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ON CONSERVATION LAWS IN ELASTOSTATICS

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1. Introduction

In paper [1] some classes of conservation laws in linearized and finite elastostatics are derived using Noether's theorem. The same paper shows that the respective coordinate and vector transformations are unique in case of linearized elastostatic states. The proof of the uniqueness is derived under the assumption that Lame constant $\lambda=0$, which is a special case that narrows the generality of above mentioned transformations even more, and leads to the suspicion that there are some other transformations to which some new conservation laws correspond.

In this paper is shown that the derived coordinate and vector transformations in [1] are unique under more general assumptions which aıe physically justified. The proof itself is a very simple and clear one and is different from that derived in [1].

2. Conservation laws for infitesimal deformations of elastic bodies

For sake of a better understanding of this paper we give the essential features of the paper [1]. Moreover, we shall strictly use the symbols and terms of the same paper. Theorems used in this paper will be cited and noted in the same way as in [1] for sake of a better orientation of the reader. The theorems are given without proofs and the reader is referred to the original paper [1]. The symbols ε and V denote respectively a threedimensional euclidean space and its associated vector space.

Letters in boldface denote tensors of positive order and matrices. Tensor components and matrix elements are defined in relation to the fixed system of cartesian coordinates. In what follows we shall deal with functions that depend on the cartesian coordinates, the components of a vector and the components of a second-order tensor.

Partial derivatives of some function $H(\mathbf{x}, \mathbf{w}, \mathbf{t}, \eta)$, $H\in C^2$ for all $(\mathbf{x}, \mathbf{w}, \mathbf{t}, \eta)\in \in\varepsilon\times V\times\tau\times L$, where $\mathbf{x}$ is position vector of point at ε, $\mathbf{w}$ vector in V and $\mathbf{t}$ tensor of second-order in space τ of second order tensors, and η a scalar parameter in L, will be denoted as:

$$F_{,i}(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)=\frac{\partial}{\partial x_i}F(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)$$

$$(1)\qquad F_{,v_i}(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)=\frac{\partial}{\partial v_i}F(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)$$

$$F_{,t_{ij}}(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)=\frac{\partial}{\partial t_{ij}}F(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)$$

$$F'(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)=\frac{\partial}{\partial \eta}F(\mathbf{x}, \mathbf{v}, \mathbf{t}; \eta)$$

Let functional Φ be defined on Ω, which is any regular subregion of open region D in ε, with

$$\Phi\{\mathbf{w}\}=\int_{\Omega} H(\mathbf{x},\ \mathbf{w}(\mathbf{x}),\ \nabla\mathbf{w}(\mathbf{x}))\,dv \tag{2}$$

for every vector field $\mathbf{w}\in C^2(D)$, where $\nabla\mathbf{w}\equiv\operatorname{grad}\mathbf{w}$. So defined functional will be named admissible functional for D generated by H. Let us introduce one-parameter family of functional Φ_η defined with

$$\Phi_\eta\{\mathbf{w}\}=\int_{\Omega_\xi} H(\xi,\ \psi(\xi;\ \eta),\ \nabla_\xi\mathbf{w}(\xi;\ \eta))\,dv_\xi, \tag{3}$$

where

$$\xi=\mathbf{f}(\mathbf{x},\ \eta)\ \text{ for every }\ (\mathbf{x},\ \eta)\in D\times L,\ \ L=(-\eta_0,\ \eta) \tag{4}$$

is regular family of coordinate mapping on D, and

$$\psi(\xi,\ \eta)=\mathbf{h}(\mathbf{w}(\mathbf{f}^{-1}(\xi,\ \eta));\ \eta) \tag{5}$$

so that $\mathbf{h}\in C^2(V\times L)$ $L=(-\eta_0,\ \eta)$ $\mathbf{h}(\mathbf{v},\ 0)=\mathbf{v}$ for all $\mathbf{v}\in V$.
According to the preceding notation it can be concluded at once that $\nabla\psi(\xi,\eta)=$
$=\operatorname{grad}_\xi\psi$
It is easy to show that

$$\Phi_\eta\{\mathbf{w}\}_{\eta=0}=\Phi\{\mathbf{w}\}\qquad\text{for all }\ \mathbf{w}\in C^2(D), \tag{6}$$

i. e. Φ is member of family Φ_η in the case when $\eta=0$.
We shall say that Φ is invariant at $\mathbf{w}$ with respect to $\mathbf{f}$ and $\mathbf{h}$ whenever

$$\Phi_\eta\{\mathbf{w}\}=\Phi\{\mathbf{w}\}\quad(-\eta_0<\eta<\eta_0)\qquad\text{for every regular }\ \Omega\in D \tag{7}$$

and that Φ is infitesimally invariant at $\mathbf{w}$ with respect to the given pair of mapping families, provided that

$$\Phi_0\{\mathbf{w}\}=\frac{\partial}{\partial\eta}\Phi_\eta\{\mathbf{w}\}_{\eta=0}=0\qquad\text{for every regular }\ \Omega\in D. \tag{8}$$

On basis of so introduced notions it is possible to state a restricted version of Noether's theorem.

T h e o r e m 2.1.[1]. *Let D be a domain in ε and let Φ be an admissible functional for D generated by H. Let* **f** *be a regular family of coordinate mappings on D and* **h** *a regular family of vector transformations. Suppose* **w** *is a vector field satisfying the Euler equation*

$$H,_{wk}(\mathbf{X})-\frac{\partial}{\partial x^j}H,_{wk,j}(\mathbf{X})=0\qquad\textit{for all }\ \mathbf{x}\in D, \tag{9}$$

where

$$\mathbf{X}=(\mathbf{x},\ \mathbf{w}(\mathbf{x}),\ \nabla\mathbf{w}(\mathbf{x})). \tag{10}$$

Then Φ is infinitesimally invariant at **w** *with respect to* **f** *and* **h** *if and only if* **w** *satisfies*

$$\frac{\partial}{\partial x^j}[H(\mathbf{X})\,a_j(\mathbf{x})+W_k(\mathbf{x})\,H,_{wk,j}(\mathbf{X})]=0\qquad\textit{for all }\ \mathbf{x}\in D \tag{11}$$

where

$$a_j(\mathbf{x}) = f_j'(\mathbf{x}; 0) \qquad \textit{for all } \mathbf{x} \in D,$$

(12) $$b_j(\mathbf{x}) = h_j'(\mathbf{v}; 0) \qquad \textit{for all } \mathbf{v} \in V,$$

$$m_k(\mathbf{x}) = b_k(\mathbf{w}(\mathbf{x})) - a_j(\mathbf{x})\, \mathbf{w}_{k,j}(\mathbf{x}) \qquad \textit{for all } \mathbf{x} \in D.$$

Moreover, (11) *is equivalent to the conservation law in integral form*

(13) $$\int_S [H(\mathbf{X})\, a_j(\mathbf{x}) + w_k(\mathbf{x})\, H,_{Wk,j}(\mathbf{X})] \cdot n_j(\mathbf{x})\, da = 0$$

for every surface S that is the boundary of a regular subregion of D, provided **n** *is the outward unit normal vector of S.*

This theorem is fundamental for deriving of conservation laws for infinitesimal deformations and finite deformations of elastic bodies. We shall further pay attention only to infinitesimal deformations of elastic bodies giving only the quantities which are essential. In this case

(14) $$\gamma_{ij} = U_{(i,j)},\ \sigma_{ij}$$

will denote the infinitesimal deformation tensor and stress tensor respectively, where **u** is displacement vector. Balance equations are given with relations

(15) $$\sigma_{ij,j} = 0,\ \sigma_{ij} = \sigma_{ji} \qquad \text{on } D.$$

The system of partial differential equations (15) is complete when the stress tensor is replaced by constitutive relations

(16) $$\sigma_{ij} = \Gamma,_{\gamma ij}(\gamma) \qquad \text{on } D,\ \Gamma,_{\gamma ij}(0) = 0,$$

where $\Gamma_{(\gamma)}$ is the elastic potentional. On the basis of elastic potentional we define the density of energy deformation $W(\mathbf{x})$ with

(17) $$\Gamma(\gamma(\mathbf{x})) = W(\mathbf{x})$$

In case that the constitutive relations (16) are linear we get the classical linear elasticity theory in which

(18) $$\sigma_{ij} = C_{ijkl}\gamma_{kl} \qquad \text{on } D,\ C_{ijkl} = C_{jikl} + C_{klij}$$

(19) $$\Gamma(\gamma) = \frac{1}{2} C_{ijkl}\gamma_{ij}\gamma_{kl}$$

Tensor *Cijkl* is called elasticity tensor. In case the material is isotropic, the tensor *Cijkl* becomes an isotropic tensor of the fourth order, so that it can be written in the form

(20) $$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Applying the Theorem 2.1 to function

(21) $$H(\mathbf{x}, \mathbf{U}, \nabla\mathbf{U}) = \Gamma(\gamma)$$

we can prove the

Theorem 3.1 [1]: *Let D be a domain in* ε *and let* $[\mathbf{u}, \gamma, \delta]$ *be an elastic state with infinitesimal deformations on D, corresponding to the elastic potential* Γ. *Then, for every surface S, with the outward unit normal vector* $\mathbf{n}$, *that is the boundary of a regular subregion of D,*

$$\int_S (W n_j - S_j U_{j,i})\, da = 0 \tag{22}$$

where $\mathbf{s}$ *is the traction vector on S, i.e.,* $S_i = \sigma_{ij} n_j$ *on S.*
In case the body is isotropic it follows that

$$\int_S \varepsilon_{ijk} (W x_k n_j) + S_j U_k - S_p U_{p,j} X_k)\, da = 0 \tag{23}$$

Finally, if the deformation state of an isotropic body is linear

$$\int_S \left(W x_i n_j - S_j U_{j,i} x_i - \frac{1}{2} S_i U_i \right) da = 0 \tag{24}$$

is valid.

We shall not give here the proofs of these theorems, but paper [1] showed that the three conservation laws of Theorem 3.1 [1] can be derived systematically by means of Noether's theorem (Theorem 2.1) on invariant variation principles. The same paper shows that each of integrals exists with respect to the special pair of mapping family. It is logical then to ask whether there exist other mappings too, and if so there would exist some other conservation laws independent of these derived in Theorem 3.1 [1].

In other words it is the question of the uniqueness of the integrals given in Theorem 3.1, or the question of existence of only three pairs of mapping to which the above mentioned integrals correspond. The proof of the completenes of theorem (3.1) is given in paper [1] and for linear elastic states is given by

Theorem 3.2 [1]. *Let* $\mathbf{f}$ *be a regular family of coordinate mappings on* ε *any let* $\mathbf{h}$ *be a regular family of vector transformations. Let D be a domain in* ε *and assume C is an invertible fourth-order tensor satisfying the symmetry relations*

$$C_{ijkl} = C_{jikl} = C_{klij}. \tag{18$_2$}$$

Suppose

$$H(\mathbf{x}, \mathbf{U}, \nabla \mathbf{U}) = \Gamma(\gamma) \qquad \textit{for each } (\mathbf{x}, \mathbf{U}, \nabla \mathbf{U}) \in \varepsilon_x V_x \varphi, \tag{21}$$

where

$$\Gamma(\gamma) = \frac{1}{2} C_{ijkl} \gamma_{ij} \gamma_{kl} \qquad \textit{for each } \mathbf{U} \in V \tag{19}$$

and let Φ *be the admissible functional for D generated by H. Suppose, further,* Φ *is infinitesimally invariant at u with respect to* $\mathbf{f}$ *and* $\mathbf{h}$ *for every linear elastic state on D corresponding to the elasticity tensor C. Then* $\mathbf{f}$ *and* $\mathbf{h}$ *must satisfy relations*

$$\mathbf{f}'(\mathbf{x}; 0) \equiv \mathbf{a}(\mathbf{x}) = \alpha + \mathbf{\Phi}\mathbf{x} + k\mathbf{x} \qquad \textit{for each } \mathbf{x} \in \varepsilon$$

$$\mathbf{h}'(\mathbf{u}; 0) \equiv \mathbf{b}(\mathbf{u}) = \beta + \mathbf{\Phi}\mathbf{u} - \frac{k}{2}\mathbf{u} \qquad \textit{for each } \mathbf{u} \in V$$

where k is scalar constant, α and β are vectorial constants, while Φ is a skew-symmetric second-order tensor. Moreover, $\Phi=0$ if C is not necessarily isotropic.

Proof of Theorem 3.2 is derived with great restrictions and from the physical point of view unjustified. The displacement field is supposed to be of the form

$$u_i(x)=\rho_{ik}\,x_k \tag{25}$$

where ρ is an arbitrary constant 3×3 matrix. In this case the infinitesimal deformation tensor is an arbitrary matrix of constant elements given by

$$\gamma_{ij}(x)=\rho_{(ij)} \tag{26}$$

For every inveıtible isotropic fourth order tensor C_{ijbl} given by (20) it is supposed that

$$\lambda=0. \tag{27}$$

Since λ and μ in (20) are Lame's constants and since they have physical meaning it follows that for elastic mateıials their values are always positive, i.e, λ and $\mu>0$ ([2] p. 69,70). According to this, restıictions introduced in paper [1], with the assumptions (25) and (27), make the completeness of theorem (3.1) questionable. We shall further prove the completeness of theorem (3.1) under most general assumptions in the way which is essentially different from the one given in [1]. Better to say, we prove theorem (3.2) with assumptions which are only physically justified.

Let H in Theorem 2.1 be a function defined with (18), (19), (20) on D, with corresponding elasticity tensor (20). The vector field $\mathbf{u}$ satisfies the Euleı equations (9) on D with respect to (10), (14), (15) and (19). Further, since, by hypothesis, the admissible functional for D generated by H is infinitesimally invariant at $\mathbf{u}$ with respect to $\mathbf{f}$ and $\mathbf{h}$, (11) must hold, provided $\mathbf{w}$ in (12) is replaced by $\mathbf{u}$.

The equations (11) then read

$$\frac{\partial}{\partial x^j}\left[\Gamma(\gamma(\mathbf{x}))\,a_j(\mathbf{x})+m_k(\mathbf{x})\frac{\partial\Gamma(\gamma)}{\partial u_{k,j}}\right]=0 \tag{28}$$

or, in more developed form,

$$\frac{\partial\Gamma}{\partial x^j}a_j+\Gamma a_{j,j}+m_{k,j}\frac{\partial\Gamma}{\partial u_{k,j}}+m_k\frac{\partial^2\Gamma}{\partial u_{k,j}\,\partial x_j}=0 \tag{29}$$

If we consider that

$$\frac{\partial\Gamma}{\partial x^j}=C_{pqkl}\gamma_{kl,j}\gamma_{pq},$$

$$\frac{\partial\Gamma}{\partial u_{k,j}}=C_{mnkj}\gamma_{mn,j},\qquad \frac{\partial^2\Gamma}{\partial u_{k,j}\,\partial x^j}=C_{mnkj}\gamma_{mn,j},$$

because of (19), then (29) gives

$$m_p\,C_{pqkj}\gamma_{kj,q}+C_{pqkj}\gamma_{pq}\left(\frac{1}{2}\gamma_{kl}a_{l,l}+b_{k,ul}u_{l,j}-a_{l,j}u_{k,l}\right)=0 \tag{30}$$

In (30) it is easy to show that

$$C_{pqkj}\gamma_{kj,q}=\sigma_{pq,q}=0 \tag{31}$$

so that (30) takes the form

$$C_{ijkl}\gamma_{kl}\left[\frac{1}{2}\gamma_{ij}a_{n,n}+b_{i,un}u_{n,j}-a_{n,j}u_{i,n}\right]=0 \tag{32}$$

If in (32) C_{ijkl} is replaced by (20) and we assume that (32) must be valid for arbitrary values $\lambda>0$ and $\mu>0$, which physically means that the relations (32) are valid for all elastic materials, then (32) is divided into two sets of equations

$$\gamma_{kk}\left[\frac{1}{2}\gamma_{jj}a_{n,n}+b_{j,un}u_{n,j}-a_{n,j}u_{j,n}\right]=0 \tag{33}$$

$$\gamma_{ij}\left[\frac{1}{2}\gamma_{ij}a_{n,n}+b_{i,un}u_{n,j}-a_{n,j}u_{i,n}\right]=0. \tag{34}$$

Further, we assume that the displacement fields are arbitrary in the elasticity domain. We require that the relations (33) and (34) are valid with respect to all displacement fields in this domain. This assumption is essential in our paper and it is far more general than the assumption introduced for the displacement fields in [1] and expressed with (25).

With this assumption, differentiating (33) and (34) with respect to displacement gradients u_{ij}, we get the following systems of equations

$$\delta_{pq}[\delta_{mu}a_{n,n}-a_{n,m}+b_{n,um}]+\delta_{mu}[b_{q,up}]=0 \tag{35}$$

$$[\delta_{pm}\delta_{nq}a_{n,n}+\delta_{qn}b_{m,}u_p-a_{q,n}\delta_{mp}]_{(m,n)}+[\delta_{uq}b_{p,}u_m-a_{u,q}\delta_{pm}]_{(p,q)}=0 \tag{36}$$

From (35) the symmetric and the skew-symmetric parts give

$$\delta_{pq}[\delta_{mu}a_{n,n}-a_{n,m}+b_{n,um}]+\delta_{mu}[b_{q,up}-a_{p,q}]_{(p,q)}=0 \tag{37}$$

$$\delta_{mu}[a_{q,p}-b_{q,up}]_{[p,p]}=0. \tag{38}$$

With regards to the value of Kronecker δ-symbols, (38) reads

$$a_{[p,q]}=b_{[p,uq]}=\Phi_{p,q} \tag{39}$$

where

$$\Phi_{pq}=-\Phi_{qp}=\text{const.} \tag{40}$$

This conclusion follows from (39) and the arbitrarity of the displacement field, so that a_p are functions of $\mathbf{x}$ and b_p are functions of $\mathbf{u}$, where $\mathbf{x}$ and $\mathbf{u}$ are two arbitrary systems of values.

If in (37) we perform contraction with respect to p and q and then to m and n, we get

$$a_{n,n}+2b_{n,un}=0. \tag{41}$$

Further, if in (37) we perform contraction with respect to m and n and use (41), we get

$$a_{(p,q)}-b_{(p,uq)}=\frac{1}{2}\delta_{pq}a_{n,n} \tag{42}$$

Using (39) and (36) it is easy to show that (36) can be written in the form

$$\begin{aligned}&(\delta_{pm}\delta_{nq}+\delta_{pn}\delta_{mq})a_{n,n}+2\delta_{nq}b_{(p,um)}-2\delta_{pm}a_{n,q}+\\&\delta_{nq}[b_{(q,um)}-a_{(q,m)}]+\sigma_{qm}[b_{(n,up)}-a_{(n,p)}]=0\end{aligned} \tag{43}$$

This expression is very much simplified by means of (42) and becomes

$$\delta_{pm}\,\delta_{nq}\,a_{n,\,n}+2\,\delta_{nq}\,b_{(p,\,um)}-2\,\delta_{pm}\,a_{(n,\,q)}=0 \tag{44}$$

Contracting the indeces n and q in (44) we get

$$b_{(p,\,um)}=-\frac{1}{6}\,\delta_{pm}\,a_{n,\,n}=\text{constant}, \tag{45}$$

since b_p (and so is $b_{(p\,,um)}$) are functions of u_m only and a_r (and so is a_{ss}) are functions of $\mathbf{x}$ only.
If we replace this in (44) we get

$$a_{(n,\,q)}=\frac{1}{3}\,\sigma_{nq}\,a_{n,\,n} \tag{46}$$

Introducing the notation

$$\frac{1}{3}\,a_{n,\,n}=k=\text{constant} \tag{47}$$

then (45) and (46) can be written as

$$a_{(p,\,q)}=k\,\delta_{pq},\quad b_{(p,\,uq)}=-\frac{k}{2}\,\sigma_{pq}, \tag{48}$$

According to (39) and well known statement that every second-order system can be uniqly represented as a sum of its symmetric and skewsymmetric parts, we get

$$\begin{aligned} a_{p,\,q}&=k\,\delta_{pq}+\Phi_{pq}\\ b_{p,\,uq}&=-\frac{k}{2}\,\delta_{pq}+\Phi_{pq} \end{aligned} \tag{49}$$

which was to be proved.

In the case when the tensor C_{ijkl} is anisotropic it is easy to show that $\Phi_{ij}=0$.

For this purpose let us suppose that C_{ijkl} is a material tensor which corresponds to materials which have a cubic symmetry. Then it can be written as [3]

$$C_{ijkl}=\lambda\delta_{ij}\,\delta_{kl}+\mu\,(\delta_{ik}+\delta_{jl}\,\delta_{jl}\,\delta_{ki})+\alpha\delta_{i\underline{m}}\,\delta_{j\underline{m}}\,\delta_{k\underline{m}}\,\delta_{l\underline{m}} \tag{50}$$

where we did not use summation convention to the underlined indeces. The expression (32) in this case give systems (35) and (36) along with the coefficients λ and μ, and with α a new set of equations

$$\gamma_{mn}\left[\frac{1}{2}\,\gamma_{mn}\,a_{n,\,n}+b_{m,\,un}\,u_{n,\,m}-a_{n,\,m}\,u_{m,\,n}\right]=0. \tag{51}$$

Replacing the solutions of the systems (35) and (36), which are given by (39), in (51) we get

$$\gamma_{mm}\,\gamma_{mn}\,\Phi_{mn}=0 \tag{52}$$

Because of the arbitrarity and the existence of the displacement field from (52) it follows that

$$\Phi_{mn}=0 \tag{53}$$

Thus we complete the proof of theorem 3.2.

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ОБ ЗАКОНАХ СОХРАНЕНИЯ В ЭЛАСТОСТАТИКЕ

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Резюме

В работе К. Кнолса и Э. Стернберга [1] некоторые класси законов сохранения в линейной эластостатике полученни помощью Теоремы Нетера.

В этой работе показано что преобразование координат и векторов однозначно в линейной эластостатике. Доказательство было получено при предположении что постояная Ламе, λ, ровна нуль. Это предположение не имеет физической основы.

Здесь мы показали что преобразования координат и векторов в работе [1] можно получить при более общим предположениям которые имеют физическую основу. Доказательство этого утверждения очень простое за разницу от доказательства в работе [1].

О ЗАКОНИМА КОНЗЕРВАЦИЈЕ У ЕЛАСТОСТАТИЦИ

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У раду Ј. К. Кнолса и Е. Стернберг изводе се неке класе закона конзервација у линеарној и коначној еластостатици користећи Нетерову теорему. У истом раду је показано да су одговарајуће координатне и векторске трансформације јединствене у случају линеарних еластостатичких стања. Доказ јединствености изведен је под претпоставком да је Ламеова константа $\lambda=0$, што, и у том уском специјалном случају, још више сужава општост горе наведених трансформација и доводи до сумње да постоје и друге трансформације којима одговарају нови закони конзервације. У овом раду се показује да су изведене координатне и векторске трансформације у споменутом раду јединствене и то под општијим претпоставкама које су физички оправдане. Сам доказ је једноставнији, прегледан и може да послужи као метод за доказивање (не) јединствености трансформација у случају нелинеарне еластичности.