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PLANE STRAIN CONSOLIDATION OF A PORO-ELASTIC LAYER ON A ROUGH BASE*

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1. Introduction

In 1970 R. E. Gibson, R. L. Schiffman and S. L. Pu [1] presented their solution of plain strain and axially symmetric consolidation of a clay layer on a smooth impervious base. They took use of the Biot's theory of three-dimensional consolidation [2—5] and adopted a simple model when the skeletal medium is perfectly elastic, the fluid (usually water) is incompressible and the relative velocity between the two phases in governed by Darcy's law.

Earlier works of a number of authors on particular problems of three-dimensional consolidation have been concerned only with a porous elastic medium occupying the half-plane or half-space. Their solutions have practical applications only to problems where the thickness of the clay stratum (layer) is great compared with the dimensions of the loaded area. When this is not so the use of the results following from the half-plane or half-space may in many cases well lead to serious errors the magnitudes of which should at least be assessed.

Thus this problem merits attention. In [1] the authors used Biot's theory to evaluate the time-dependent displacements in a loaded clay layer of thickness h $(o>z>h)$ of infinite lateral extent resting on a rigid impervious medium. At the interface they presumed the contact to be perfectly smooth. They recognized however that a condition of complete adherence might reflect more closely the restraint likely to be present in nature.

In this paper we shall make use of the method developed in earlier papers [1, 6] and consider such boundary conditions at the interface that the layer will adhere to the rough rigid base which can be impervious or pervious. Only the plain strain consolidation will be treated and the results for the displacements of the loaded clay layer will be developed.

2. The governing equations

The equations governing the problem of plain strain consolidation were formulated in terms of two displacement functions $E(x, z, t)$ and $S(x, z, t)$ each of which is dependent on two space variables x, z (see Fig. 1) and time t and obeying the equations

$$(1) \qquad c\nabla^4 E=\frac{\partial}{\partial t}(\nabla^2 E), \qquad \nabla^2 S=0$$

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where $c=2\,G\eta\,k/\gamma_w$ [m²/s] is the coefficient of consolidation, G [kp/m²] shear modulus of the porous skeleton $\eta=(1-\nu)/(1-2\nu)$ an auxilliary elastic constant, ν being the Poisson's ratio, k [m/s] coefficient of permeability of the medium (Darcy's Law) and γ_w [kp/m³] unit weight of fluid. We shall proceed with seeking the solution for the plane strain problems by means of these displacement functions for those cases where the surface loading f [kp/m²] is distributed uniformly over a limited area D of the plane boundary ($z=0$).

We make use of the equations which govern displacements of an isotropic, homogeneous, fully saturated porous elastic medium [2, 4] for conditions of plane strain [6], restating them briefly and with a minimum of comment. To conform with present usage in soil mechnics we regard compressive stress as positive. The components u, w of displacement in the x — and z — coordinate directions and the excess pore fluid pressure σ [kp/m²] must satisfy the equations

$$
(2)\qquad \left.\begin{aligned}
\nabla^2 u-(2\,\mu-1)\frac{\partial e}{\partial x}-\frac{1}{G}\frac{\partial \sigma}{\partial x}&=0\\
\nabla^2 w-(2\,\eta-1)\frac{\partial e}{\partial z}-\frac{1}{G}\frac{\partial \sigma}{\partial z}&=0\\
c\,\nabla^2 e&=\frac{\partial e}{\partial t}
\end{aligned}\right\}
$$

where the dilatation e is given by

$$
e=-\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)
$$

and the operator ∇^2 by $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$.

It has been shown [6] that problems of the type we are concerned with can be reduced to the determination of two displacement functions E and S, related to the displacement components and the excess pore water pressure by

$$
(3)\qquad \left.\begin{aligned}
u=-\frac{\partial E}{\partial x}+z\frac{\partial S}{\partial x},\qquad w=-\frac{\partial E}{\partial z}+z\frac{\partial S}{\partial z}-S\\
\frac{\sigma}{2G}=\frac{\partial S}{\partial z}-\eta\nabla^2 E
\end{aligned}\right\}
$$

and governed by the field equations (1).

The primed components of effective stress acting in the skeleton are connected with the components of total stress in the bulk material (skeleton and fluid) by equations

$$
(4)\qquad \sigma_{xx}=\sigma'_{xx}+\sigma,\quad \sigma_{zz}=\sigma'_{zz}+\sigma.
$$

The strains of the skeleton are connected with the effective stresses through Hooke's law. It follows that the total stresses are given by

$$\left.\begin{aligned}\frac{\sigma_{xx}}{2G}&=\frac{\partial^2 E}{\partial x^2}-\nabla^2 E-z\frac{\partial^2 S}{\partial x^2}+\frac{\partial S}{\partial z}\\ \frac{\sigma_{zz}}{2G}&=\frac{\partial^2 E}{\partial z^2}-\nabla^2 E-z\frac{\partial^2 S}{\partial z^2}+\frac{\partial S}{\partial z}\\ \frac{\sigma_{xz}}{2G}&=\frac{\partial^2 E}{\partial x\,\partial z}-z\frac{\partial^2 S}{\partial x\,\partial z}\end{aligned}\right\}\tag{5}$$

3. Solutions of the governing equations

It has been shown [1], that the solution of field equations (1) can be achieved by introduction of new functions $\bar{E}$ and $\bar{S}$ which are related to E and S by the repeated Fourier cosine and Laplace transforms. Their form is suggested by the symmetry of the problems we are concerned with

$$\left.\begin{aligned}E(x,z,t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty \bar{\bar{E}}(\xi,z,p)\,K(x,\xi)\,e^{pt}\,d\xi\,dp\\ S(x,z,t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty \bar{\bar{S}}(\xi,z,p)\,K(x,\xi)\,e^{pt}\,d\xi\,dp\end{aligned}\right\}\tag{6}$$

where the kernel is $K(x,\xi)=(2/\pi)\cos(x\xi)$ and the Bromwich-Wagner contour in the p — plane is taken to the right of all the poles of the integrands. By the use of these transforms it can readily be verified that the partial differential equations (1) are reduced to the following ordinary differential equations for $\bar{\bar{E}}$ and $\bar{\bar{S}}$

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\left(\frac{d^2}{dz^2}-\zeta^2\right)\bar{\bar{E}}=0\tag{7}$$

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\bar{\bar{S}}=0\tag{8}$$

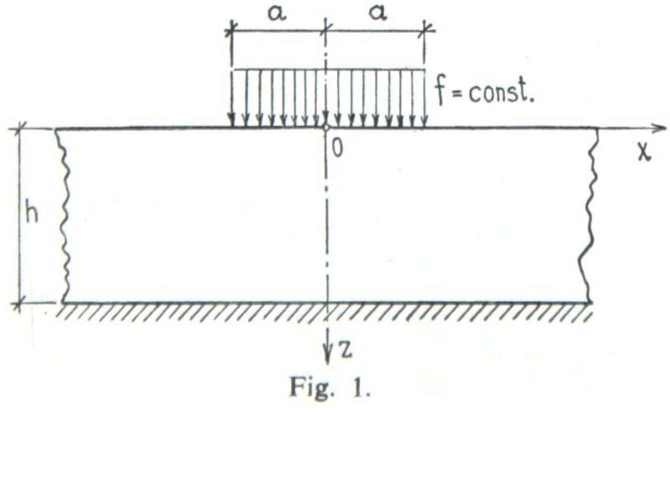

Fig. 1.

where $\zeta^2 = \xi^2 + p/c$ is an auxilliary variable. We set the condition that initially (at $t<0$) the dilatation e of the medium is everyhwere zero. The solutions of these equations are

$$\bar{E} = A_1\, sh\, \xi z + A_2\, ch\, \xi z + A_3\, sh\, \zeta z + A_4\, ch\, \zeta z \tag{9}$$

$$\bar{S} = A_5\, sh\, \xi z + A_6\, ch\, \xi z \tag{10}$$

The parameters $A_1, \ldots, A_6$ are functions of ξ and p. They must now be determined from the boundary conditions imposed on the plane surfaces $z=0, h$ (see Fig. 1).

4. Transformed stresses, displacements and excess pore pressure

From equations (3), (5) and (6) these quantities can be expressed as follows

$$u = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \bar{u} K'(x, \xi)\, e^{pt}\, d\xi\, dp \tag{11a}$$

and

$$\bar{u} = -\xi(-\bar{E} + z\bar{S}) \tag{12a}$$

where

$$K'(x, \xi) = \frac{2}{\pi} \sin(x\xi) \tag{13}$$

We proceed in the same way and find

$$w = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \bar{w} K(x, \xi)\, e^{pt}\, d\xi\, dp \tag{11b}$$

$$\bar{w} = -\frac{d\bar{E}}{dz} + z\frac{d\bar{S}}{dz} - \bar{S} \tag{12b}$$

$$\frac{\sigma}{2G} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \left[\frac{\bar{\sigma}}{2G}\right] K(x, \xi)\, e^{pt}\, d\xi\, dp \tag{11c}$$

$$\frac{\bar{\sigma}}{2G} = \frac{d\bar{S}}{dz} - \eta\left(\frac{d^2\bar{E}}{dz^2} - \xi^2 \bar{E}\right) \tag{12c}$$

$$\frac{\sigma_{zz}}{2G} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \left[\frac{\bar{\sigma}_{zz}}{2G}\right] K(x, \xi)\, e^{pt}\, d\xi\, dp \tag{11d}$$

$$\frac{\bar{\sigma}_{zz}}{2G} = \xi^2 \bar{E} + \frac{d\bar{S}}{dz} - z\frac{d^2\bar{S}}{dz^2} \tag{12d}$$

(11e) $$\frac{\sigma_{xz}}{2G}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\int_{0}^{\infty}\left[\frac{\overline{\sigma}_{xz}}{2G}\right]K'(x,\xi)\,e^{pt}\,d\xi\,dp$$

(12e) $$\frac{\overline{\sigma}_{xz}}{2G}=-\xi\left(\frac{d\overline{E}}{dz}-z\frac{d\overline{S}}{dz}\right)$$

5. The boundary conditions

We are seeking solutions to the problems of plane strain, shown in Fig. 1. The layer $0>z>h$ is loaded at $t=0$ over a pressed area D of its plane surface $z=0$ by a uniform pressure f which is maintained thereafter. The region D occupies the strip $|x|<a$. The entire surface remains free from shear stress and the excess pore fluid pressure there is zero. These conditions on $z=0$ can be written as follows

(14a) $$\sigma_{zz}=0 \quad \text{over} \quad |x|\geqslant 0 \quad \text{for} \quad t<0$$

(14b) $$\sigma_{zz}=\begin{cases} f & \text{in} \quad |x|<a \\ 0 & \quad\;\; |x|>a \end{cases} \quad \text{for} \quad t>0$$

(14c) $$\sigma_{xz}=0, \qquad \sigma=0 \quad \text{everywhere}$$

The base ($z=h$) of the layer rests in contact with a rough, rigid and impervious or pervious medium. These conditions on $z=h$ can be stated as follows

(15) $$u=0, \quad w=0 \quad \text{over} \quad |x|\geqslant 0 \quad \text{for} \quad t>0$$

(16) $$\frac{\partial\sigma}{\partial z}=0 \quad \text{over} \quad |x|\geqslant 0 \quad \text{impervious base}$$

(17) $$\sigma=0 \quad \text{over} \quad |x|\geqslant 0 \quad \text{pervious base}$$

Next we have to express these boundary conditions in transform space. From (14c), (12e) and (12c) we find

(18a) $$-\xi\left[\frac{d\overline{E}}{dz}\right]_{z=0}=0$$

(18b) $$\left[\frac{d\overline{S}}{dz}-\eta\left(\frac{d^2\overline{E}}{dz^2}-\xi^2\overline{E}\right)\right]_{z=0}=0$$

From (14a), (14b) and (12d) however

(18c) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{0}^{\infty}\left[\xi^2\overline{E}+\frac{d\overline{S}}{dz}\right]_{z=0}K(x,\xi)\,e^{pt}\,d\xi\,dp=\begin{cases}\dfrac{f}{2G} & \text{within } D\\[2ex] 0 & \text{outside } D\end{cases}$$

On the base ($z=h$) of the layer transformed boundary conditions are: from (15), (12a) and (12b)

(19a) $$-\xi[-\bar{E}+z\bar{S}]_{z=h}=0$$

(19b) $$\left[-\frac{d\bar{E}}{dz}+z\frac{d\bar{S}}{dz}-\bar{S}\right]_{z=h}=0$$

For the impervious and for pervious base respectively we consider (16) and (17) together with (12c) and find

(20) $$\left[\frac{d^2\bar{S}}{dz^2}-\eta\frac{d}{dz}\left(\frac{d^2\bar{E}}{dz^2}-\xi^2\bar{E}\right)\right]_{z=h}=0$$

(21) $$\left[\frac{d\bar{S}}{dz}-\eta\left(\frac{d^2\bar{E}}{dz^2}-\xi^2\bar{E}\right)\right]_{z=h}=0$$

6. The solution for parameters

Now we shall write the boundary conditions in terms of $\bar{E}$ and $\bar{S}$ by means of solutions (9) and (10). This way we get for a particular problem (impervious base or pervious base) six algebraic equations, which yield the parameters $A_1, \ldots, A_6$.

For the case of impervious base we find for the boundary conditions (18a), (18b), (19a), (19b) and (20) the following five equations

(22) $$A_1\xi+A_3\zeta=0$$

(23) $$A_5\xi=\frac{\eta p}{c}A_4=0$$

(24) $$A_1\,sh\,\xi h+A_2\,ch\,\xi h+A_3\,sh\,\zeta h+A_4\,ch\,\zeta h-h(A_5\,sh\,\xi h+A_6\,ch\,\xi h)=0$$

(25) $$A_1\xi\,ch\,\xi h+A_2\xi\,sh\,\xi h+A_3\zeta\,ch\,\zeta h+A_4\zeta\,sh\,\zeta h-\xi h(A_5\,ch\,\xi h+A_6\,sh\,\xi h)+$$
$$+A_5\,sh\,\xi h+A_6\,ch\,\xi h=0$$

(26) $$\xi^2(A_5\,sh\,\xi h+A_6\,ch\,\xi h)-\frac{\zeta\eta p}{c}(A_3\,ch\,\zeta h+A_4\,sh\,\zeta h)=0$$

For the case of pervious base we shall prime the parameters $A', \ldots A'_6$. The first four conditions are analoguous to the conditions (22), (23), (24) and (25).The fifth condition (21) takes the form

(27) $$\xi(A_5'\,ch\,\xi h+A_6'\,sh\,\xi h)=\frac{\eta p}{c}(A_3'\,sh\,\zeta h+A_4'\,ch\,\zeta h)=0$$

In calculating the parameters it proves convenient to express them as ratios of A_5, A'_5 respectively. After some calculation we find for the case of impervious base

$$\frac{A_1}{A_5}=\frac{1}{\eta p}\left[\frac{\xi c\, sh\,\xi h\, ch\,\zeta h-\dfrac{\zeta(\eta p+\xi^2 c)}{\xi^2}ch\,\xi h\, sh\,\zeta h+\eta ph}{1+\dfrac{\xi}{\zeta}sh\,\xi h\, sh\,\zeta h-\dfrac{\xi^2 c+\eta p}{\xi^2 c}ch\,\xi h\, ch\,\zeta h}\right] \tag{28}$$

$$\frac{A_3}{A_5}=\frac{\xi}{\eta p\zeta}\left[\frac{\xi c\, sh\,\xi h\, ch\,\zeta h-\dfrac{\zeta(\eta p+\xi^2 c)}{\xi^2}ch\,\xi h\, sh\,\zeta h+\eta ph}{1+\dfrac{\xi}{\zeta}sh\,\xi h\, sh\,\zeta h-\dfrac{\xi^2 c+\eta p}{\xi^2 c}ch\,\xi h\, ch\,\zeta h}\right] \tag{29}$$

$$\frac{A_2}{A_5}=\frac{\left[\begin{array}{l}1-2\xi h\dfrac{sh\,\xi h}{ch\,\xi h}+\dfrac{(\xi^2+\zeta^2)h}{\zeta}\dfrac{sh\,\zeta h}{ch\,\xi h}-\dfrac{\eta ph^2}{c}\dfrac{ch\,\zeta h}{ch\,\xi h}+\\[2ex] +\dfrac{\zeta(\xi^2 c+\eta p)}{\xi\eta p}sh\,\xi h\, sh\,\zeta h+\dfrac{\xi^2 c}{\eta p}(1-ch\,\xi h\, ch\,\zeta h)\end{array}\right]}{\xi\left(1+\dfrac{\xi}{\zeta}sh\,\xi h\, sh\,\zeta h-\dfrac{\xi^2 c+\eta p}{\xi^2 c}ch\,\xi h\, ch\,\zeta h\right)} \tag{30}$$

$$\frac{A_6}{A_5}=-th\,\xi h+\frac{\xi\left(\dfrac{\zeta^2}{\xi^2}\dfrac{sh\,\zeta h}{ch\,\xi h}-\dfrac{\eta ph\zeta}{\xi^2 c}-\dfrac{\zeta}{\xi}th\,\xi h\right)}{\zeta\left(1+\dfrac{\xi}{\zeta}sh\,\xi h\, sh\,\zeta h-\dfrac{\xi^2 c+\eta p}{\xi^2 c}ch\,\xi h\, ch\,\zeta h\right)} \tag{31}$$

In a similar manner we find for the case of pervious base

$$\frac{A'_1}{A'_5}=\frac{\dfrac{1}{sh\,\xi h}+\dfrac{\xi h}{ch\,\xi h}-\dfrac{\xi\zeta c}{\eta p}sh\,\zeta h-\dfrac{ch\,\xi h}{sh\,\xi h}ch\,\zeta h+\dfrac{\xi^2 c}{\eta p}\dfrac{sh\,\xi h}{ch\,\xi h}ch\,\zeta h}{\xi\left(\dfrac{1}{ch\,\xi h}+\dfrac{\xi}{\zeta}\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\eta p}{\xi\zeta c}\dfrac{ch\,\xi h}{sh\,\xi h}sh\,\zeta h-ch\,\zeta h\right.} \tag{32}$$

$$\frac{A'_3}{A'_5}=\frac{\dfrac{1}{sh\,\xi h}+\dfrac{\xi h}{ch\,\xi h}-\dfrac{\xi\zeta c}{\eta p}sh\,\zeta h=\dfrac{ch\,\xi h}{sh\,\xi h}ch\,\zeta h+\dfrac{\xi^2 c}{\eta p}\dfrac{sh\,\xi h}{ch\,\xi h}ch\,\zeta h}{\zeta\left(\dfrac{1}{ch\,\xi h}+\dfrac{\xi}{\zeta}+\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\eta p}{\xi\zeta c}\dfrac{ch\,\xi h}{sh\,\xi h}sh\,\zeta h-ch\,\zeta h\right)} \tag{33}$$

$$\frac{A'_2}{A'_5}=\frac{\left[\begin{array}{l}-2\xi h\dfrac{ch\,\xi h-ch\,\zeta h}{sh\,\xi h}+\dfrac{\xi^2 c-\eta p}{\eta p\, ch\,\xi h}+\dfrac{\xi}{\zeta}\dfrac{c-\eta p h^2}{c}\dfrac{sh\,\zeta h}{sh\,\xi h\, ch\,\xi h}+\\[2ex] +\dfrac{\xi\zeta c}{\eta p}\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\xi^2 c-\eta p}{\eta p}ch\,\zeta h\end{array}\right]}{\xi\left(\dfrac{1}{ch\,\xi h}+\dfrac{\xi}{\zeta}\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\eta p}{\xi\zeta c}\dfrac{ch\,\xi h}{sh\,\xi h}sh\,\zeta h-ch\,\zeta h\right)} \tag{34}$$

$$(35)\qquad \frac{A'_6}{A'_5}=\frac{\dfrac{ch\,\xi h\,ch\,\zeta h-2}{sh\,\xi h}+\dfrac{1}{sh\,\xi h\,ch\,\xi h}\left(ch\,\zeta h-\dfrac{\eta ph}{\zeta c}\right)+\left(\dfrac{\eta p}{\xi\zeta c}-\dfrac{\xi}{\zeta}\right)sh\,\zeta h}{\left(\dfrac{1}{ch\,\xi h}+\dfrac{\xi}{\zeta}\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\eta p}{\xi\zeta c}\dfrac{ch\,\xi h}{sh\,\xi h}sh\,\zeta h-ch\,\zeta h\right)}$$

It remains to express the last boundary condition (18c) through (9) and (10). Its integrand takes the same form in the cases of impervious and pervious base (in the latter case we take the primed parameters)

$$(36)\qquad \left[\xi^2\bar{E}+\frac{d\bar{S}}{dz}\right]_{z=0}=\xi A_5\left[\xi\frac{A_2+A_4}{A_5}+1\right]$$

With (23) and (30) we find the expression in square brackets for case of impervious base

$$\left[\xi\frac{A_2+A_4}{A_5}+1\right]=$$

$$(37)\qquad =\frac{\left[\begin{aligned}&2\left(ch\,\xi h-\xi h\,sh\,\xi h+\frac{\xi^2 c}{\eta p}ch\,\xi h\right)+\frac{\xi^2+\zeta^2}{\xi\zeta}\left(\xi h+sh\,\xi h\,ch\,\xi h+\right.\\&\left.+\frac{\xi^2 c}{\eta p}sh\,\xi h\,ch\,\xi h\right)sh\,\zeta h-2\left(\frac{\eta ph^2}{2c}+\frac{\eta p}{2\xi^2 c}ch^2\,\xi h+ch^2\,\xi h+\right.\\&\left.+\frac{\xi^2 c}{\eta p}ch^2\,\xi h\right)ch\,\zeta h\end{aligned}\right]}{\left(ch\,\xi h+\dfrac{\xi}{\zeta}sh\,\xi h\,ch\,\xi h\,sh\,\zeta h-\dfrac{\eta p}{\xi^2 c}ch^2\,\xi h\,ch\,\zeta h-ch^2\,\xi h\,ch\,\zeta h\right)}$$

With (23) and (34) however we find the expression in square brackets for the case of pervious base

$$\left[\xi\frac{A'_2+A'_4}{A'_5}+1\right]=$$

$$(38)\qquad =\frac{\left[\begin{aligned}&2\left(\frac{\xi^2 c}{\eta p}\frac{1}{ch\,\xi h}-\frac{\xi h}{sh\,\xi h}\right)+2\left(\frac{\xi h}{sh\,\xi h\,ch\,\xi h}-\frac{\xi^2 c}{\eta p}\right)ch\,\zeta h+\\&+\left(\frac{\xi^2+\zeta^2}{\xi\zeta}\frac{\xi^2 c}{\eta p}\frac{sh\,\xi h}{ch\,\xi h}-\frac{\xi}{\zeta}\frac{\eta ph^2}{c}\frac{1}{sh\,\xi h\,ch\,\xi h}-\frac{\eta p}{\xi\zeta c}\frac{ch\,\xi h}{sh\,\xi h}\right)sh\,\zeta h\end{aligned}\right]}{\left(\dfrac{1}{ch\,\xi h}+\dfrac{\xi}{\zeta}\dfrac{sh\,\xi h}{ch\,\xi h}sh\,\zeta h-\dfrac{\eta p}{\xi\zeta c}\dfrac{ch\,\xi h}{sh\,\xi h}sh\,\zeta h-ch\,\zeta h\right)}$$

For the temporary convenience we denote the numerators and denominators in (37) and (38) by $[N]$, $[N_1]$ and $[D]$, $[D_1]$ respectively. Now we write the complete boundary condition (18c). Multiplying it by $2\,G/f$ we find for the case of impervious base

(39) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{0}^{\infty}\frac{2\,G}{f}\xi A_5 \frac{[N]}{[D]} K(x,\ \xi)\, e^{pt}\, d\xi\, dp = \begin{cases} 1 & \text{within } D \\ 0 & \text{outside } D \end{cases}$$

For the case of pervious base we find the same way a very similar integrand as in (39). Instead of $[N]$, $[D]$ we must insert $[N_1]$, $[D_1]$ as stated above.

The Fourier transform of a step function at a is

$$\int_{0}^{\infty}\frac{1}{\xi}\sin\xi a\cos\xi x\, d\xi = \begin{cases} \pi/2 & \text{at} \quad x<a \\ 0 & \text{at} \quad x>a \end{cases}$$

or, considering the kernel $K(x,\ \xi)=(2/\pi)\cos(x\,\xi)$

(40) $$\int_{0}^{\infty}\frac{1}{\xi}\sin a\xi\, K(x,\ \xi)\, d\xi = \begin{cases} 1 & \text{at} \quad x<a \\ 0 & \text{at} \quad x>a \end{cases}$$

The Bromwich integral at unity is

(41) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{pt}}{p}\, dp = 1$$

Then it is

(42) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{0}^{\infty}\frac{1}{\xi p}\, e^{pt}\, \sin a\xi\, K(x,\xi)\, d\xi\, dp = \begin{cases} 1 & \text{at} \quad x<a \\ 0 & \text{at} \quad x>a \end{cases}$$

Now we can compare the integrands at (39) and (42). For the case of impervious base we find

(43) $$\frac{2G}{f} A_5\, \xi \frac{[N]}{[D]} = \frac{1}{\xi p}\sin a\xi$$

for the case of pervious base we find a very similar equation. We only have to insert in (43) $[N_1]$, $[D_1]$ instead of $[N]$, $[D]$. This way we find the solutions for A_5 and A'_5

(44) $$A_5 = \frac{f}{2G}\,\frac{[D]}{[N]}\,\frac{\sin a\xi}{\xi^2 p}$$

Through (9) and (10) the displacement functions (6) may now be recovered. It is obvious that the resulting formulae will be complicated and unconvenient for computational purposes.

7. The surface settlement

In the engineering the main concern is given to the settlement of the loaded surface $z=0$ and the manner of its development with time. In seeking for this solution it proves that a considerable reduction of formulae occurs.

The vertical displacement of any point of the layer (see section 2) is

$$w(x, z, t) = z\frac{\partial S}{\partial z} - S - \frac{\partial E}{\partial z} \tag{45}$$

On the surface of the layer ($z=0$) this quantity is

$$w(x, o, t) = w_0(x, t) = -\left[\frac{\partial E}{\partial z} + S\right]_{z=0} \tag{46}$$

or, by means of (11b) and (12b)

$$w_0(x, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^{\infty}\left[-\frac{d\bar{E}}{dz} - \bar{S}\right]_{z=0} K(x, \xi)\, e^{pt}\, d\xi\, dp \tag{47}$$

Considering (9), (10), (28) and (29) we find for the case of impervious base

$$\left[-\frac{d\bar{E}}{dz} - \bar{S}\right]_{z=0} = -A_6 \tag{48}$$

In the case of pervious base we have to consider (9), (10), (32), and (33) and find (48) with the only difference, that on the right hand side stands $-A'_6$. Now we consider further the surface settlement (47). Taking into account (31) and (44) for the case of impervious base and (35) with (44) (this time with $[N_1]$, $[D_1]$) for the case of pervious base we find the formulae for $w_0(x, t)$.

It is convenient to introduce new variables

$$\lambda = \xi h, \qquad s = \zeta^2 h^2 \tag{49}$$

After some calculation the surface settlement can be expressed in the form

$$\frac{2G}{f} w_0(x, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^{\infty}\Gamma(x, \lambda)\frac{\varphi(\lambda, s)}{\psi(\lambda, s)}\, ds\, d\lambda \tag{50}$$

where

$$\Gamma(x, \lambda) = \frac{2h^5}{\pi c \lambda^4}\sin\frac{a\lambda}{h}\cos\frac{x\lambda}{h} \tag{51}$$

We can carry out the integration in the complex s—plane along the Bromwich — Wagner contour. The surface settlement (50) may now be expressed in the form

$$\frac{2G}{f} w_0(x, t) = \int_0^{\infty}\Gamma(x, \lambda)\, \Delta(t, \lambda)\, d\lambda \tag{52}$$

Where the integral along the Bromwich—Wagner contour in the s—plane is

$$\Delta(t, \lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\varphi(\lambda, s)}{\psi(\lambda, s)}\, ds \tag{53}$$

For the case of impervious base the numerator $\varphi(\lambda, s)$ and the denominator $\psi(\lambda, s)$ in (53) can be expressed in the following form

(54) $$\varphi(\lambda, s)=\left\{2sh\,\lambda+\left[A(\lambda)-\frac{s}{\lambda}\right]\frac{sh\sqrt{s}}{\sqrt{s}}+\left[B(\lambda)+C(\lambda)\,s\right]ch\sqrt{s}\right\}\times$$
$$\times\exp\frac{c}{h^2}(s-\lambda^2)\,t$$

(55) $$\psi(\lambda, s)=D(\lambda)+E(\lambda)\,s+\left[F(\lambda)+G(\lambda)\,s+H(\lambda)\,s^2\right]\frac{sh\sqrt{s}}{\sqrt{s}}+$$
$$+\left[I(\lambda)+J(\lambda)\,s+K(\lambda)\,s^2\right]ch\sqrt{s}$$

where

$$A(\lambda)=\lambda\,sh^2\lambda,\quad B(\lambda)=-\eta\lambda+(\eta-1)\,sh\,\lambda\,ch\,\lambda,\quad C(\lambda)=\eta\frac{\lambda-sh\,\lambda\,ch\,\lambda}{\lambda^2}$$

$$D(\lambda)=2\lambda^2\left(\lambda\,sh\,\lambda-ch\,\lambda+\frac{ch^2\lambda}{\eta}\right),\quad E(\lambda)=2\,(ch\,\lambda-\lambda\,sh\,\lambda)$$

(56) $$F(\lambda)=-\lambda^3\left(\lambda+\frac{\eta-1}{\eta}sh\,\lambda\,ch\,\lambda\right),\quad G(\lambda)=\frac{\lambda\,sh\,\lambda\,ch\,\lambda}{\eta}$$

$$H(\lambda)=\frac{\lambda+sh\,\lambda\,ch\,\lambda}{\lambda},\quad I(\lambda)=2\lambda^2\left[\frac{\eta-1}{\eta}ch^2\lambda-\eta\,(\lambda^2+ch^2\lambda)\right]$$

$$J(\lambda)=-2\,[\eta\lambda^2+(\eta+1)\,ch^2\lambda],\quad K(\lambda)=-\eta\frac{\lambda^2+ch^2\lambda}{\lambda^2}$$

For the case of pervious base we must put in (53) the numerator $\varphi_1(\lambda, s)$ and the denominator $\psi_1(\lambda, s)$ as follows

(57) $$\varphi_1(\lambda, s)=\left\{2ch\,\lambda+[A_1(\lambda)+B_1(\lambda)\,s]\frac{sh\sqrt{s}}{\sqrt{s}}+C_1(\lambda)\,ch\sqrt{s}\right\}\times$$
$$\times\exp\frac{c}{h^2}(s-\lambda^2)\,t$$

(58) $$\psi_1(\lambda, s)=D_1(\lambda)+E_1(\lambda)\,s+[F_1(\lambda)+G_1(\lambda)\,s+H_1(\lambda)\,s^2]\frac{sh\sqrt{s}}{\sqrt{s}}+$$
$$+[I_1(\lambda)+2\lambda\,s]\,ch\sqrt{s}$$

where

(59) $$A_1(\lambda)=\lambda\,[-\eta\lambda+(1+\eta)\,sh\,\lambda\,ch\,\lambda],\quad B_1(\lambda)=\eta\frac{\lambda-sh\,\lambda\,ch\,\lambda}{\lambda}$$

$$C_1(\lambda)=-(1+ch^2\lambda),\quad D_1(\lambda)=2\lambda^2\left(\lambda\,ch\,\lambda+\frac{sh\,\lambda}{\eta}\right),\quad E_1\lambda=-2\lambda\,ch\,\lambda$$

$$F_1(\lambda)=\lambda^3\left[\frac{sh^2\lambda}{\eta}-\eta(\lambda^2+ch^2\lambda)\right],\quad G_1(\lambda)=\lambda\left[\frac{sh^2\lambda}{\eta}+2\eta(\lambda^2+ch^2\lambda)\right]$$

(59)

$$H_1(\lambda)=-\eta\frac{\lambda^2+ch^2\lambda}{\lambda},\quad I_1(\lambda)=-2\lambda^2\left(\lambda+\frac{sh\lambda\, ch\lambda}{\eta}\right)$$

The evaluation of the integral (53) requires the locating of the poles of its integrand for both cases under consideration (impervious and pervious base). It is readily shown, that there is an isolated pole at $s=\lambda^2$. The existence of other poles will not be proved in this paper. On the basis of physical similarity of problems under consideration with those shown in [1] which assume the smooth base of the layer it will be speculated that the denominators $\psi(\lambda, s)$ or $\psi_1(\lambda, s)$ in (53) have besides at $s=\lambda^2$ an infinity of zeros along the negative real axis of s at $s=-\alpha_n^2$, where α_n are the roots of the characteristic equation.

The integral (53) equals to the infinite sum of residues at all the poles of its integrand. The residue at $s=\lambda^2$ can be found by applying the rule [7]

$$\text{Res}=\frac{\varphi(\lambda^2)}{\psi'(\lambda^2)} \tag{60}$$

Since for both cases under consideration $\varphi(\lambda^2)=0$ and $\psi'(\lambda^2)=0$ (60) becomes au indefinite form $0/_0$. So we must apply L'Hospital's Rule by seeking $\varphi'(s)$ and $\psi''(s)$ and their limits when $s\rightarrow\lambda^2$. For both cases under consideration we find

$$[\varphi'(s)]_{s=\lambda^2}=\frac{sh\lambda}{2\lambda}[(2\eta-1)\lambda-(2\eta+1)\,sh\lambda\, ch\lambda] \tag{61}$$

$$[\psi''(s)]_{s=\lambda^2}=-\frac{sh\lambda}{2\lambda}\frac{(1-\nu)}{(1-2\nu)}\left[4ch^2\lambda-\frac{(1-2\nu)^2}{(1-\nu)^2}sh^2\lambda+\frac{1}{(1-\nu)^2}\lambda^2\right] \tag{62}$$

It is obvious, that the residue (60) represents in (52) the ultimate settlement ($t=\infty$) which is the same for both cases under consideration. Therefore we can write

$$w_0(x,\infty)=\frac{hf}{2(1-\nu)G}\int_0^\infty\frac{(3-4\nu)\,sh\lambda\, ch\lambda-\lambda}{4ch^2\lambda-\left(\frac{1-2\nu}{1-\nu}\right)^2 sh^2\lambda+\frac{\lambda^2}{(1-\nu)^2}}\times \tag{63}$$

$$\times\frac{\sin\frac{a}{h}\lambda}{\lambda}\,\frac{2}{\pi}\,\frac{\cos\frac{x}{h}\lambda}{\lambda}\,d\lambda$$

There is a possibility to compare this result with the similar result for an elastic layer in plane stress given by K. Marguerre [8]. In his paper the expression (67) on p. 117 for the settlement (omitting several misprints) should read

$$Ev=\frac{2P}{\pi}(1+\nu)\int_0^\infty\frac{-(3+\nu)\,sh z\, ch z+(1+\nu)z}{4ch^2z-(1-\nu)^2sh^2z+(1+\nu)^2z^2}\,\frac{\cos tz}{z}\,dz \tag{64}$$

For the loading in our case instead of P/π there must be the factor $\sin \lambda a/\lambda a$ under the integral sign. With $\nu=0$ plane stress and plane strain become identical. This is true for the expressions (63) and (64). Thus (63) represents the ultimate settlement in our problem.

There still remain to determine the infinite sets of zeroes of $\psi(s)$ and $\psi_1(s)$ respectively and the residues of (53). In this case $s=-\alpha_n^2$, $s^2=\alpha_n^4$, $\sqrt{s}=\pm i\alpha_n$, $sh\sqrt{s}=\pm i \sin\alpha_n$ and $ch\sqrt{s}=\cos\alpha_n$,

For the case of imprevious base the characteristic equation from (55) becomes

$$D(\lambda)-E(\lambda)\alpha_n^2+[F(\lambda)-G(\lambda)\alpha_n^2+H(\lambda)\alpha_n^4]\frac{\sin\alpha_n}{\alpha_n}+ +[I(\lambda)-J(\lambda)\alpha_n^2+K(\lambda)\alpha_n^4]\cos\alpha_n=0 \tag{65}$$

and for the case of pervious base from (58) the characteristic equation is

$$D_1(\lambda)-E_1(\lambda)\alpha_n^2+[F_1(\lambda)-G_1(\lambda)\alpha_n^2+H_1(\lambda)\alpha_n^4]\frac{\sin\alpha_n}{\alpha_n}+ +[I_1(\lambda)-2\lambda\alpha_n^2]\cos\alpha_n=0 \tag{66}$$

where α_n are the roots of these equations for each value of λ. The corresponding residues of (53) are determined by the rule [7]

$$\mathrm{Res}_n=\frac{\varphi(-\alpha_n^2)}{\psi'(-\alpha_n^2)} \tag{67}$$

For the cose of impervious base we differentiate (55) and form the residues (67). After some calculation we find the surface settlement (52) in the final form

$$w_0(x,t)=w_0(x,\infty)+$$

$$+\frac{hf}{2G}\int_0^\infty\sum_{n=1}^\infty\frac{\left\{2\,sh\,\lambda+\left[A(\lambda)+\frac{\alpha_n^2}{\lambda}\right]\frac{\sin\alpha_n}{\alpha_n}+[B(\lambda)-C(\lambda)\alpha_n^2]\cos\alpha_n\right\}e^{-(\alpha_n^2+\lambda^2)\frac{ct}{h^2}}}{\left[\begin{array}{l}E(\lambda)+\frac{1}{2}\left\{F(\lambda)\frac{1}{\alpha_n^2}+[G(\lambda)+I(\lambda)]-[3H(\lambda)+J(\lambda)]\alpha_n^2+\right.\\ \left.+K(\lambda)\alpha_n^4\right\}\frac{\sin\alpha_n}{\alpha_n}+\frac{1}{2}\left\{-F(\lambda)\frac{1}{\alpha_n^2}+[G(\lambda)+2J(\lambda)]-\right.\\ \left.-[H(\lambda)+4K(\lambda)]\alpha_n^2\right\}\cos\alpha_n\end{array}\right]}\times$$

$$\times\frac{\sin\frac{a}{h}\lambda}{\lambda}\,\frac{2}{\pi}\,\frac{\cos\frac{x}{h}h}{\lambda}\,d\lambda \tag{68}$$

For the case of pervious base we differentiate (58) and form the residues (67) with (57) as numerators. After some calculation we find

$$
\begin{aligned}
w_0(x,t) &= w_0(x,\infty) + \\
&+ \frac{hf}{2G}\int_0^\infty \sum_{n=1}^{\infty} \frac{\left\{2\, ch\,\lambda + [A_1(\lambda) - B_1(\lambda)\,\alpha_n^2]\,\dfrac{\sin\alpha_n}{\alpha_n} + C_1(\lambda)\cos\alpha_n\right\} e^{-\left(\alpha_n^2+\lambda^2\right)\frac{ct}{h^2}}}{\left[\begin{aligned}&E_1(\lambda) + \frac{1}{2}\left\{F_1(\lambda)\frac{1}{\alpha_n^2} + [G_1(\lambda)+I_1(\lambda)] - [3H_1(\lambda)+2\lambda]\,\alpha^2\right\}\frac{\sin\alpha_n}{\alpha_n} + \\ &+ \frac{1}{2}\left\{-F_1(\lambda)\frac{1}{\alpha_n^2} + [G_1(\lambda)+4\lambda] - H_1(\lambda)\,\alpha_n^2\right\}\cos\alpha_n\end{aligned}\right]} \times \\
&\times \frac{\sin\frac{a}{h}\lambda}{\lambda}\,\frac{2}{\pi}\,\frac{\cos\frac{x}{h}\lambda}{\lambda}\,d\lambda
\end{aligned}
\tag{69}
$$

To reach the results of engineering interest, the expressions (68) and (69) for the surface settlement should be evaluated by numerical methods.

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EBENE FORMÄNDERUNG-KONSOLIDATION EINER PORO-ELASTISCHEN SCHICHT AUF EINER RAUHER STARRER UNTERLAGE

M. Muršić

Zusammenfassung

Eine poro-elastische Schicht von endlicher Dicke h auf einer rauher starrer Unterlage wird belastet mit einer Streifen von Breite $2a$ und im ebenen Formänderungszustand behandelt. Der Ausdruck für die Oberfläche-Konsolidation als Funktion der Zeit wird ermittelt in einer Form, die für die numerische Behandlung geeignet ist.

KONSOLIDACIJA POROELASTIČNEGA SLOJA NA HRAPAVI PODLAGI PRI RAVNINSKEM DEFORMACIJSKEM STANJU

M. Muršić

Rezime

V tej razpravi je prikazano obravnavanje poroelastičnega sloja končne debeline *h*, ki leži na hrapavi, togi, propustni ali nepropustni podlagi. Uporabljena je metoda, ki so jo podali v letu 1970 R. E. GIBSON, R. L. SCHIFFMAN in S. L. PU, ko so obravnavali enak poroelastičen sloj na gladki, togi in nepropustni podlagi. Osnovne enačbe slede iz Biot-jeve teorije tridimenzionalne konsolidacije, pri čemer se predpostavi, da je porozno ogrodje sloja idealno elastično, vmesna tekočina nestisljiva, relativna hitrost med obema fazama pa sledi Dary-jevemu zakonu.

Na kratko so povzete osnovne enačbe in prikazana je uporaba zaporednih Fourier-jeve in Laplace-ove transformacije. Nato so formulirani robni pogoji za oba primera (propustna ali nepropusta podlaga). Končno je deduciran izraz za konosolidacijo površine poroelastičnega sloja, ki je obremenjen s trakasto obremenitvijo konstantne velikosti in širine 2a. Rezultat je prikazan v takšni obliki, ki je primerna za nadaljne izvrednotenje na računalniku.

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