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SOME REMARKS ON SOLVING PLANE ELASTOSTATIC BOUNDARY--VALUE PROBLEMS WITH CUTS IN ONE ROW

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1. It is well-known that for solving plane boundary-value problems with cuts in one row the method of solving the Hilbert-Riemann's problem [1] can be applied. The functions $\varphi(z)$ and $\psi(z)$ appearing in this problem must satisfy the boundary condition

$$-\varkappa\varphi(z)+z\,\overline{\varphi'(z)}+\overline{\psi(z)}=g(z),\quad z\in C \tag{1—1}$$

where C denotes the boundary curve of the domain D in which our problem should be solved, and for a multiply connected domain D they are multivalued. Therefore we introduce the functions $\Phi(z)=\varphi'(z)$ and $\Psi(z)=\psi'(z)$, which are already holomorphic in D. Now, the problem can be attacked in the form:

$$-\varkappa\varphi'(z)+\overline{\varphi'(z)}+z\,\overline{\varphi''(z)}+\overline{\psi'(z)}=\frac{dg(z)}{dx},\quad z\in C \tag{1—2}$$

which is possible to treat as a Hilbert-Riemann's problem. If we are interested in stresses only, it suffices to know the functions $\Phi(z)$ and $\Psi(z)$. But in order to find the displacements, these two functions have to be integrated. In this article will be indicated a possibility of treating this problem in the unchanged form (1—1) in passing over to Hilbert-Riemann's problem.

2. Expressions of functions $\varphi(z)$ and $\psi(z)$ for a multiply connected domain D are known [1], [2], but it is not appropriate to use them in solving our problem. Therefore, another changed expression will be introduced.

Let the plane z be cut along finite number of separated und unclosed curves C_k $1\leq k\leq N$, not intersecting themselves. The initial point of the curve C_k is denoted by α_k, the final one by β_k. The left side of the cut C_k is called the positive C_k^+ and the right one the negative C_k^-. If s denotes the arc length on $\cup C_k$ then let be

$$\left.\begin{array}{r} z(s_k)=\alpha_k \\ s_k\leqslant s\leqslant\sigma_k \;\Rightarrow\; z(s)\in C_k \\ z(\sigma_k)=\beta_k \end{array}\right\}$$

and $s_{k+1} > \sigma_k$. Now it is possible to define the functions $\varphi(z)$ and $\psi(z)$ in the equivalent form

$$\varphi(z) = \sum_{k=1}^{k=N} A_k \int_{s_k}^{\sigma_k} \ln(z-\zeta)\, ds + \varphi_0(z) \tag{2—1}$$

$$\psi(z) = -\varkappa \sum_{k=1}^{k=N} \bar{A}_k \int_{s_k}^{\sigma_k} \ln(z-\zeta)\, ds + \psi_0(z) \tag{2—1'}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are holomorphic in the whole plane z outside of the cuts $\cup C_k$ except at $z=\infty$, where we have eventually a simple pole. For the constants A_k we have

$$2\pi(1+\varkappa)(\sigma_k - s_s) A_k = -P_k \tag{2—2}$$

where P_k denotes the main vector on the whole cut $C_k = C_k^+ \cup C_k^-$.

The functions $\varphi_0(z)$ and $\psi_0(z)$ can be expressed in the form:

$$\begin{aligned} &\varphi_0(z) = \Gamma_1 z + K_1 + \tilde{\varphi}(z) \\ &\psi_0(z) = \Gamma_2 z + K_2 + \tilde{\psi}(z) \\ &\tilde{\varphi}_0(z) = O(z^{-1}), \ \tilde{\psi}_0(z) = O(z^{-1}) \end{aligned} \tag{2—3}$$

Fuıther let us introduce the constant A_0:

$$A_1 + A_2 + \cdots + A_N = A_0 . \tag{2—4}$$

The uniqueness theorem of displacements boundary-value problem, which will be taken as known, states:

If we have

(2—5)

1. $A_0 = \Gamma_1 = \Gamma_2 = 0$

2. $g(z) = 0, \ z \in C = (\cup C_k^+) \cup (\cup C_k^-), \ 1 \leqslant k \leqslant N$

then

$$\begin{aligned} &\varphi(z) = K_1 \\ &\psi(z) = x \bar{K}_1, \ z \in D \cup C \end{aligned} \tag{2—6}$$

where K_1 denotes an arbitrary complex constant.

Then it follows

$$A = 0, \ 1 \leqslant k \leqslant N \tag{2—6'}$$

$$K_2 = x \bar{K}_1 \tag{2—6''}$$

$$\tilde{\varphi}_0(z) = \tilde{\psi}_0(z) = 0, \ z \in D \cup C \tag{2—6'''}$$

If we introduce the notations

$$G[\varphi(z),\ \psi(z)] = -\varkappa\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \tag{2—7}$$

$$\tilde{\varphi}(z) = \sum_{k=1}^{k=N} A_k \int_{s_k}^{\sigma_k} \ln(z-\zeta)\, ds \tag{2—8}$$

$$\tilde{\psi}(z) = -\varkappa \sum_{k=1}^{k=N} \bar{A}_k \int_{s_k}^{\sigma_k} \ln(z-\zeta)\, ds \tag{2—8'}$$

$$g_0(z) = g(z) - G[\tilde{\varphi}(z),\ \tilde{\psi}(z)] - G[\Gamma_1 z,\ \Gamma_2 z] \tag{2—8''}$$

then the boundary value condition (1—1) in respect to (2—6″) can be expressed as follows:

$$G[\tilde{\varphi}(z),\ \tilde{\psi}_0(z)] = g_0(z),\ z \in C. \tag{2—9}$$

Now, we have to find the functions $\tilde{\varphi}_0(z)$ and $\tilde{\psi}_0(z)$, holomorphic outside of the cuts (and at the point $z=\infty$ too) and the constants A_k, $1 \leq k \leq N$, which are contained in $g_0(z)$. The constants A_0, Γ_1 and Γ_2 are arbitrary given complex numbers and $g(z)$ is an arbitrary given function (displacements) on $C = (\cup C_k^+) \cup (\cup C_k^-)$ which has to be smooth enough [1], [2].

3. Further, the discussion will be continued for the cuts along the x-a only. In this case we introduce a new function

$$\tilde{\chi}_0(z) = \overline{\tilde{\varphi}}_0(z) + z\,\overline{\tilde{\varphi}}_0'(z) \tag{3—1}$$

so that the equation (1—1) in respect to (2—9) takes the form

$$-\varkappa\tilde{\varphi}_0(z) + \tilde{\chi}_0(\bar{z}) + (z-\bar{z})\,\overline{\tilde{\varphi}_0'(z)} = g_0(z),\ z \in C \tag{3—2}$$

The limiting process $y \to +0$ gives now

$$-\varkappa\tilde{\varphi}_0^+(x) + \tilde{\chi}_0^-(x) = \tilde{g}_0^+(x),\ z = x \in C \tag{3—3}$$

and $y \to -0$

$$-\varkappa\tilde{\varphi}_0^-(x) + \tilde{\chi}_0^+(x) = \tilde{g}_0^-(x),\ z = x \in C \tag{3—3'}$$

From the above equations we have at once

$$\begin{aligned}
&[-\varkappa\tilde{\varphi}_0(x) + \tilde{\chi}_0(x)]^+ + [-\varkappa\tilde{\varphi}_0(x) + \tilde{\chi}_0(x)]^- = g_0^+(x) + g_0^-(x) \\
&[-\varkappa\tilde{\varphi}_0(x) - \tilde{\chi}_0(x)]^+ - [-\varkappa\tilde{\varphi}_0(x) - \tilde{\chi}_0(x)]^- = g_0^+(x) - g_0^-(x).
\end{aligned} \tag{3—4}$$

In comparison with the methodology of solving this problem in [1] here, we have to find the solution of the equation (3—4) in the class $h[\alpha_k, \beta_k, 1 \leq k \leq N]$, that means a solution bounded at all points α_k and β_k. Now if follows [3]:

$$(3-5)\qquad -\varkappa\tilde{\varphi}_0(x)+\tilde{\chi}_0(z)=\frac{\sqrt{R(z)}}{2\pi i}\int\limits_{\cup[\alpha_k,\beta_k]}\frac{[g_0^+(x)+g_0^-(x)]}{\sqrt{R(x)}\cdot(x-z)}\,dx$$

where

$$(3-5')\qquad R(z)=\prod_{k=1}^{k=N}(z-\alpha_k)(z-\beta_k)$$

and

$$(3-6)\qquad -\varkappa\tilde{\varphi}_0(z)-\tilde{\chi}_0(z)=\frac{1}{2\pi i}\int\limits_{\cup[\alpha_k,\beta_k]}\frac{[g_0^+(x)-g_0^-(x)]}{x-z}\,dx$$

where the necessary and sufficient conditions for existing such a solution of (3—4), bounded at $z=\infty$ are [3]

$$(3-7)\qquad \int\limits_{\cup[\alpha_k,\beta_k]}\frac{x^k[g_0^+(x)+g_0^-(x)]}{\sqrt{R(x)}}\,dx=0,\quad 0\leqslant k\leqslant N-2.$$

These conditions are sufficient, because the eventual constant $\tilde{\varphi}_0(\infty)$ we can add to the constant K_1. So, from (3—7) and (2—4) we get a system of $2N$ linear equations for the $2N$ real unknowns Re $[A_k]$ and Im $[A_k]$, $1\leqq k\leqq N$. If this system takes an usually ordered form, then it is easy to establish that the structure of the left sides of these equations do not change if we take on the right sides $A_0=\Gamma_1=\Gamma_2=0$ and $g(x)=0$, $x\in C$. From (2—5) it follows now that such a homogenous system has only the trivial solution $A_1=A_2=\ldots=A\ =0$. But this is possible only in the case of Cramer's system. Hence, the system (3—7) and (2—4) is solvable uniquely for arbitrary given right sides.

4. The bending problem for the thin and moderately thick plate can be similarly treated [4]. The bending of the middle plane is with a nonessential supposition goverened by a biharmonic function $w=w(x,y)$, and it can be expressed in the known form

$$(4-1)\qquad w=\bar{z}\,\varphi(z)+z\,\overline{\varphi(z)}+\omega(z)+\overline{\omega(z)}$$

$$(4-2)\qquad \omega'(z)=\psi(z)$$

and

$$(4-3)\qquad G[\varphi(z),\ \psi(z)]=\varphi(z)+z\,\overline{\varphi'(z)}+\overline{\psi(z)}=\frac{1}{2}\left[\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right].$$

For a multiply connected domain D the functions $\varphi(z)$ and $\psi(z)$ are multivalued. It is possible to express them in the case of a infinite plate cut along the intervals $[\alpha_k,\beta_k]$ on the x-axis, in the form

$$(4-4)\qquad \varphi(z)=\sum_{k=1}^{k=N}(A_k z+B_k)\int\limits_{s_k}^{\sigma_k}\ln(z-\zeta)\,ds+\Gamma_1 z+K_1+\tilde{\varphi}_0(z)$$

$$(4-5)\qquad \psi(z)=\sum_{k=1}^{k=N}\bar{B}_k\int\limits_{s_k}^{\sigma_k}\ln(z-\zeta)\,ds+\Gamma_2 z+K_2+\tilde{\psi}(z)$$

where

$$\overline{A}_k = A_k, \quad \overline{\Gamma}_1 = \Gamma_1 \tag{4—6}$$

$$\tilde{\varphi}_0(z) = O(z^{-1}), \quad \tilde{\psi}_0(z) = O(z^{-1}) \tag{4—7}$$

The constants A_k and B_k have here an analogous meaning as they have in the classical expressions of the functions $\varphi(z)$ and $\psi(z)$.

Let us introduce the notations

$$A_1 + A_2 + \cdots + A_N = A_0 \tag{4—8}$$

$$B_1 + B_2 + \cdots + B_N = B_0 \tag{4—8'}$$

So we can define the basic boundary-value problem of bending plates as follows: we seek a biharmonic function $w = w(x, y)$, uniform in the domain D, which corresponds on the boundary curves C to the conditions

$$w(x, y) = f_1(x, y), \frac{\partial w}{\partial n}(x, y) = f_2(x, y), \quad T(x, y) \in C. \tag{4—9}$$

Where f_1 and f_2 are two arbitrary functions on C, which are smooth enough.

With regard to (4—1) and (4—8'), we can define this problem in the form

$$G[\varphi(z), \psi(z)] = g(z), \quad z \in C \tag{4—10}$$

where $g(z)$ is a known function on C, which we can compose of f_1 and f_2. $\varphi(z)$ and $\psi(z)$ are of the form (4—4) and (4—5), where A_0, B_0, Γ_1 and Γ_2 are arbitrary given constants, restricted only by $\overline{A}_0 = A_0$ and $\overline{\Gamma}_1 = \Gamma_1$. But with regard to (4—9) we have to add the conditions

$$w(\alpha_{k+1}) - w(\alpha_k) = f_1(\alpha_{k+1}) - f_1(\alpha_k) \qquad 1 < k \leqslant N - 1 \tag{4—10'}$$

The uniqueness theorem of our boundary-value problem states:
If we take:

(4—11)

1. $A_0 = B_0 = \Gamma_1 = \Gamma_2 = 0$
2. $f_1(x, y) = f_2(x, y) = 0, \quad T(x, y) \in C$

then it follows

$$\varphi(z) = K_1, \qquad \psi(z) = K_2 = -\overline{K}_1, \quad z \in D \cup C \tag{4—12}$$

and from the above equations we get

$$A_k = B_k = 0, \quad 1 < k < N, \qquad \tilde{\varphi}_0(z) = \tilde{\psi}_0(z) = 0, \quad z \in D \cup C \tag{4—13}$$

If we proceed in this case analogously as at (3—1) we get similarity equations as at (3—4), and the solution is similar at (3—5) and (3—5'). The solving conditions in the class

$$h[\alpha_k, \beta_k, 1 \leqslant k \leqslant N] \text{ are } [3]$$

(4—14)
$$\int\limits_{\cup[\alpha_k, \beta_k]} \frac{x^k [g_0^+(x) + g_0^-(x)]}{\sqrt{R(x)}} dx = 0, \quad 0 \leqslant k \leqslant N-2$$

which together with (4—8), (4—8') and (4—10') give a sistem of $3N$ linear equations for the $3N$ real unknowns A_k, Re $[B_k]$ and Im $[B_k]$, $1 \leqq k \leqq N$. Like in the previous problem we establish that this system is always uniquely solvable. The solution $w_1(x, y)$ found in this way can differ from the requested one $w(x, y)$ for a constant γ. This constant we can find at once:

(4—15)
$$\gamma = w(x_0, y_0) - w_1(x_0, y_0)$$

where $T(x_0, y_0)$ is an arbitrary point on C, where $w(x, y)$ is known. Finally we get

(4—16)
$$w(x, y) = w_1(x, y) + \gamma$$

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EINIGE BEMERKUNGEN ZUR LÖSUNG DER ELASTOSTATISCHEN RANDWERTPROBLEME IN DER HALBEBENE MIT SCHNITTEN AN EINER GERADE

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Zusammenfassung

In diesem Aufastz werden einige Randwertprobleme in der Halbebene behandelt, die in ein Hilbert-Riemannsches Problem überführt werden können. Im Unterschied zu diesem klassischen Beitritt, wo die Gleichung (1—2) mit den Funktionen $\Phi(z)$ und $\Psi(z)$ behandelt wird, bleibt man hier bei Funktionen $\varphi(z)$ und $\psi(z)$. Deswegen müssen wir eine solche Lösung des Hilbert-Reimannschen Problems suchen, die in den Endpunkten der Intervale $[\alpha_k, \beta_k]$ endlich bleibt und in $z=\infty$ beschränkt ist.

PRILOG REŠAVANJA PROBLEMA GRANIČNIH VREDNOSTI ELASTIČNE RAVNI SA PROREZIMA NA JEDNOJ PRAVI

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Rezime

U ovom radu tretirani su neki problemi graničnih vrednosti elastične ravni, koje možemo svesti na Hilbert-Riemannov problem. U klasičnom postupku radi se na određivanju dvaju funkcija $\Phi(z)$ i $\Psi(z)$ dok je u ovom radu prikazan postupak kad ostajemo kod određivanja prvobitnih funkcija $\varphi(z)$ i $\psi(z)$, za koje tražimo takvo rešenje, koje je u međama intervala $[\alpha_k, \beta_k]$ konačno i u tački $z=\infty$ ograničeno.

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