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A VARIATIONAL APPROACH TO THE PROBLEM OF FREE DAMPED LONGITUDINAL OSCILLATIONS OF A PRISMATIC BEAM WITH A NONLINEAR ELASTIC CHARACTERISTIC

Đula Mešter

Introduction

The purpose of this paper is to describe the free longitudinal oscillations of a prismatic beam with nonlinear elastic characteristic and viscous damping where small nonlinearity is considered. The problem is solved by two methods; first by application of a classical Hamilton's principle, and later by applying Vujanović's variational principle with noncommutative variations.

Basic Suppositions

Longitudinal displacement in the direction of the x-axis is represented by $u=u(x, t)$. The cross sections of the beam during oscillation remain plane, and the particles inside these cross sections move only longitudinally. The material of the rod is homogenous, isotropic and behaves by the nonlinear elastic law [4]:

$$\sigma = E\left(1 - a_3 E^2 e_x^2\right) e_x$$

$$e_x = \frac{\partial u}{\partial x}$$

where: σ is normal stress, e_x —strain in the direction of x, E — elastic modulus, a_3 — small parameter. The rod is fixed on both ends, damping is achieved by forces of viscous friction which are proportional to the first degree of velocity.

Differential Equation of Oscillation

The differential equation of this problem, based on previous statements reads as follows:

$$\frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} = \frac{E}{\rho} \frac{\partial}{\partial x} \left\{ \left[1 - a_3 E^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] \frac{\partial u}{\partial x} \right\} \tag{1}$$

Obviously, this is a second order nonlinear partial differential equation, ρ is material density, μ — viscous damping coefficient.

A. Solution by application of a classical Hamilton's principle

The Lagrangian function L is defined in the form of:

$$L=\frac{1}{2}\int_{0}^{1}\left\{\rho\left(\frac{\partial u}{\partial t}\right)^2-E\left[\left(\frac{\partial u}{\partial x}\right)^2-\frac{1}{2}a_3 E^2\left(\frac{\partial u}{\partial x}\right)^4\right]\right\}e^{\mu t}\,dx \tag{2}$$

Let the corresponding action integral be:

$$I=\int_{t_1}^{t_2} L\,dt \tag{3}$$

By all laws of variational calculus, from the conditions of stationarity of action integral the differential equation of oscillation (1) is obtained as well as the corresponding boundary conditions of the problem. Consequently, it is proved that the problem under consideration can be formulated variationally and that the Lagrangian function is set correctly in the form of (2).

In order to obtain an approximate solution of the problem, Kantorovich's method of partial integration is applied. Introducing a dimensionless variable: $\xi=\frac{x}{1}\pi$, where 1 is the length of the beam, the action integral (3) will be transformed into the form:

$$I=\frac{\rho 1}{2\pi}\int_{t_1}^{t_2}\int_{\xi=0}^{\pi}\left\{\left(\frac{\partial u}{\partial t}\right)^2-\alpha^2\left[1+\frac{1}{6}\lambda\left(\frac{\partial u}{\partial \xi}\right)^2\right]\left(\frac{\partial u}{\partial \xi}\right)^2\right\}e^{\mu t}\,d\xi dt \tag{4}$$

where the constants are:

$$\alpha^2=\frac{E\pi^2}{\rho 1^2} \qquad \lambda=-3\frac{E^2 a_3 \pi^2}{1^2}$$

For a rod fixed on both ends, it will be assumed that the approximate solution of the equation (1) can be written as:

$$u(\xi, t)=p(\xi)\cdot q(t)=\sin n\xi\, q(t) \tag{5}$$

The known function $p(\xi)=\sin n\xi$ satisfies the boundary conditions of the problem, because

$$p(o)=0$$
$$p(\pi)=0$$

are the geometrical boundary conditions.

Replacing (5) into (4) after the completed integration of ξ from 0 to π, a reduced action integral will be obtained:

$$I=\int_{t_1}^{t_2}\left[\frac{\pi}{2}\dot{q}^2-\frac{\pi}{2}\alpha^2 n^2 q^2\left(1+\frac{3}{24}\lambda n^2 q^2\right)\right]e^{\mu t}\,dt=\int_{t_1}^{t_2} L\left(t, q, \frac{\partial q}{\partial t}\right)dt \tag{6}$$

The condition of stationarity of a reduced action integral is $\delta I=0$, which is equivalent to the Euler-Lagrangian equations:

$$\frac{\partial L}{\partial q}-\frac{\partial}{\partial t}\left\{\frac{\partial L}{\partial\left(\frac{\partial q}{\partial t}\right)}\right\}=0 \tag{7}$$

If necessary partial derivations are formed and if the obtained results are replaced into (7), one has:

$$\ddot{q}+\mu\,\dot{q}+\omega_1^2\,q\left(1+\frac{1}{4}\lambda\,n^2\,q^2\right)=0 \tag{8}$$

where: $\omega_1^2=\alpha^2\,n^2$.

Applying this procedure, the solution of nonlinear partial differential equations is reduced to the solution of an ordinary nonlinear differential equation. An approximate solution of (8), by the Krilov-Bogoliubov method [5] can be written in the form:

$$q(t)=A\,e^{-\frac{\mu}{2}t}\cos\left\{\omega_1\left[t+\frac{3}{32}\,\frac{A^2\lambda\,n^2}{\mu}\,(1-e^{-\mu t})\right]\right\} \tag{9}$$

Substituting the term (9) into (5) the law of oscillation of the considered beam is obtained in the form:

$$u(\xi,\,t)=A\,\sin n\,\xi\,e^{-\frac{\mu}{2}t}\cos\left\{\omega_1\left[t+\frac{3}{32}\,\frac{A^2\,\lambda\,n^2}{\mu}\,(1-e^{-\mu t})\right]\right\} \tag{10}$$

B. Solution of the problem by application of the variational principle with noncommutative variations

This variational principle was formulated by B. Vujanović for nonconservative systems [2] and [3]. The essential idea of that principle is that variations of generalized velocities are not completely defined only with the variations of generalized coordinates, but with generalized dissipative forces, too.

According to this principle, for a conservative part of the problem the Lagrangian function will be formed:

$$L=\frac{1}{2}\int_0^1\left\{\rho\left(\frac{\partial u}{\partial t}\right)^2-E\left[\left(\frac{\partial u}{\partial x}\right)^2-\frac{1}{2}\,a_3\,E^2\left(\frac{\partial u}{\partial x}\right)^4\right]\right\}dx \tag{11}$$

and the variation rules as:

$$\delta\,\frac{\partial}{\partial x}=\frac{\partial}{\partial x}\,\delta \tag{12}$$

$$\frac{\partial u}{\partial t}\,\delta\,\frac{\partial u}{\partial t}=\frac{\partial u}{\partial t}\,\frac{\partial}{\partial t}\,\delta u-\mu\,\frac{\partial u}{\partial t}\,\delta\,u \tag{13}$$

From (13) it is obvious, that if $\mu=0$, this rule becomes a commutative variation of classical variational calculus. Forming an action integral (3) after an accomplished variation, using variational rules (12) and (13), the differential equation (1) and the boundary conditions of the problem will be obtained; which proves that the terms (11) and (13) have been determined correctly.

In order to obtain an approximate solution of the problem using Kantorovich's methods, an assumed solution in the form of (5) will be replaced into an action integral (3) where the Lagrangian function is now determined in (11), an integration after the variable ξ from 0 to π will be accomplished.

A stationarity condition of a reduced action integral, together with the variational rule (13) results a differential equation (8).

Conclusions

After the whole problem has been discussed, it can be stated that the application of Vujanović's principle in order to solve the problem, leads to the entirely same results as the application of the classical Hamilton's principle.

Moreover, it is well known, that the application of a classical Hamilton's principle to nonconservative problems does not contain certain generality (universality) and only a limited number of problems can be solved in this way. When there is a case of damping in a complex form at oscillation problems, an exact Lagrangian function cannot be written and then the variational principle with noncommutative variations can be advantageously applied because of its simplification.

LITERATURE

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DIE LÖSUNG DER FREIEN GEDÄMPFTEN LÄNGSSCHWINGUNGEN VON PRISMATISCHEN STÄBEN BEI NICHTLINEAREM ELASTIZITÄTSGESETZ MIT VARIATIONS-METHODE

Mešter Đula

Zusammenfassung

Diese Arbeit betrachtet die Ausfindung der Schwingungsgesetze bei freien gedämpften Längsschwingungen von prismatischen Stäben bei nichtlinearem Elastizitätsgesetz mit Variations-Methode.

ВАРИЈАЦИОНИ ПРИСТУП ПРОБЛЕМУ СЛОБОДНО ПРИГУШЕНИХ ЛОНГИТУДИНАЛНИХ ОСЦИЛАЦИЈА ПРИЗМАТИЧНЕ ГРЕДЕ СА НЕЛИНЕАРНОМ ЕЛАСТИЧНОМ КАРАКТЕРИСТИКОМ

Ђула Мештер

Резиме

Посматра се проналажење закона осциловања призматичног штапа при слободним пригушеним уздужним осцилацијама са нелинеарном еластичном карактеристиком

$$\sigma = E\left(1 - a_3 E^2 e_x^2\right) e_x .$$

При томе се узима да је нелинеарност слаба и да се пригушење остварује силама вискозног трења које су пропорционалне првом степену брзине. Задатак је решен на два начина, коришћењем Хамилтоновог и Вујановићевог варијационог принципа са некомутативним варијацијама. Применом обе методе добијени су исти резултати. Указано је на једноставност, ефикасност и широку могућност примене варијационог принципа са некомутативним варијацијама.

Mešter Đula
Institut za elektro
i mašinsku automatiku
24000 Subotica