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THERMODIFFUSION IN ELASTIC, MAGNETICALLY SATURATED, CURRENT CONDUCTING MEDIA. II. FIELD EQUATIONS

Natalija Naerlović-Veljković

In the Part I of this paper [1] there were derived the equations of motion and the energy equation for a simple magnetizable and current conducting solid, submitted to the process of thermodiffusion. The stress tensor, the magnetic exchange tensor, the local material magnetization field vector, the chemical potential and the entropy were determined from the free energy function as a thermodynamic potential. Taking into account the restriction imposed upon the response of the body in the form of the dissipation inequality and introducing the notion of the dissipation function, the corresponding constitutive equations for the heat flux vector and convection current have been obtained. At least, the conditions of the objectivity of the constitutive equations were derived in terms of the free energy function, as well as the condition of the invariance of the spin system in a rigid rotation of the material lattice [2].

In the Part II of this work we are going to derive the constitutive equations which are in agreement with mentioned conditions of objectivity and invariance. Beside the material form, the spatial form of constitutive equations is done too. The constitutive relations for the irreversible part of the process given in Part I are generalized in this part of the work. The obtained results are compared with known relations of the classical theory.

1. Sequence of the conditions of objectivity and invariance

In the Part I [1] it was shown that the free energy represented a singlevalued function of following arguments:

$$\psi=\psi(x^{k}_{;K},\ \mu^{k},\ \mu^{k}_{;K},\ T,\ c), \tag{1.1}$$

where $x^{k}_{;k}$ is displacement gradient, μ^k-magnetization vector density, T-absolute temperature, c-concentration of diffused mass. The appropriate equations for stress tensor, magnetic exchange tensor, local material magnetization field vector, chemical potential and entropy were derived in the form:

$$t^{ij}=\rho\, g^{il}\frac{\partial\psi}{\partial x^{l}_{;K}}\,x^{j}_{;K},$$

$$A^{ij}=\left(g^{il}-\frac{\mu^{i}\mu^{l}}{\mu_s^{2}}\right)\frac{\partial\psi}{\partial\mu^{l}_{;K}}\,x^{j}_{;K},$$

(1.2)
$$B^i_{(L)}=-\left(g^{il}-\frac{\mu^i\mu^l}{\mu_s^2}\right)\frac{\partial\psi}{\partial\mu^l}+\frac{\partial\psi}{\partial\mu^l_{;K}}\frac{\mu^l\mu^i_{;K}}{\mu_s^2},$$
$$M=\frac{\partial\psi}{\partial c},\qquad \eta=-\frac{\partial\psi}{\partial T}.$$

The condition of objectivity of response functions yielded a system of three differential equations:

(1.3)
$$\left(g^{il}\frac{\partial\psi}{\partial x^l_{;K}}x^j_{;K}-\mu^i g^{jl}\frac{\partial\psi}{\partial\mu^l}+\mu^i\mu^j_{;K}\frac{\partial\psi}{\partial\mu^l_{;K}}\frac{\mu^l}{\mu_s^2}\right)_{[i,j]}=0.$$

The condition of invariance of the spin system with respect to the material frame [2] was expressed by the relation:

(1.4)
$$(A_i^{;l}\mu_{j,l})_{[i,j]}=0.$$

The number of independent arguments of the free energy function is through conditions (1.3) and (1.4) reduced from 23 to 17.

It may be easily shown that conditions (1.3) and (1.4) are satisfied by introducing following set of arguments in the free energy function:

(1.5)
$$E_{KL}=\frac{1}{2}g^{ij}x_{i;K}x_{j;L}\qquad \theta=T-T_0$$
$$m_K=g^{ij}\mu_i x_{j;K}\qquad \zeta=c-c_0.$$
$$m_{KL}=\frac{1}{2}g^{ij}\mu_{i;K}\mu_{j;L}.$$

The last two variables are introduced in accordance with prescribed values of T and c in the referent configuration. It was namely assumed that in the referent configuration, the field of temperature $T=T_0$ and the field of concentration $c=c_0$ were uniform fields. Hence, instead of (1.1), we may write:

(1.6)
$$\psi=\psi(E_{KL},m_K,m_{KL},\zeta,\theta).$$

After introducing the arguments (1.5) in Equ. (1.2) , we obtain following results:

(1.7)
$$t^{ij}=\rho\left(x^i_{;K}x^j_{;L}\frac{\partial\psi}{\partial E_{KL}}+\mu^i x^j_{;K}\frac{\partial\psi}{\partial m_K}\right)$$
$$A^{ij}=\mu^i_{;K}x^j_{;L}\frac{\partial\psi}{\partial m_{KL}}\qquad M=\frac{\partial\psi}{\partial\zeta}$$
$$B^i_{(L)}=-\left(g^{il}-\frac{\mu^i\mu^l}{\mu_s^2}\right)\frac{\partial\psi}{\partial m_K}x_{l;K}\qquad \eta=-\frac{\partial\psi}{\partial\theta}.$$

2. The dissipation inequality

In the Part I we suggested the equation of balance of entropy:

(2.1)
$$\rho T\dot{\eta}=q^i_{,i}+\rho h+j^{;i}_{(e)}\varepsilon_i.$$

yielding the dissipation inequality:

$$\frac{q^i T_{,i}}{T} + j^i_{(e)}\varepsilon_i \geqq 0, \tag{2.2}$$

in the form expressing the irreversibility of the processes of heat flow and electric current conduction (with $j^i_{(e)}$ — convection current and ε_i — electromotive intensity at a point moving with the particles of the body). Let us remark that, at the present state of things, the only thing we can do is to postulate the form of the dissipation inequality and the only explanation we may offer is, that different forms of equations of balance together with some particular form of the dissipation inequality just assert the diversity of modes of behaviour of thermodynamic bodies.

From (2.2) it may be immediately concluded that the heat flux vector and the convection current depend both on the temperature gradient and the electromotive intensity. Hence, following constitutive equations have been proposed in Part I:

$$q^i = \hat{q}^i\left(\frac{1}{T}\,\mathrm{grad}\,T, \underset{\sim}{\varepsilon}\right), \qquad j^i_{(e)} = \hat{j}^i_{(e)}\left(\frac{1}{T}\,\mathrm{grad}\,T, \underset{\sim}{\varepsilon}\right). \tag{2.3}$$

These relations were further elaborated by using the notion of the dissipation function:

$$\rho\,\Phi = \frac{q^i T_{,i}}{T} + j^i_{(e)}\,\varepsilon_i \geqq 0 \tag{2.4}$$

and applying the Ziegler's principle [3]. Of course, the suggested relations (2.3) are not the general that could have been derived in the frames of the present theory. Returning to the energy equation:

$$\rho\dot{\psi} = t^{ij} v_{i,j} + \rho\, A^{ij}\,\dot{\mu}_{i,j} - \rho\, B^i_{(L)}\,\dot{\mu}_i + \rho\, M\,\dot{c} - \rho\eta\,\dot{T} \tag{2.5}$$

and to the equation of the balance of entropy (2.1), we may present the dissipation inequality in the form:

$$\begin{aligned} \rho\Phi = & -\rho\dot{\psi} + t^{ij} X^K_{;ij}\,\dot{x}_{i;K} + \rho\, A^{ij} X^K_{;j}\dot{\mu}_{i;K} - \rho\, B^i_{(L)}\,\dot{\mu}_i + \\ & + \rho\, M\dot{c} - \rho\eta\,\dot{T} + \frac{q^i T_{,i}}{T} + j^i_{(e)}\,\varepsilon_i \geqq 0. \end{aligned} \tag{2.6}$$

According to the Truesdell's principle of equipresence [4], we are going to suppose that all response functions in (2.6) depend on:

$$x^k_{;K},\ \mu^k,\ \mu^k_{;K},\ T,\ c,\ \frac{T_{,i}}{T},\ \varepsilon_i \tag{2.7}$$

or, after accounting of (1.3) and (1.4), on:

$$E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ \frac{T_{,i}}{T},\ \varepsilon_i. \tag{2.8}$$

Aftei inserting the arguments (2.8) in the free energy function, we readily obtain on the ground of (2.6) that:

$$\frac{\partial \psi}{\partial (T_{,i}/T)}=0, \quad \frac{\partial \psi}{\partial \varepsilon_i}=0 \tag{2.9}$$

For the response functions (2.3) it nevertheless holds:

$$\begin{aligned} q^i &= q^i\left(E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ \frac{T_{,i}}{T},\ \varepsilon_i\right) \\ j^i_{(e)} &= j^i_{(e)}\left(E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ \frac{T_{,i}}{T},\ \varepsilon_i\right). \end{aligned} \tag{2.10}$$

Equations (2.10) may be understood as nonlinear relations for the heat flux vector and convection current.

We remark that the dissipation inequality (2.2) does not contain any term describing the irreversibility of the process of thermodiffusion. On the contrary, the term connected with thermodiffusion is available in the energy equation (2.5). As a result of such approach, we have got the concentration c as appropriate independent variable in the list (2.7).

3. An alternative model

We are going to replace the free energy equation (2.1), by the following one:

$$\rho\dot{\psi}=t^{ij}\,v_{i,j}+\rho\,A^{ij}\,\dot{\mu}_{i,j}-B^i_{(L)}\,\dot{\mu}_i-\rho\,M\,\dot{c}-\rho\eta\,\dot{T}, \tag{3.1}$$

where the term ($-\rho M\dot{c}$) is coming from the entropy equation. Namely, the equation of entropy balance contains now, beside terms due to heat flow and conduction of current, one more, connected with thermodiffusion:

$$\rho\,T\dot{\eta}=q^i_{,i}+\rho\,h+j^i_{(e)}\,\varepsilon_i+Mj^i_{(m),\,i} \tag{3.2}$$

As it was stated by Equ. (4.3) of Part I, the flux vector of the diffused mass $j^i_{(m)}$ is, in absence of body sources of mass production, related to the rate of concentration through the following equation of balance:

$$\rho\,\dot{c}=j^i_{(m),\,i}\,. \tag{3.3}$$

In this case the dissipation inequality takes one of following forms of presentation:

$$\rho\Phi=\rho\,T\dot{\eta}-T\left(\frac{q^i}{T}\right)_{,\,i}-\rho\,h-T\left(\frac{M}{T}j^i_{(m)}\right)_{,\,i}\geq 0, \tag{3.4}$$

or

$$\begin{aligned} \rho\Phi= &-\rho\dot{\psi}+t^{ij}\,X^K_{;\,j}\,\dot{x}_{i;\,K}+\rho\,A^{ij}\,X^K_{;j}\,\dot{\mu}_{i;\,K}-\rho\,B^i_{(L)}\,\dot{\mu}_i- \\ &-\rho\,M\dot{c}-\rho\eta\,\dot{T}+\frac{q^i\,T_{,i}}{T}+j^i_{(e)}\,\varepsilon_i-j^i_{(m)}\,T\left(\frac{M}{T}\right)_{,\,i}\geq 0. \end{aligned} \tag{3.5}$$

Analysing the last expression we come to the list of arguments by which (2.7) is to be replaced:

$$x^k_{;K},\ \mu^k,\ \mu^k_{;K},\ T,\ c,\ \frac{T_{,i}}{T},\ \varepsilon_i,\ \left(\frac{M}{T}\right)_{,i} \tag{3.6}$$

Again, after inserting the arguments (3.6) in the inequality (3.5), we get:

$$\frac{\partial\psi}{\partial(T_{,i}/T)}=0,\qquad \frac{\partial\psi}{\partial\varepsilon_i}=0,\qquad \frac{\partial\psi}{\partial\left(\frac{M}{T}\right)_{,i}}=0. \tag{3.7}$$

In that way we realize that the constitutive equations (1.7/1—3) as well as (1.7/5) are still valid, equation (1.7/4) is replaced by:

$$M=-\frac{\partial\psi}{\partial\zeta} \tag{3.8}$$

and instead of relations (2.10) we obtain now:

$$\begin{aligned} q^i&=q^i(E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i})\\ j^i_{(e)}&=j^i_{(e)}(E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i})\\ j^i_{(m)}&=j^i_{(m)}(E_{KL},\ m_K,\ m_{KL},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i}). \end{aligned} \tag{3.9}$$

The difference between the model represented by relations (1.7) and (2.10) and that represented through relations (1.7/1—3,5), (3.8) and (3.9) consists in the place of the working due to the thermodiffusion. By the first approach it was placed as a contribution to the rate of change of the internal energy. By the second approach, we regarded thermodiffusion as a irreversible flow.

4. Isotropic material

The constitutive equations (1.7) are written for thermomechanically anisotropic solid. If we suppose the solid being isotropic, then we may write the constitutive equations in the spatial form. We introduce following spatial tensors:

$$\begin{aligned} &c^{-1\,kl}=G^{KL}\,x^k_{;K}\,x^l_{;L}, \qquad m^{-1\,kl}=G^{KL}\,\mu^k_{;K}\,\mu^l_{;L}.\\ &m^k=c^{-1\,kl}\,\mu_l, \end{aligned} \tag{4.1}$$

The corresponding spatial form of constitutive equations (1.7), (3.8) and (3.9) is:

$$\begin{aligned} t^{ij}&=\rho\frac{\partial\psi}{\partial(c^{-1}_{ik})}c^{-1\,j}{}_k+\rho\frac{\partial\psi}{\partial m_i}m^j,\\ A^{ij}&=2\left(g^{il}-\frac{\mu^i\mu^l}{\mu_s^2}\right)\frac{\partial\psi}{\partial m^{-1\,kl}}d^{-1\,jk}, \qquad \left(d^{-1\,jk}\overset{\text{def}}{=}G^{KL}\,x^j_{;K}\,\mu^k_{;L}\right)\\ B^i_{(L)}&=-\left(g^{il}-\frac{\mu^i\mu^l}{\mu_s^2}\right)\frac{\partial\psi}{\partial m_k}c^{-1}_{kl}+2\frac{\mu^l}{\mu_s^2}\frac{\partial\psi}{\partial(m^{-1\,kl})}m^{-1\,ki}\\ M&=-\frac{\partial\psi}{\partial\zeta},\qquad \eta=-\frac{\partial\psi}{\partial\theta} \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
&q^i = q^i\,(c^{-1kl},\ m^k,\ m^{-1kl},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i}),\\
(4.3)\qquad &j^i_{(e)} = j^i_{(e)}\,(c^{-1kl},\ m^k,\ m^{-1kl},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i}),\\
&j^i_{(m)} = j^i_{(m)}\,(c^{-1kl},\ m^k,\ m^{-1kl},\ \theta,\ \zeta,\ T_{,i}/T,\ \varepsilon_i,\ (M/T)_{,i}).
\end{aligned}
$$

5. Field equations

The complete set of field equations consists of the equation of balance of mass (4.1), Part I, balance of diffused mass (4.3), Part I, balance of magnetization (4.7), Part I, Maxwell's equations (1.1), Part I, equations of motion (5.11) and (5.12), Part I, balance of entropy (2.1) or (3.2), Part II, together with the whole set of constitutive equations.

We are not going to write the set of field equations in terms of displacements, magnetization, electric charge density, temperature and concentration of diffused mass. A few remarks will still be done.

Equation (4.3/1) may be obviously understood as a generalization of the Fourier's law of heat conduction. We may represent equations (4.3/1,2) in the linearized form. But if we neglect in these two equations all contributions exept the influence of temperature gradient and gradient of chemical potential, we obtain relations which coincide with the *phenomenological equations* for heat flux and flux of diffused mass in frames of the classical theory, Equ. (7—3) and (7—4) of [5]:

$$
\begin{aligned}
&j_{(m)i} = -L_{11}\,M_{,i} - L_{12}\frac{T_{,i}}{T}\\
(5.1)\qquad\\
&q_i = -L_{21}\,M_{,i} - L_{22}\frac{T_{,i}}{T}.
\end{aligned}
$$

Equation (4.3/3) yields a generalization of the Ohm's law. After linearization and neglecting all coupled effects exept that of heat flow on the conduction of current, we obtain the relation coinciding with that, given by Landau and Lifschitz, Equ. (25.2) of [6]:

$$
(5.2)\qquad j_{(e)i} = \sigma\,(\varepsilon_i - \alpha\,T_{,i})
$$

σ being current conductivity and α-quantity characterizing thermoelectrical properties of the body. In the same way, neglecting all other influences exept the influence of diffusion on the conduction of current and using Equ. (4.2/4) and (4.3/3), we may get the relation coinciding with Equ. (26.1) of [6]:

$$
(5.3)\qquad j_{(e)i} = \sigma\,(\varepsilon_i - \beta\xi_{,i}).
$$

In the light of these remarks we may conclude, that the given theory of thermodiffusion in magnetizable conductors contains certain classical relations as special cases.

REFERENCES

[1] N. Naerlović-Veljković, *Thermodiffusion in elastic, magnetically saturated, current conducting media. I. Constitutive equations*, Teorijska i primenjena MEHANIKA, pp. 101—109, **2**, 1976.

[2] H. F. Tiersten, *Variational principle for saturated magnetoelastic insulators*, J. Math. Phys. **6**, pp. 779—787, May 1965.

[3] H. Ziegler, *Progress in solid mechanics*, vol. IV, 1963.

[4] C. Truesdell, R. A. Toupin, *The classical field theories*, Encyclopaedia of physics, III/1. Springer Verlag 1960.

[5] P. G. Shewmon, *Diffusion in solids*. McGraw-Hill Book Co, 1963.

[6] L. D. Landau, E. M. Lifšic, *Elektrodinamika splošnih sred*, Gos. izd. fiz-mat lit., Moskva 1959.

ТЕРМОДИФУЗИЈА У ЕЛАСТИЧНОМ МАГНЕТНО ЗАСИЋЕНОМ ПРОВОДНИКУ СТРУЈЕ

Н. Наерловић-Вељковић

Резиме

Настављајући се на први део овог рада (1), у делу II су конститутивне једначине усклађене са условом објективности и условом инваријантности (1.4). Дата је дискусија једначина којима се одређује топлотни флукс и конвективна струја; за алтернативан модел у коме се поред процеса провођења топлоте и струје и термодифузија јавља као иреверзибилан процес, дате су одговарајуће конститутивне једначине које се разликују од предходних у деловима које се односе на иреверзибилне флуксеве. За изотропан модел материјала приказане су конститутивне једначине у просторном облику. Коначно, задржавајући само по пар спрегнутих ефеката (дифузија и провођење топлоте, струја и провођење топлоте, струја и дифузија) показано је да се резултати овога рада поклапају са одговарајућим класичним релацијама.

ТЕРМОДИФФУЗИЯ В УПРУГОМ МАГНИТНО ЗАСЬЩЕННОМ ПРОВОДНИКЕ

Н. Наерлович-Велькович

Резюме

Продолжая работу (1), в этой части выведены определяющие уравнения в согласии с условями обьективности и условьем инвариянтности (1.4). Совершена дискуссия определяющих уравнений для ирреверсибильных потоков (тепла, электричества и диффузии массы) вместе с дискуссией формы диссипативного уравнения Для изотропного материяла урав-

нения приведенны к просторной форме. Учитывая лишь по пару сопряженных еффектов (теплота и диффузия, теплота и ток электричества, ток электричества и диффузия) показано согласие этой теории со классическими уравнениями.

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Адреса аутора:

Н. Наерловић-Вељковић,
11000 Београд,
Високог Стевана 5,
Југославија