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ON STABILITY OF MOTION OF NON-HOLONOMIC SYSTEMS

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A general criterion on the stability of the motion of scleronomic systems, based upon the application of Liapounoff's direct method, has been worked out in the paper [2]. The criterion suggested in this paper assumes the existence of a function having certain properties, this function being dependent on the displacement coordinates and time only, but not upon the velocities. Therefore its evaluation is considerably simpler than the evalution of Liapounoff's functions. The present paper is in fact an attempt to apply the suggested method to non-holonomic scleronomic systems.

1. Let us consider a mechanical system $\mathcal{M}$ described by:

D_1) A set of particles $M_i (i=1,, \ldots, N)$ each having a mass m_i, the radii vector of which are $\vec{r}_i$, and

D_2) The motion of these particles is constrained by K holonomic constraints

$$f_a(\vec{r}_1, \ldots, \vec{r}_n)=0 \qquad (a=1, \ldots, K) \tag{1}$$

and L non-holonomic, scleronomic, constraints of the form

$$\sum_{i=1}^{n} \vec{l}_{bi} \cdot \vec{V}_i + l_b = 0 \qquad (\mathrm{b}=1, \ldots, L) \tag{2}$$

where

$$\vec{l}_{bi}=\vec{l}_{bi}(\vec{r}_1, \ldots, \vec{r}_n); \quad l_b=l_b(\vec{r}_1, \ldots, \vec{r}_n).$$

D_3) The system is in motion as acted upon by active forces

$$\vec{F}_i=\vec{F}_i(t; \vec{r}_1, \ldots, \vec{r}_n; \vec{v}_1, \ldots, \vec{v}_n)$$

and the forces of constrain reactions

$$\vec{R}_i=\sum_{a=1}^{k} \lambda_a \operatorname{grad}_i f_a+\sum_{b=1}^{L} \mu_b \vec{l}_{bi}$$

D_4) It is possible to describe the motion involved by differential equations (Lagrange's equations of the first kind)

$$m_i \vec{W}_i=\vec{F}_i+\sum_{a=1}^{K} \lambda_a \operatorname{grad}_i f_a+\sum_{b=1}^{L} \mu_b \vec{l}_{bi} \quad \left(\vec{W}_i=\frac{d^2 \vec{r}_i}{dt^2}\right), \tag{3}$$

in which λ_a and μ_b are the constraint multipliers.

If $n=3N-K$ Lagrange's coordinates q^μ independent of holonomic constraints are introduced, the equations of non-holonomic constraints can be rewritten in the form of

$$\Phi_{(b)\mu}\dot{q}^\mu+\Phi_{(b)}=0 \qquad (\mu=1, \ldots, n) \tag{4}$$

where

$$\Phi_{(b)\mu}=\sum_{i=1}^{N}\vec{l}_{bi}\frac{\partial\vec{r}_i}{\partial q^\mu}, \qquad \Phi_{(b)}=l_b[\vec{r}_i(q^1, \ldots, q^n)].$$

The system $\Phi_{(b)\mu}$, when b is fixed, with respect to the permitted transformations of the variables

$$\overline{q}^\mu=\overline{q}^\mu(q^1, \ldots, q^n)$$

has the character of a covariant vector. The equations (3) can be written in the following way, provided they are multiplied ordinarily by vectors $\partial\vec{r}_i/\partial q^\mu$ and after carrying out all necessary transformations required:

$$\frac{dq^\mu}{dt}=a^{\mu\nu}p_\nu \tag{5}$$

$$\frac{Dp_\mu}{dt}=Q_\mu+\sum_{b=1}^{L}\mu_b\Phi_{(b)\mu}.$$

where

$$a_{\mu\nu}=\sum_{i=1}^{N}m_i\frac{\partial\vec{r}_i}{\partial q^\mu}\cdot\frac{\partial\vec{r}_i}{\partial q^\nu}; \quad p_\mu=\sum_{i=1}^{N}m_i\vec{V}_i\cdot\frac{\partial\vec{r}_i}{\partial q^\mu}; \quad a_{\mu\nu}a^{\mu\tau}=\delta_\mu^\tau; \quad Q_\mu=\sum_{i=1}^{N}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q^\mu}.$$

The system (5) comprises $2n$ equations, and together with L equations (4) makes up a system of $2n+L$ equations with an equal number of unknown functions $q^\mu=q^\mu(t)$, $p_\mu=p_\mu(t)$, $\mu_b=\mu_b(t)$. When non-holonomic constraint multipliers are eliminated from (5), we have

$$\frac{dq^\mu}{dt}=a^{\mu\nu}p_\nu \tag{6}$$

$$\frac{Dp_\mu}{dt}=P_\mu-\sum_{b,c=1}^{L}\mathcal{F}^{bc}\Phi_{(b)\mu}\left(\frac{D\Phi^\sigma_{(c)}}{dt}+\frac{\partial\Phi_{(c)}}{\partial q^\mu}a^{\mu\sigma}\right)p_\sigma$$

where

$$P_\mu=Q_\mu-\sum_{b,c=1}^{L}Q_\sigma\Phi^\sigma_{(c)}\Phi_{(b)\mu}\mathcal{F}^{bc}$$

$$\mathcal{F}_{bc}=a^{\mu\nu}\Phi_{(b)\mu}\Phi_{(c)\nu}; \qquad \mathcal{F}_{bc}\mathcal{F}^{cd}=\delta_b^d.$$

The system of equations (6) has $2n$ equations with an equal number of unknown functions and defines the motion of the mechanical system $\mathcal{M}$, which, with reference to the assumptions introduced, has only $2n-L$ degrees of freedom of motion.

2. Consider the solution

$$(7)\qquad \begin{aligned} q^{\mu} &= q^{\mu}(t;\ \overset{1}{q}_0, \ldots, \overset{n}{q}_0, p_{10}, \ldots, p_{p0}) \\ p_{\mu} &= p_{\mu}(t;\ \overset{1}{q}_0, \ldots, \overset{n}{q}_0, p_{10}, \ldots, p_{n0}) \end{aligned}$$

of equations (6) arrived at for the initial conditions

$$(8)\qquad \begin{aligned} \overset{\mu}{q}_0 &= q^{\mu}(t_0;\ \overset{1}{q}_0, \ldots, p_{n0}) \\ p_{\mu 0} &= p_{\mu}(t_0;\ \overset{1}{q}_0, \ldots, p_{n0}). \end{aligned}$$

The functions (7) describe the motion of the system $\mathcal{M}$ for the given initial conditions (8) which can be presented also in the vectorial form

$$(9)\qquad \vec{r}_i = \vec{r}_i(q^1(t), \ldots, q^n(t)) = \vec{r}_i(t).$$

We shall assume the motion described by scalar functions (7) or by vector functions (9) as an unperturbed motion. Let the motion

$$(10)\qquad \vec{r}^{\,*}_i = \vec{r}_i(\overset{*}{q}{}^1(t), \ldots, \overset{*}{q}{}^n(t)) = \vec{r}^{\,*}_i(t),$$

be a perturbed motion as compared to the former.
The perturbation vector

$$\vec{\rho}_i = \vec{r}^{\,*}_i(t) - \vec{r}_i(t)$$

can be presented in the form [3]

$$(11)\qquad \vec{\rho}_i = \xi^{\mu} \frac{\partial r_i}{\partial q^{\mu}}.$$

We shall evaluate, first, the conditions set upon this vector (or, to put it more precisely, upon its time derivative) by the non-holonomic constraints. Since the perturbed motion must satisfy the equations of non-holonomic constraints as well, we have

$$(12)\qquad \sum_{i=1}^{N} \vec{l}^{\,*}_{ib} \cdot \vec{V}^{\,*}_i + \overset{*}{l}_b = 0$$

where

$$\vec{l}^{\,*}_{ib} = \vec{l}_{ib}(\vec{r}^{\,*}_1, \ldots, \vec{r}^{\,*}_n),\quad \overset{*}{l}_b = l_b(\vec{r}^{\,*}_1, \ldots, \vec{r}^{\,*}_n),\quad \vec{V}^{\,*}_i = \vec{V}_i + \frac{D\xi^{\mu}}{dt}\frac{\partial \vec{r}_i}{\partial q^{\mu}}.$$

If we find the difference between the equations (12) and (2), and limit ourselves to the linear approximation of the equation thus obtained. we shall have

$$(13)\qquad \Phi_{(b)\mu}\frac{D\xi^{\mu}}{dt} + R_{(b)\mu}\xi^{\mu} = 0$$

where

$$R_{(b)\mu}=\sum_{i=1}^{N}\frac{\partial\vec{l}_{bi}}{\partial q^{\mu}}\cdot\frac{\partial\vec{r}_i}{\partial q^{\nu}}\dot{q}^{\nu}+\frac{\partial l_b}{\partial q^{\mu}}=\Phi_{(b)\mu\nu}\dot{q}^{\nu}+\frac{\partial l_b}{\partial q^{\mu}}.$$

The equations (13) yield the conditions we have been looking for.

For the perturbed motion equations we can take

$$m_i(\vec{W}_i^{*}-\vec{W}_i)=\vec{F}_i^{*}-\vec{F}_i+\sum_{a=1}^{K}(\overset{*}{\lambda}_a \operatorname{grad}_i \overset{*}{f}_a-\lambda_a \operatorname{grad}_i f_a)$$

(14)

$$+\sum_{b=1}^{L}(\overset{*}{\mu}_b \overset{*}{\vec{l}}_{bi}-\mu_b \vec{l}_{bi})$$

where an asterisk denotes that the item involved belongs to the perturbed motion. If the equation (14) is multiplied by $\partial\vec{r}_i/\partial q^{\mu}$, and an addition in terms of i is carried out, we shall have

$$\sum_{i=1}^{N}m_i(\vec{W}_i^{*}-\vec{W}_i)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}=\sum_{i=1}^{N}(\vec{F}_i^{*}-\vec{F}_i)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}+\sum_{i=1}^{N}\sum_{a=1}^{K}(\overset{*}{\lambda}_a \operatorname{grad}_i \overset{*}{f}_a$$

(15)

$$-\lambda_a \operatorname{grad}_i f_a)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}+\sum_{i=1}^{N}\sum_{b=1}^{L}(\overset{*}{\mu}\overset{*}{\vec{l}}_{bi}-\mu_b\vec{l}_{bi})\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}.$$

We propose to transform this equation in the following manner. First, the term on its left hand side can be rewritten as follows:

$$\sum_{i=1}^{N}m_i(\vec{W}_i^{*}-\vec{W}_i)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}=\frac{D\eta_{\mu}}{dt}. \tag{16}$$

We shall call the terms $\eta_{\mu}=\overset{*}{p}_{\mu}-p_{\mu}$ as perturbation impulses. On the right hand side of Eq. (15) we have: first, perturbations of the generalized forces

$$\Psi_{\mu}=\sum_{i=1}^{N}(\vec{F}_i^{*}-\vec{F}_i)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}=\overset{*}{Q}_{\mu}-Q_{\mu}.$$

Should this function be expanded in the vicinity of the unperturbed motion, and limiting ourselves to the linear approximation, we shall have

$$\Psi_{\mu}=\xi^{\nu}\nabla_{\nu}Q_{\mu}+\eta_{\nu}\frac{\partial Q_{\mu}}{\partial p_{\nu}}. \tag{17}$$

The second term vanishes in the first approximation

$$\sum_{i=1}^{N}\sum_{a=1}^{K}(\overset{*}{\lambda}_a \operatorname{grad}_i \overset{*}{f}_a-\lambda_a \operatorname{grad}_i f_a)\cdot\frac{\partial\vec{r}_i}{\partial q^{\mu}}=0 \tag{18}$$

since the perturbed motion itself is being derived consistently with the equations of the holonomic constraints. Now, there remains the last term in (15) which represents the difference of reactions between non- holonomic constraints in both the perturbed and unperturbed motions, to be transformed. When the function $\overset{\rightarrow *}{l_{bi}}$ is expanded into a power series in the vicinity of the unperturbed motion, we shall obtain

$$(19)\quad \sum_{i=1}^{N}\sum_{b=1}^{L}(\overset{*}{\mu}_b \overset{\rightarrow *}{l_{bi}} - \mu_b \overrightarrow{l_{bi}})\cdot\frac{\partial \vec{r}_i}{\partial q^{\mu}} = \sum_{b=1}^{L}(\overset{*}{\mu}_b \Phi_{(b)\mu} + \overset{*}{\mu}_b \Phi_{(b)\mu\nu}\xi^{\nu} - \mu_b \Phi_{(b)\mu} + \cdots).$$

If the coefficients $\overset{*}{\mu}_b$ are presented in the form of the summation $\overset{*}{\mu}_b = \mu_b + \mu'_b$ in which $\mu_b' = \mu_b'(\xi^1, \ldots, \xi^n, \eta_1, \ldots, \eta_n)$ are the unknown functions which vanish for $\xi^{\mu} = 0$, and $\eta_{\mu} = 0$, the expression (19) in its linear approximations becomes

$$(20)\qquad \sum_{i=1}^{N}\sum_{b=1}^{L}(\overset{*}{\mu}_b \overset{\rightarrow *}{l_{bi}} - \mu_b \overrightarrow{l_{bi}})\cdot\frac{\partial \vec{r}_i}{\partial q^{\mu}} = \sum_{b=1}^{L}(\mu_b \Phi_{(b)\mu\nu}\xi^{\nu} + \mu'_b \Phi_{(b)\mu}).$$

If equations (16) to (20) are taken into account, the equation (15) can be transformed into

$$\frac{D\eta_{\mu}}{dt} = \xi^{\nu}\nabla_{\nu} Q_{\mu} + \frac{\partial Q_{\mu}}{\partial p_{\nu}}\eta_{\nu} + \sum_{b=1}^{L}(\mu_b \Phi_{(b)\mu\nu}\xi^{\nu} + \mu'_b \Phi_{(b)\mu}).$$

After introducing the function

$$\Xi_{\mu} = \left(\nabla_{\nu} Q_{\mu} + \sum_{b=1}^{L}\mu_b \Phi_{(b)\mu\nu}\right)\xi^{\nu} + \frac{\partial Q_{\mu}}{\partial p_{\nu}}\eta_{\nu}.$$

the perturbed motion equations can now, be rewritten in the following way

$$\frac{D\xi^{\mu}}{dt} = a^{\mu\nu}\eta_{\nu}$$

$$(21)\qquad \frac{D\eta_{\mu}}{dt} = \Xi_{\mu} + \sum_{b=1}^{L}\mu'_b \Phi_{(b)\mu}$$

$$a^{\mu\nu}\Phi_{(b)\mu}\eta_{\nu} + R_{(b)\mu}\xi^{\mu} = 0.$$

Upon the elimination of the coefficient μ'_b from (21), the equations of the perturbed motion now are

$$\frac{D\xi^{\mu}}{dt} = a^{\mu\nu}\eta_{\nu}$$

$$(22)\qquad \frac{D\eta_{\mu}}{dt} = H_{\mu}$$

where

$$H_{\mu} = \Xi_{\mu} - \mathcal{T}^{bc}\left[\Phi_{(c)\nu}\Xi^{\nu} + \frac{D\overset{\nu}{\Phi}_{(c)}}{dt}\eta_{\nu} + \overset{\nu}{R}_{(c)}\eta_{\nu} + \frac{DR_{(c)\nu}}{dt}\xi^{\nu}\right]\Phi_{(b)\mu}.$$

3. We now propose to investigate the stability of the unperturbed motion (9) of the system $\mathcal{M}$ in Liapounoff's sense of the word. The following two theorems provide sufficient conditions for the unperturbed motion to either be stable, asymptotically stable or unstable.

Theorem 1. *Should within the configuration space V_n there exist a positively definite scalar function*

$$W = W(t, \xi^1, \ldots, \xi^n) \in C^{(1,1)}_{t,\xi^\nu}(\mathcal{H}) \tag{23}$$

$$\mathcal{H} = \{t_0 \leqslant t < +\infty, \ |\xi^\nu| < h = \text{const.}\}$$

for which the expression

$$\mathcal{J} = \frac{\partial W}{\partial t} + \left(H_\mu + \frac{\partial W}{\partial \xi^\mu}\right) a^{\mu\nu} \eta_\nu \tag{24}$$

put together in the way of the equations (22) for the perturbed motion, is

a) *negative or identically equal to nil, the trivial solution, $\xi^\mu = 0$, $\eta_\mu = 0$, of the equations* (22) *is stable;*

b) *negatively definite, the trivial solution $\xi^\mu = 0$, $\eta_\mu = 0$, of the equations* (22) *is asymptotically stable.*

Proof. Let us consider the function

$$V = \frac{1}{2} a^{\mu\nu} \eta_\mu \eta_\nu + W. \tag{25}$$

Its time derivative is

$$\frac{dV}{dt} = \frac{D}{dt}\left(\frac{1}{2} a^{\mu\nu} \eta_\mu \eta_\nu\right) + \frac{DW}{dt} = a^{\mu\nu} \frac{D\eta_\mu}{dt} \eta_\nu + \frac{\partial W}{\partial \xi^\mu} \frac{D\xi^\mu}{dt} + \frac{\partial W}{\partial t}. \tag{26}$$

If we substitute (22) for the corresponding terms of (26), we shall have

$$\frac{dV}{dt} = a^{\mu\nu}\left(H_\mu + \frac{\partial W}{\partial \xi^\mu}\right) \eta_\nu + \frac{\partial W}{\partial t}. \tag{27}$$

With respect to the assumptions made for this theorem, and the equalities (25) and (27), we may conclude that the function V satisfies, in the case a), the conditions of Liapounoff's first theorem [1], and, in the case b), the conditions of Liapounoff's second theorem. Whence there follows the statement of the present theorem.

Theorem 2. *Should exist within the configuration space V_n a scalar function*

$$W = W(t, \xi^1, \ldots, \xi^n) \in C^{(1,1)}_{t,\xi^\nu}(\mathcal{H})$$

$$\mathcal{H} = \{t_0 \leqslant t < +\infty, \ |\xi^\nu| < h = \text{const.}\}$$

which in arbitrary vicinity of the origin of coordinates has negative values, and if

1° *it is limited in* $\mathcal{H}$

2° *in* $\mathcal{H}$, *it has continuous derivatives in terms of* t *and* ξ^ν, *which satisfy the inequalities*:

a) $$a^{\mu\nu}\left(H_\mu+\frac{\partial W}{\partial \xi^\mu}\right)\eta_\nu+\frac{\partial W}{\partial t}\leqslant 0$$

b) $$\xi^\tau H_\tau>0$$

then the trivial solution $\xi^\mu=0$, $\eta_\mu=0$, *of the equations* (22) *is unstable.*

P r o o f. Let us consider the function

$$V=-\left(\frac{1}{2}a^{\mu\nu}\eta_\mu\eta_\nu+W\right)\xi^\tau\eta_\tau \tag{28}$$

and if we write its time derivative, made up in the way of equations (22), then we shall obtain

$$\frac{dV}{dt}=-\left[a^{\mu\nu}\left(H_\mu+\frac{\partial W}{\partial \xi^\mu}\right)\eta_\nu+\frac{\partial W}{\partial t}\right]\xi^\tau\eta_\tau-\left(\frac{1}{2}a^{\mu\nu}\eta_\mu\eta_\nu+W\right)(a^{\tau\sigma}\eta_\tau\eta_\sigma+\xi^\tau H_\tau)$$

whence, on the grounds of the assumptions made for this theorem, and equations (28), it is possible to see that the function V satisfies the conditions of Chetayev's theorem [1], wherefrom there follows the statement of the present theorem.

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ОБ УСТОЙЧИВОСТИ ДВИЖЕНИЯ НЕГОЛОНОМНЫХ СИСТЕМ

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Резюме

В статье [2] получено общее утверждение об устойчивости движения голономных склерономных систем в смисле Ляпунова, базированное на втором методе Ляпунова. Предложенным критерием не нужно определять дифференциальнные уравнения движения и вместо функции Ляпунова достаточно выбрать некоторую функцию зависящую только од обобщенных координат. В настоящей статье на основе этово метода исследуется устойчивость движения неголономых систем.

О СТАБИЛНОСТИ КРЕТАЊА НЕХОЛОНОМНИХ СИСТЕМА

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Посматра се механички систем који чини скуп од N материјалних тачака чији су положаји одређени векторима $\vec{r}_i$. Кретање система ограничавају K холономних веза (1) и L нехолономних, линеарних по брзинама, веза (2). За изучавање кретања система користе се једначине (6), у којима су q^μ независне Лагранжеве координате а p_μ одговарајући импулси·

Уочено је неко непоремећено кретање (7) /односно, написано у векторском облику — (9)/ и изучава се његова стабилност у Љапуновљевом смислу. Најпре је показано да, ако се вектор поремећаја узме у облику (11), једначине поремећеног кретања се могу написати у облику (22), где су ξ^μ поремећаји генералисаних координата q^μ а η_μ су поремећаји генералисаних импулса p_μ.

Наредна два става, формулисана и доказана у овом раду, дају довољне услове да непоремећено кретање буде стабилно — асимптотски стабилно, односно нестабилно.

Став 1. Ако у конфигурационом простору V_n постоји позитивно дефинитна скаларна функција (23) за коју је израз (24), састављен у смислу једначина непоремећеног кретања (22),

а/ негативан или идентички једнак нули, тривијално решење $\xi^\mu=0$, $\eta_\mu=0$ једначина (22) је стабилно;

б/ негативно дефинитан, тривијално решење је асимптотски стабилно.

Став 2. Ако у конфигурационом простору V_n, постоји скаларна функција облика (23) која у произвољној околини координатног почетка има негативне вредности и ако је

1° ограничена у области $\mathscr{H}$

2° има у $\mathscr{H}$ непрекидне изводе по t и ξ^μ који задовољавају неједнакости а/ и б/, онда је тривијално решење $\xi^\mu=0$ и $\eta_\mu=0$ једначина (22) нестабилно.

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