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APPLICATION OF THE CORRESPONDENCE PRINCIPLE TO THE DETERMINATION OF STRESSES AND DISPLACEMENTS IN COMPOSITE STRUCTURES[1)]

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We introduce the integral operator $\widehat{r}$, associated with the function $r(t, \tau)$, by the definition

$$h = h(t, \tau) = \int_{\tau}^{t} r(t, \theta)\, g(\theta, \tau)\, d\theta = \widehat{r} g. \tag{1}$$

The indentity operator $\widehat{1}$ is associated with the Dirac function $1 = \delta(\tau - t)$

$$h = \int_{\tau}^{t} \delta(\theta - t)\, g(\theta, \tau)\, d\theta = \widehat{1} g = g. \tag{2}$$

The one-dimensional stress-strain relation for concrete, as an ageing linear viscoelastic, is given by Boltzmann's principle of superposition. Integrating by parts yields an inhomogeneous integral equation which by a simple transformation reduces to a Volterra integral equation of the second kind. In terms of operators it can be written

(3) a) $\sigma_b = E_b \widehat{K} (\varepsilon - \varepsilon_S)$

or b) $\varepsilon - \varepsilon_S = \dfrac{1}{E_b} \widehat{R}\, \sigma_b.$

E_b is the modulus of elasticity of the concrete at the time of action of the first exitation $\tau°$; ε_S is a known function expressing the shrinkage of the concrete;

$$\widehat{K} = k\widehat{1} - \widehat{\Psi}, \quad \widehat{R} = \frac{1}{k}\widehat{1} + \widehat{\Phi}. \tag{4}$$

$E_b k(t)$ is the modulus of elasticity of the concrete at time t; $\Psi(t, \tau)$ and $\Phi(t, \tau)$ are the kernel and resolvent kernel of the integral equation (3a). $\widehat{K}$ and $\widehat{R}$ are inverse operators, i.e.

$$\widehat{K}\widehat{R} = \widehat{R}\widehat{K} = \widehat{1}. \tag{5}$$

We consider a coplanar bar of variable cross section, consisting of concrete, which behaves according to (3), and several kinds of steel, which obey Hooke's law

$$\sigma_a = E_a \varepsilon, \qquad a = 1, 2, \ldots \tag{6}$$

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1) This also includes prestressed structures.

The cross sections have an axis of symmetry (y) and the loading acts in the plane of symmetry of the bar. The ratio between the height of the cross section and the radius of curvature of the bar axis is small, so that Navier's assumption

$$\varepsilon = \eta + \varkappa y. \tag{7}$$

is valid, where ε is the normal strain at a arbitrary point of the bar, η the normal strain of the bar axis, and $\varkappa$ the change in the curvature of the bar axis.

From (3a), (6), (7) and the condition of equilibrium of the external and internal forces acting in an arbitrary cross section we get the system of inhomogeneous integral equations

$$\begin{aligned} N + N_S &= EA\widehat{K}_{11}\eta + ES\widehat{K}_{12}\varkappa \\ M + M_S &= ES\widehat{K}_{12}\eta + EJ\widehat{K}_{22}\varkappa. \end{aligned} \tag{8}$$

E is some relative modulus of elasticity. The geometrical parameters of the cross section are reduced by the factors $\frac{E_a}{E}$ ($a=1, 2, \ldots$) and $\frac{E_b}{E}$. A is the reduced area, J the reduced moment of inertia of the cross section about the principal centroidal axis z in $\tau°$, $S=\sqrt{AJ}$, N is the axial force and M the bending moment about z — in the general case these are functions of time; N_s and M_s with respect to z, are known functions of time — they are the axial force and bending moment due to shrinkage of the concrete.

The operators $\widehat{K}_{hl}$ are of the form

$$\widehat{K}_{hl} = k_{hl}\widehat{1} - \beta_{hl}\widehat{\Psi}, \qquad h, l = 1, 2. \tag{9}$$

The functions k_{hl} are expressed in terms of $k(t)$ and the quantities β_{hl}, known reduced geometrical characteristics of the cross section which depend on the coordinate s along the axis of the bar.

The principal values of the operator matrix $\| \widehat{K}_{hl} \|$ consists of the operators

$$\widehat{K}_h = k_h\widehat{1} - \beta_h\widehat{\Psi}, \qquad h = 1, 2 \tag{10}$$

where k_h depends on $k(t)$ and β_h, the principal value of the matrix $\| \beta_{hl} \|$.

The inverse operators are

$$\widehat{K}_h\widehat{R}_h = \widehat{R}_h\widehat{K}_h = \widehat{1}, \qquad h = 1, 2. \tag{11}$$

$$\widehat{R}_h = \frac{1}{k_h}\widehat{1} + \beta_h\widehat{\Phi}_h \qquad h = 1, 2. \tag{12}$$

From (11) it follows that the function Φ_h ($h=1, 2$) is the solution of the inhomogeneous integral equation

$$k_h\Phi_h - \Psi\frac{1}{k_h} = \beta_h\widehat{\Psi}\Phi_h, \qquad h = 1, 2. \tag{13}$$

The solution of system (8) has the foım

(14)
$$\eta = \frac{1}{EA}\widehat{R}_{11}(N+N_S) + \frac{1}{ES}\widehat{R}_{12}(M+M_S)$$
$$\varkappa = \frac{1}{ES}\widehat{R}_{12}(N+N_S) + \frac{1}{EJ}\widehat{R}_{22}(M+M_S).$$

The operators $\widehat{R}_h$ (12) are the principal values of the operator matrix $\|\widehat{R}_{hl}\|$. In the light of the relations

(15)
$$\widehat{R}_{11} = \frac{1}{\beta_1-\beta_2}[(\beta_{11}-\beta_2)\widehat{R}_1 + (\beta_1-\beta_{11})\widehat{R}_2],$$
$$\widehat{R}_{22} = \frac{1}{\beta_1-\beta_2}[(\beta_1-\beta_{11})\widehat{R}_1 + (\beta_{11}-\beta_2)\widehat{R}_2],$$
$$\widehat{R}_{12} = \frac{\beta_{12}}{\beta_1-\beta_2}(\widehat{R}_1-\widehat{R}_2), \qquad 1>\beta_1>\beta_2>0,$$

solution (14) is determined by the two independent quadratures (13). We introduce the corresponding bar of a homogeneous elastic material (E_c) in whose cross sections we introduce the coordinate system y, z: y is the axis of symmetry and z is parallel to the second principal centroidal axis of the section.

For an arbitrary cross section we have

(16)
$$N_c = E_c A_c \eta + E_c S_c \varkappa$$
$$M_c = E_c S_c \eta + E_c J_c \varkappa$$

A_c is its area and S_c and J_c its first moment of the area and moment of inertia about the z axis. The bending moment M_c is taken about the same axis.

The solution of this system is

(17)
$$\eta = \frac{1}{E_c} J_c d^{-1} N_c - \frac{1}{E_c} S_c d^{-1} M_c$$
$$\varkappa = -\frac{1}{E_c} S_c d^{-1} N_c + \frac{1}{E_c} A_c d^{-1} M_c,$$

where

(18)
$$d = A_c J_c - S_c^2.$$

By means of the formal substitution

(19)
$$A_c \to A\widehat{K}_{11}, \qquad S_c \to S\widehat{K}_{12}, \qquad J_c \to J\widehat{K}_{22},$$
$$N_c \to N+N_S, \quad M_c \to M+M_S, \quad E_c \to E,$$

eqs. (16) become eqs. (8), referring to the cross section of the composite bar. The substitution

(20)
$$J_c d^{-1} \to \frac{1}{A}\widehat{R}_{11}, \quad S_c d^{-1} \to -\frac{1}{S}\widehat{R}_{12}, \quad A_c d^{-1} \to \frac{1}{J}\widehat{R}_{22},$$
$$N_c \to N+N_S, \qquad M_c \to M+M_S$$

and

(21) $$E_c \to E$$

in (17) yields the quantities η and $\varkappa$ (14) for the composite bar. The generalized displacement ζ of the corresponding bar is given by the principle of virtual forces[2)]

(22) $$\zeta = \zeta(s) = \int_L [\hat{N}(u, s)\, \eta(u) + \hat{M}(u, s)\, \varkappa(u)]\, du.$$

$\hat{N}$ is the axial force and $\hat{M}$ the bending moment due to a unit generalized force acting at point s; η and $\varkappa$ are due to the given loading and are determined by (17).

The formal substitutions (20) and (21) in (17) and (22) yield the generalized displacement ζ for the composite structure[3)].

The stress in the corresponding bar is

(23) $$\sigma = E_c\, \varepsilon = E_c(\eta + \varkappa y),$$

where η and $\varkappa$ are given by (17).

From (6), by the formal substitution (20) and

(24) $$E_c \to E_a, \qquad a = 1, 2, \ldots$$

in (23), we get the stress σ_a in the steel parts of the bar ($a=1, 2, \ldots$). The stress in the concrete is obtained from (3a) as

(25) $$\sigma_b + E_b \widehat{K} \varepsilon_S = E_b \widehat{K}(\eta + \varkappa y).$$

It may be seen from (14) that the right hand side involves the operators

(26) $$\widehat{B}_{hl} = \widehat{K}\widehat{R}_{hl}, \qquad h, l = 1, 2.$$

The principal values of the operator matrix $\| \widehat{B}_{hl} \|$ are the operators $\widehat{B}_h$, which satisfy

(27) $$\beta_h \widehat{B}_h + (1-\beta_h)\widehat{R}_h = \widehat{1}, \text{ i.e. } \widehat{B}_h = \frac{k}{k_h}\widehat{1} - (1-\beta_h)\widehat{\Phi}_h, \qquad h = 1, 2.$$

Between the operators $\widehat{B}_{hl}$ and $\widehat{B}_h$ there is a relation analogous to (15), so that to determine σ_b it is not necessary to carry out another quadrature. From (23), (20), (25) and (26) it follows that $\sigma_b + E_b K \varepsilon_s$ can be obtained from the stress in the corresponding bar by the formal substitution

(28) $$J_c d^{-1} \to \frac{1}{A}\widehat{B}_{11}, \quad S_c d^{-1} \to -\frac{1}{S}\widehat{B}_{12}, \quad A_c d^{-1} \to \frac{1}{J}\widehat{B}_{22},$$
$$N_c \to N + N_S, \qquad M_c \to M + M_S, \qquad E_c \to E_b.$$

The expressions for the generalized displacement (22) and stress (23) of the corresponding bar yield the corresponding expressions for the composite bar after the substitutions (20) and (21), (20) and (24), and (28), expressing thereby the principle of correspondence.

[2)] The shear deformation is neglected.

[3)] The principle of virtual forces is also applicable to the composite structures because the virtual forces only have to satisfy the equilibrium conditions, so that for the composite structures $\hat{N}$ and $\hat{N}$ are time independent.

Conclusion

For a composite coplanar bar of which the concrete is assumed to behave according to the general relationship for a homogeneous linear viscoelastic material subject to ageing, whose cross sections have arbitrary geometry, arbitrarily loaded in the plane of symmetry, with both $N \neq 0$, $M \neq 0$ and $\varepsilon_s \neq 0$, expressions have been derived for the stresses and the generalized displacement, thereby permitting in principle the derivation of a solution for statically indeterminate composite structures.

In the literature available to us only expressions for the stresses have been derived, and only for special cases of the cross section geometry and loading for statically determinate structures. These are in fact those special cases when integral equations (8) reduce to one or break down into two independent inhomogenous integral equations.

By considering the principal values of the operator matrix $\| \widehat{K}_{-hl} \|$, it is shown that the inversion of system (8) always reduces to the solution of the two independent integral equations (13). Thanks to this it is possible to derive expressions for the stresses and displacements for an arbitrary composite structure under arbitrary loading. Thereby we have also demonstrated the shortest route for solving the problem of composite structures, because the two independent quadratures (13) are sufficient to determine the stresses in the given cross section, and also for finding the generalized displacements of the bar.

Applying the principle of correspondence the expressions for the stresses and displacements are obtained directly from the known expressions from Strenght of Materials derived for the corresponding elastic bar.

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ПРИЛОЖЕНИЕ ПРИНЦИПА КОРРЕСПОНДЕНЦИИ К НАХОЖДЕНИЮ НАПРЯЖЕНИЙ И ПЕРЕМЕЩЕНИЙ В СОПРЯЖЕННЫХ СООРУЖЕНИЯХ

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Резюме

Применением линейных интегральных операторов можно определить выражения для напряжений и перемещений в сопряженных сооружениях, когда бетон рассматривается как однородный вязкоупругий материял, с учётом старости.

Для любого сечения образуется система двух неоднородных интегральных уравнений, которые представляют собой связь между нормальной силой и изгибающим моментом и величин характеризующих его деформацию.

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Рассматривая эту систему уравнений и её решения с одной стороны и соответствующие уравнения для соответствующего однородного упругого сооружения с другой стороны, восстановляется корресподенция таким образом, что геометрические характеристики упругого сечения, формально заменяются интегральными операторами.

Функции которым ассоцираны эти операторы однозначно определены решениями двух независимых интегральных уравнений Вольтерра второго класса. Воспользованием принципа корресподенции, можно непосредствено написать выражения для напряжений и перемещений в сопряженом сооружении, из соответствующих выражений исвестных в Сопротивлении материялов, относящихся на соответственое упругое сооружение.

Принцип корресподенции относящийся на однородный вязкоупругий материял с учётом старости, был восстановлен Манделем. В настоящей работе, распространяется принцип Манделя на случай стержневого сооружения, в котором однородный вязкоупругий материял с учётом старости сопряжается с одным или несколькими упругими материялами.

ПРИМЕНА ПРИНЦИПА КОРЕСПОНДЕНЦИЈЕ НА ОДРЕЂИВАЊЕ НАПОНА И ПОМЕРАЊА У СЛОЖЕНИМ СТРУКТУРАМА

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Резиме

Мандел је успоставио принцип коресподенције, помоћу кога се из једначина линеарне еластичности добијају једначине за линеарне вискоеластичне материале са старањем. У раду је дато проширење Манделовог принципа на случај линијског носача у коме се хомоген линеаран вискоеластичан материал са старањем спреже са једним или више еластичних материала.

Примена линеарних интегралних оператора омогућава да се изведу изрази за напоне и померања код спрегнутих линијских носача, при чему је за бетон уведен линеаран вискоеластичан материал са старењем. За уочени попречни просек поставља се систем од две нехомогене интегралне једначине као веза између пресечних сила и деформацијских величина. Из овог система једначина и његовог решења, успоставља се коресподенција са одговарајућим једначинама за носач који је од хомогеног еластичног материала тако, што се геометријске карактеристике коресподентног пресека формално замењују операторима. Функције којима су асоцирани ови оператори су једнозначна решења две независне интегралне једначине Волтера II врсте.

Коришћењем принципа коресподенције, добијају се изрази за напоне и померања за спрегнути носач, непосредно из израза познатих у Отпорности материјала, који важе за носач од хомогеног еластичног материала.

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