \varepsilon^{ijl} m_{l\,,k}^{\cdot k} + \rho\, l^{ij},$$

or

$$\rho_0 I \ddot{\varphi}_k = \frac{1}{2} \varepsilon_{ijk} t^{ij} + m_{k,l}^{\cdot l} + \rho\, l_k, \tag{4.6}$$

where

$$l_k = \frac{1}{2} \varepsilon_{ijk} l^{ij}.$$

Now, using the linear constitutive equations (3.10)—(3.13), we obtain the system of linear partial differential equations (4.6) in the form

$$\begin{aligned} &\nu_2 \nabla^2 \varphi_k + (\nu_1 + \nu_3) \varphi^{l}_{,lk} + k(\omega_k - \varphi_k) + \\ &+ \frac{1}{2}(e_3 - e_2 + 2f) \varepsilon_{ijk} p^{j,i} + \rho\, l_k - \rho_0 I \ddot{\varphi}_k = 0, \end{aligned} \tag{4.7}$$

or, in the vectorial form,

$$\begin{aligned} &\nu_2 \nabla^2 \vec{\varphi} + (\nu_1 + \nu_3) \operatorname{grad} (\operatorname{div} \vec{\varphi}) - k\left(\vec{\varphi} + \frac{1}{2} \operatorname{rot} \vec{u}\right) + \\ &+ \frac{1}{2}(e_3 - e_2 + 2f) \operatorname{rot} \vec{p} + \rho \vec{l} - \rho_0 I \ddot{\vec{\varphi}} = 0. \end{aligned} \tag{4.8}$$

The system of partial differential equations (1.3), written in the linearized form, is

$$\rho_0 \nu \ddot{p}^i = r^{ij}_{,j} - r^i + \varepsilon^i, \tag{4.9}$$

where use has been made of (1.8) and (4.2). However, using the linear constitutive equations (3.12) and (3.13), this system becomes

$$\begin{aligned} &d_2 \nabla^2 p^i + e_2 \nabla^2 u^i + (d_1 + d_3) p^{k\cdot i}_{,k} + (e_1 + e_3) u^{k\cdot i}_{,k} - \\ &- (e_2 - e_3 + 2f) \varepsilon^{ijk} \varphi_{k,j} - c_1 p^i + \varepsilon^i - \rho_0 \nu \ddot{p}^i = 0, \end{aligned} \tag{4.10}$$

or, in the vectorial form,

$$(4.11)\quad d_2\nabla^2\vec{p}+e_2\nabla^2\vec{u}+(d_1+d_3)\operatorname{grad}(\operatorname{div}\vec{p})+(e_1+e_3)\operatorname{grad}(\operatorname{div}\vec{u})-$$
$$-(e_2-e_3+2f)\operatorname{rot}\vec{\varphi}-c_1\vec{p}+\vec{\varepsilon}-\rho_0\nu\ddot{\vec{p}}=0,$$

where $\vec{\varepsilon}=\{\varepsilon^i\}$ is the dynamical electric field vector, determined by the expression (1.7).

The system of nine linear partial differential equations (4.4), (4.8) and (4.11), together with the conservation mass equation (4.2), represent the complete system of linear partial differential equations for the determination of unknown functions $\vec{u}$, $\vec{\varphi}$, $\vec{p}$ and ρ.

REFERENCE

[1] Plavšić M., S. Đurić and M. Gligorić; *Micropolar elastic dielectrics*, Ibidem.

DIE FELDGLEICHUNGEN FÜR ISOTROPE MIKROPOLARE ELASTISCHE DIELEKTRISCHE MATERIALEN

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Zusammenfassung

Das Modell des istotropen mikropolaren elastischen Dielektrikum unter dem Einfluss des äusseren elektromagnetischen Feldes wird untersucht. Die nichtlinearen Materialgleichungen werden ausgeführt. Nach durchgesetzer Linearisierung der Materialgleichungen, werden die entsprechenden Feldgleichungen abgeleitet.

ЈЕДНАЧИНЕ ПОЉА ЗА ИЗОТРОПНИ МИКРОПОЛАРНИ ЕЛАСТИЧНИ ДИЕЛЕКТРИК

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Резиме

У овом раду се посматра модел изотропног микрополарног еластичног диелектрика у електромагнетном пољу. Добијене су нелинеарне конститутивне једначине за посматрани модел. Затим су линеаризоване конститутивне једначине и добијене одговарајуће једначине поља.

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