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THERMOELASTIC MATERIALS WITH MICROSTRUCTURE

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Abstract

We consider, in this paper, a model of thermoelastic material with microstructure, under the assumption of nonuniformity of temperature distribution in the domain of macroelement. We obtain the nonlinear constutive equations for all mechanical and thermodynamic constitutive variables, which are linearized in the case of an isotropic material.

1. Introduction

It is supposed in the theory of a continuum with microstructure, that the body is composed of particles, i.e., the macroelements of the body, which can be deformed independetly of the displacements of their mass centres.

In the theory of a thermoelastic continuum with microstucture, formulated in papers [1] and [2], it is supposed that a nonuniform temperature distribution exists in the domain of a macroelement, so that we must take into consideration the influence of the temperature and microtemperature on the deformation, as well as the influence of the gradients of deformation and microdeformation.

Wozniak [1] takes into consideration the influence of the gradients of microdeformation and microtemperature of n-oder, in fact, he assumes that the macroelement are not deformed homogeneously. Theormodynamic limitations are given in the first and second principle of thermodynamics, with the assumpton that the material is perfectly elastic and that the thermodynamic processes at each point of a macroelement of the body are locally reversible. Microtemperature have been considered by Wozniak who assumes that it is known function of the temperature and temperature gradients. Howerer, equation determining the fields of temprature and microtemperature is not obtained.

R. A. Grot [2] considered, at the difference of [1], homogeneous deformation of the macroelement, in fact, he observes the influence of the deformation gradient of first order, and supposes for the temperature that it is a linear function of microcoordinates (coordinates of some point of the macroelement with respect to its mass centre). The second law of thermodynamics is modified as to include microtemperature. After the formulation of a constitutive theory of thermoelastic material with microstructure, the first moment energy equations yield conduction for the microtemperature.

We consider the same model of thermoelastic material as in [2], in fact, we suppose that the deformation of a macroelement is homogeneous and that the mass of the macroelement remains unchanged during deformation. Further, we assume that the material of the body with microstructure is perfectly elastic and that the thermodynamic processes are locally reversible. The first and second law of thermodynamic are modified as to include microtemperature. The second law of thermodynamic is given to the dissipation function of system.

We shall obtain the nonlinear constitutive relations for stresses and, at the difference of [2], obtain the nonlinear constitutive relations for heat fluxes using Ziegler's principle of least irreversible force.

2. Kinematics

The equations of motion are

$$\chi^k = \chi^k (X^K, t), \quad \chi^{k}_{\cdot K} = \chi^{k}_{\cdot K} (X^K, t) \tag{2.1}$$

where $x^k (X^K, t)$ determines the motion of the mass centre of the macroelement. The microdeformation gradients $\chi^{k}_{\cdot K}$ determine the deformation of the macroelement and they are independent of the motion of the mass centre. We have from [4]

$$\chi^{k}_{\cdot K} \cdot \chi^{K}_{\cdot l} = \delta^k_l, \quad \chi^{k}_{\cdot K} \cdot \chi^{L}_{\cdot K} = \delta^L_K \tag{2.2}$$

As we have seen in [3], the microdeformation gradients are completely determined by deformation of three noncomplanar vectors, the so called directors, in fact

$$\chi^{k}_{\cdot K} = d^{k}_{\cdot (\alpha)} D^{(\alpha)}_{\cdot K}, \quad d^{k}_{\cdot (\alpha)} = \chi^{k}_{\cdot K} D^{k}_{\cdot (\alpha)} \tag{2.3}$$

The velocity of a point of the macroelement is

$$v'^k = v^k + b^{k}_{\cdot l} d^l \tag{2.4}$$

where v^k is the velocity of the mass centre of the macroelement, and d^l is the vector of the position of some point of the macroelement with respect to its mass centre. The quantites $b^{k}_{\cdot l}$ are given in the form

$$b^{k}_{\cdot l} = \dot{\chi}^{k}_{\cdot K} \chi^{K}_{\cdot l} = \dot{d}^{k}_{\cdot (\alpha)} d^{(\alpha)}_{\cdot l} \tag{2.5}$$

3. Temperature field

Let $\vartheta (x^k, d^k, t)$ be some continuous and differentiable function which determines, in the moment t, the temperature of some point of the macroelement. Then, it can be given in the following way

$$\vartheta (x^k, d^k, t) = \theta (x^k, t) + \mu (x^k, d^k, t) \tag{3.1}$$

where $\theta (x^k, t)$ is the temperature of the mass centre, and $\mu (x^k, d^k, t)$ is microtemperature. It means, that we must assume that we have a nonuniform distribution of temperature in the domain of the macroelement. Further, we must assume that the microtemperature is continuous and differentiable function of its variables, so that we can expand them series, neighbourhood of point $d^k=0$. In the linear approximation we have

$$\mu=\theta_k d^k \tag{3.2}$$

where

$$\theta_k(x^k, t)=\left(\frac{\partial\mu}{\partial d^k}\right)_{d^k=0} \tag{3.3}$$

denotes the gradient of microtemperature determined at the mass centre of macroelement. The relation (3.1) could be written in the following form

$$\vartheta=\theta(x^k, t)+\theta_k(x^k, t)\,d^k \tag{3.4}$$

wherefrom we can see that the temperature is determined by equations

$$\theta=\theta(x^k, t), \quad \theta_k=\theta_k(x^k, t) \tag{3.5}$$

The density of the body for unit mass is $\varphi(x^k, t)$, the surface density is $\psi_{(\mathbf{n})}(x^k, t)$, which are scalar or vector fields, determined in the region v, respectively on the surface s with the normal unit vector $\mathbf{n}$. They are defined in the following way [1],

$$\rho\varphi(x^k, t)\,dv=\int_{dv}\rho'\varphi'(x^k, d^k, t)\,dv', \quad \psi_{(\mathbf{n})}=\int_{ds}\frac{\psi'_{(n')}(x^k, d^k, t)}{\Delta s(\mathbf{n}, \mathbf{n}')}\,ds' \tag{3.6}$$

4. The balance of energy. Equations of motion

The Law of the balance of energy for a thermoelastic material with microstructure has the form

$$\int_v \rho(\dot{\varepsilon}+\dot{v}^i v_i+\Gamma^{ij}b_{ij})\,dv=\oint_s(T^i v_i+H^{ij}b_{ij}+q_{(\mathbf{n})})\,ds+\int_v\rho(f^i v_i+l^{ij}b_{ij}+h)\,dv \tag{4.1}$$

where, according to (3.6)

$$\rho\varepsilon\,dv=\int_{dv}\rho'\varepsilon'\,dv' \quad q_{(\mathbf{n})}=\int_{ds}\frac{q'_{(\mathbf{n}')}}{\Delta s}\,ds', \quad \rho h\,dv=\int_{dv}\rho' h'\,dv'. \tag{4.2}$$

ρ', ε' and h' are the density of mass, the density of internal energy and the density of heat sources in a point of the macroelement. $q'_{(\mathbf{n}')}$ is the heat flux through the surface ds'. Γ^{ij} is the internal spin, T^i is the surface force or stress, H^{ij} is the first surface moment, f^i is the body force and l^{ij} is the first body moment.

From relation (4.1), performing the identical transformation, and using the theorem of tetrahedron, we obtain boundary conditions in the form

$$(4.3) \qquad T^i = t^{ik} n_k, \quad H^{ij} = h^{ijk} n_k, \quad q_{(n)} = q^k n_k.$$

t^{ik} being the nonsummetric stress tensor, h^{ijk} the first surface stress moment.

Relation (4.1) can be written in the form

$$(4.4) \qquad \int_v \rho (\dot{\varepsilon} + \dot{v}^i \dot{x}_i + \Gamma^{i(\alpha)} \dot{d}_{i(\alpha)})\, dv = \int_v \rho (f^i \dot{x}_i + l^{i(\alpha)} \dot{d}_{i(\alpha)} + h)\, dv + $$

$$+ \int_v (t^{ij}{}_{,j} \dot{x}_i + t^{ij} \dot{x}_{i,j} + h^{i(\alpha)k}{}_{,k} \dot{d}_{i(\alpha)} + h^{i(\alpha)k} \dot{d}_{i(\alpha),k} + q^k{}_{,k})\, dv$$

where

$$(4.5) \qquad \Gamma^{i(\alpha)} = \Gamma^{ij} d^{(\alpha)}_{\cdot j}, \quad l^{i(\alpha)} = l^{ij} d^{(\alpha)}_{\cdot j}, \quad h^{i(\alpha)k} = h^{ijk} d^{(\alpha)}_{\cdot j}.$$

From the condition of invariance of internal energy, as well as from quantitis $q^k{}_{,k}$ and h, with respect to superposed rigid body motions, we get from relation (4.4) the first and the second Cauchy's law of motion in the form

$$(4.6) \qquad t^{ij}{}_{,j} + \rho f^i + \rho \dot{v}^i$$

$$(4.7) \qquad \tau^{[ij]} = 0$$

where

$$(4.8) \qquad \tau^{ij} = \rho l^{ij} + t^{ij} + h^{ijk}{}_{,k} - \rho \Gamma^{ij}$$

is the so called microstress average.

By using (4.6), (4.7), (4.8) and (2.5), relation (4.4) redused to

$$(4.9) \qquad \rho \dot{\varepsilon} = t^{ij} \dot{x}_{i,j} + [(\tau^{ij} - t^{ij}) \chi^K_{\cdot j} + h^{ijk} \chi^K_{\cdot j,k}] \dot{\chi}_{iK} + h^{ijk} \chi^K_{\cdot j} \dot{\chi}_{iK,k} + \rho h + q^k{}_{,k}.$$

This is the expression for the rate of specific internal energy which is formin-variant with respect to superposed rigid motion.

5. The second law of thermodynamics

As we suppose that the material of the body with microstructure is perfectly elastic and that the thermodynamic process at each point of the macroelement is locally reversible, the equation of entropy production in any point of the macroelement can be written in the form

$$(5.1) \qquad \rho' v \dot{\eta}' = \rho' h' + q'^k{}_{|k}$$

wherefrom, taking in account the structure of the material, we get the global form of the equation production of entropy

$$(5.2) \qquad \int_v \rho (\theta \dot{\eta} + \theta_i \dot{\eta}^i - \theta_i \eta^j b^i_{\cdot j})\, dv = \int_v (\rho h + q^k{}_{,k})\, dv$$

where, according to (3.6)

$$\rho\eta\, dv = \int_{dv} \rho' \eta' dv', \qquad \rho \eta^i dv = \int_{dv} \rho' \eta' d^i dv'. \tag{5.3}$$

η^i is the first entropy moment. It is easy to see that

$$\rho \dot{\eta}^i - \rho \eta^j b^i_{\cdot j} = \frac{1}{\theta} (q^{ij}{}_{,j} + \rho h^i - \delta^i) \tag{5.4}$$

where

$$q^{ij} = \int_{ds} \frac{q'^i d^j}{\Delta s} ds', \qquad \rho h^i dv = \int_{dv} \rho' h' d^i dv', \qquad \delta^i dv = \int_{dv} q'^i dv'. \tag{5.5}$$

δ^i is the body flux, h^i is the first moment of heat sources [4], and q^{ij} will be the first moment of heat flux.

As we have seen previously, we must suppose that the second law of thermodynamics has the form

$$\rho' \Phi' \geqslant 0 \tag{5.6}$$

where Φ' is the dissipation function in any point of the macroelement. Taking into consideration our supposition that the mechanic process is fully reversible, we have

$$\rho' \Phi = \frac{q'^k \vartheta_{|k}}{\vartheta} \tag{5.7}$$

wherefrom we get

$$\int_v \rho \Phi\, dv = \int_v \left[\frac{q^k \theta_{,k}}{\theta} + q^{kl} \left(\frac{\theta_l}{\theta} \right)_{,k} + \delta^k \frac{\theta_k}{\theta} \right] dv \tag{5.8}$$

in fact this is

$$\rho \Phi = q^k \frac{\theta_{,k}}{\theta} + q^{kl} \left(\frac{\theta_l}{\theta} \right)_{,k} + \delta^k \frac{\theta_k}{\theta} \tag{5.9}$$

where

$$\rho \Phi\, dv = \int_{dv} \rho' \Phi' dv'.$$

By using (5.9) we see that the dissipation function for the thermoelastic materials with microstructure is given as the sum of products of heat gradients of temperature and microtemperature. As we have seen in [5], we must consider that the heat fluxes are irreversible thermodynamic forces and the gradients of temperature and microtemperature are generalized velocities, so that we can write

$$\Phi = \Phi \left[\frac{\theta_{,k}}{\theta}, \left(\frac{\theta_l}{\theta} \right)_{,k}, \frac{\theta_k}{\theta} \right] \tag{5.10}$$

6. Constitutive equations

The free energy function of thermoelastic materials with microstructure has the form

(6.1) $$\psi=\varepsilon-\theta\eta-\theta_l\eta^l$$

and taking into consideration (4.1) and (5.2) we get

(6.2) $$\rho\dot{\psi}=t^{ij}\dot{x}_{i,j}+[(\tau^{ij}-t^{ij}-\rho\theta^i\eta^j)\,x^K_{\cdot j}+h^{ijk}\chi^{K}_{\cdot\, j,k}]\,\dot{\chi}_{iK}+h^{ijk}\chi^K_{\cdot\, j}\dot{\chi}_{iK,k}-\rho\eta\dot{\theta}-\rho\eta\dot{\theta}^i{}_i$$

wherefrom it can be seen that the free energy is a function of the form

(6.3) $$\psi=\psi(x^l_{;,K};\ \chi^l_{\cdot\, K};\ \chi^l_{\cdot\, K;L};\ \theta;\ \theta_l).$$

From (6.2) and (6.3) we get

(6.4)
$$t^{ij}=\rho g^{il}\frac{\partial\psi}{\partial x^l_{;K}}x^j_{;K}$$
$$\tau^{ij}=\rho g^{il}\left(\frac{\partial\psi}{\partial x^l_{;K}}x^j_{;K}+\frac{\partial\psi}{\partial\chi^l_{\cdot\, K}}x^j_{\cdot\, K}+\frac{\partial\psi}{\partial\chi^l_{\cdot\, L,K}}\chi^j_{\cdot\, L,K}-\frac{\partial\psi}{\partial\theta_j}\theta_l\right)$$
$$h^{ijk}=\rho g^{il}\frac{\partial\psi}{\partial\chi^l_{\cdot\, L,K}}\chi^j_{\cdot\, L}x^k_{;K},\quad \eta=-\frac{\partial\psi}{\partial\theta},\quad \eta^i=-\frac{\partial\psi}{\partial\theta_i}$$

these are nonlinear constitutive equations for nonisotropic thermoelastic materials with microstructure.

The objectivity condition is of the form

(6.5) $$\left[\rho g^{il}\left(\frac{\partial\psi}{\partial x^l_{;K}}x^j_{,K}+\frac{\partial\psi}{\partial\chi^l_{\cdot\, K}}\chi^j_{\cdot\, K}+\frac{\partial\psi}{\partial\chi^l_{\cdot\, L,K}}\chi^j_{\cdot\, L,K}-\frac{\partial\psi}{\partial\theta_j}\theta_l\right)\right]_{[ij]}=0$$

and we get the system of three differential equations which has 45 mutualy independent integrals. We shall take the following ones

(6.6) $$\Gamma_{KL}=g_{kl}\chi^k_{\cdot\, K}\chi^l_{\cdot\, L},\quad \Sigma_{KL}=\chi_{Kk}x^k_{;L},\quad D_{LKM}=\chi_{Kk}\chi^k_{\cdot\, L,M},\quad \theta_K=\chi^k_{\cdot\, K}\theta_k$$

so that the system (6.5) has the general solution

(6.7) $$\psi=\psi(\Gamma_{KL},\ \Sigma_{KL},\ D_{KLM},\ \theta,\ \theta_K).$$

Substituting (6.7) into (6.4) we obtain

(6.8)
$$t^{ij}=\rho\frac{\partial\psi}{\partial\Sigma_{KL}}\chi^i_{\cdot\, K}x^j_{;L},\quad \tau^{ij}=2\rho\frac{\partial\psi}{\partial\Gamma_{KL}}\chi^i_{\cdot\, K}\chi^j_{\cdot\, L},$$
$$h^{ijk}=\rho\frac{\partial\psi}{\partial D_{KLM}}\chi^{\cdot i}_{\cdot\, K}\chi^j_{\cdot\, L}x^k_{;M},\quad \eta=-\frac{\partial\psi}{\partial\theta},\quad \eta^i=-\frac{\partial\psi}{\partial\theta_K}\chi^i_{\cdot\, K}.$$

These are the nonlinear constitutive equations of nonisotropic thermoelastic materials with microstructure which are form-invariant with respect superposed rigid motion.

We get nonlinear consitutive equations for irreversible thermodynamic forces-heat fluxes, by using Ziegler's principle of the least irreversible force. Its mathematical formulation is

(6.9) $$X_\alpha^{(i)} = \lambda \frac{\partial \Phi}{\partial \dot{x}^\alpha}, \quad \lambda = \Phi \cdot \left(\frac{\partial \Phi}{\partial \dot{x}^\beta} \dot{x}^\beta \right)^{-1}$$

where $X_\alpha^{(i)}$ are the rectangular coordinates of forces and $\dot{x}^\alpha$ are the rectangular coordinates of velocities into corresponding N-dimensional spaces of forces and velocities, taking into consideration that Φ is a function of the form

(6.10) $$\Phi = \Phi(\dot{x}^\alpha),$$

Now, according to (5.9), (5.10), (6.9) and (6.10), we have

(6.11) $$q^k = \lambda \frac{\partial \Phi}{\partial \left(\frac{\theta_{,k}}{\theta} \right)}, \quad \delta^k = \lambda \frac{\partial \Phi}{\partial \left(\frac{\theta_k}{\theta} \right)}, \quad q^{kl} = \lambda \frac{\partial \Phi}{\partial \left[\left(\frac{\theta_l}{\theta} \right)_{,k} \right]}$$

where

(6.12) $$\lambda = \Phi \left[\frac{\partial \Phi}{\partial \left(\frac{\theta_{,m}}{\theta} \right)} \frac{\theta_{,m}}{\theta} + \frac{\partial \Phi}{\partial \left(\frac{\theta_m}{\theta} \right)} \frac{\theta_m}{\theta} + \frac{\partial \Phi}{\partial \left[\left(\frac{\theta_n}{\theta} \right)_{,m} \right]} \cdot \left(\frac{\theta_n}{\theta} \right)_{,m} \right]^{-1}.$$

Equations (6.11) are nonlinear constitutive equations for heat fluxes in the case of nonisotropic thermoelastic materials with microstructure.

7. Isotropic thermoelastic materials. The linear theory

In the case of thermoelastic materials with microstructure, elastically and thermically isotropic, the free energy function has the following form

(7.1) $$\psi = \psi(\gamma_{kl}, \sigma_{kl}, d_{klm}, \theta, \theta_l)$$

where γ_{kl}, σ_{kl} and d_{klm} are following spatial deformation tensors

(7.2) $$\gamma_{kl} = G_{KL} \chi^{K}_{\cdot k} \chi^{L}_{\cdot l}, \quad \sigma_{kl} = \chi_{kK} X^K_{;l}, \quad d_{klm} = \chi_{kK,m} \chi^{K}_{\cdot l}.$$

Substituting (7.1) into (6.4) and using (7.2) we obtain nonlinear contitutive equations for the isotropic materials

$$t^{ij} = -\rho \frac{\partial \psi}{\partial \sigma_{kj}} \sigma_k^{\cdot i} - \rho \frac{\partial \psi}{\partial d_{klj}} d_{kl}^{\cdot\cdot i},$$

(7.3) $$\tau^{ij} = \rho \left(-2 \frac{\partial \psi}{\partial \gamma_{kj}} \gamma_k^{\cdot i} + \frac{\partial \psi}{\partial \sigma_{ik}} \sigma^j_{\cdot k} - \frac{\partial \psi}{\partial \sigma_{kj}} \sigma_k^{\cdot i} - \frac{\partial \psi}{\partial d_{klj}} d_{kl}^{\cdot\cdot i} - \right.$$
$$\left. - \frac{\partial \psi}{\partial d_{kjm}} d_{k\cdot m}^{\ i} - \frac{\partial \psi}{\partial d_{ilm}} d^j_{\cdot lm} - \frac{\partial \psi}{\partial \theta_j} \theta^i \right),$$

$$h^{ikl} = \rho \frac{\partial \psi}{\partial d_{ijk}}, \quad \eta = -\frac{\partial \psi}{\partial \theta}, \quad \eta^i = -\frac{\partial \psi}{\partial \theta_i}$$

where the following condition of objectivity must be satisfied

$$(7.4)\qquad \left(-2\frac{\partial\psi}{\partial\gamma_{kj}}\gamma_k^{\cdot i}+\frac{\partial\psi}{\partial\sigma_{ik}}\sigma^{j}_{\cdot k}-\frac{\partial\psi}{\partial\sigma_{kj}}\sigma_k^{\cdot i}-\frac{\partial\psi}{\partial d_{klj}}d_{kl}^{\cdot\cdot i}-\right.$$
$$\left.-\frac{\partial\psi}{\partial d_{kjm}}d_{k\ \cdot m}^{\ i}-\frac{\partial\psi}{\partial d_{ilm}}d^{j}_{\cdot lm}-\frac{\partial\psi}{\partial\theta_j}\theta^i\right)_{[ij]}=0.$$

If we introduce, instead of γ_{kl} and σ_{kl}, following spatial measures of deformation

$$(7.5)\qquad 2f_{kl}=g_{kl}-\gamma_{kl},\quad \varepsilon_{kl}=g_{kl}-\sigma_{kl}$$

nonlinear constitutive equations (7.3) take the form

$$t^{ij}=\rho\left(\frac{\partial\psi}{\partial\varepsilon_{ij}}-\frac{\partial\psi}{\partial\varepsilon_{kj}}\varepsilon_k^{\cdot i}-\frac{\partial\psi}{\partial d_{klj}}d_{kl}^{\cdot\cdot i}\right),$$

$$(7.6)\qquad \tau^{ij}=\rho\left(\frac{\partial\psi}{\partial f_{ij}}-2\frac{\partial\psi}{\partial f_{jk}}f^{i}_{\cdot k}+\frac{\partial\psi}{\partial\varepsilon_{ik}}\varepsilon^{j}_{\cdot k}-\frac{\partial\psi}{\partial\varepsilon_{kj}}\varepsilon_k^{\cdot i}+\right.$$
$$\left.+\frac{\partial\psi}{\partial d_{ilm}}d^{j}_{\cdot lm}-\frac{\partial\psi}{\partial d_{kjm}}d_{k\ \cdot m}^{\ i}-\frac{\partial\psi}{\partial d_{klj}}d_{kl}^{\cdot\cdot i}-\frac{\partial\psi}{\partial\theta_j}\theta^i\right),$$

$$h^{ijk}=\rho\frac{\partial\psi}{\partial d_{ijk}},\qquad \eta=-\frac{\partial\psi}{\partial\theta},\qquad \eta^i=-\frac{\partial\psi}{\partial\theta_i}$$

and the condition objectivity is

$$(7.7)\qquad \left(\frac{\partial\psi}{\partial f_{ij}}-2\frac{\partial\psi}{\partial f_{jk}}f^{i}_{\cdot k}+\frac{\partial\psi}{\partial\varepsilon_{ik}}\varepsilon^{j}_{\cdot k}-\frac{\partial\psi}{\partial\varepsilon_{kj}}\varepsilon_k^{\cdot i}+\frac{\partial\psi}{\partial d_{ilm}}d^{j}_{\cdot lm}-\right.$$
$$\left.-\frac{\partial\psi}{\partial d_{kjm}}d_{k\ \cdot m}^{\ i}-\frac{\partial\psi}{\partial d_{klj}}d_{kl}^{\cdot\cdot i}-\frac{\partial\psi}{\partial\theta_j}\theta^i\right)_{[ij]}=0.$$

The deformation tensors f_{kl}, ε_{kl}, d_{klm} and θ_k may be expressed in the form

$$(7.8)\qquad 2f_{kl}=\varphi_{kl}+\varphi_{lk}-\varphi_{mk}\varphi^m_l,\qquad \varepsilon_{kl}=u_{k,l}-\varphi_{kl}+(u_{m,l}-\varphi_{ml})\varphi_{kK}g^{mK}$$
$$d_{klm}=\varphi_{kl,m}+\varphi_{sl,m}\varphi_{kK}g^{sK},\qquad \theta_k=g^K_k\theta_K-\varphi^K_{\cdot k}\theta_K.$$

u_k is the vector of displacement, φ_{kl} are spatial coordinates of the gradients of microdisplacements which satisfy the conditions

$$(7.9)\qquad \varphi^{k}_{\cdot l}=\varphi^{K}_{\cdot L}\chi^{L}_{\cdot l},\qquad \varphi^{K}_{\cdot L}=\varphi^{K}_{\cdot l}\chi^{l}_{\cdot L}$$

where

$$(7.10)\qquad \chi^{k}_{\cdot K}=g^{k}_{\cdot K}+\varphi^{k}_{\cdot K},\qquad \chi^{K}_{\cdot k}=g^{K}_{\cdot k}-\varphi^{K}_{\cdot k}.$$

In the linear theory, for infinitesimal deformations, tensors (7.8) have the form

(7.11) $$2f_{kl}=\varphi_{kl}+\varphi_{lk},\quad \varepsilon_{kj}=u_{k,l}-\varphi_{kl},\quad d_{klm}=\varphi_{kl,m},\quad \theta_k=g_k^K\theta_K.$$

The linear constitutive equations are

(7.12) $$t^{ij}=\rho_0\frac{\partial\psi}{\partial\varepsilon_{ij}},\quad \tau^{ij}=\rho_0\frac{\partial\psi}{\partial f_{ij}},\quad h^{ijk}=\rho_0\frac{\partial\varphi}{\partial d_{ijk}},\quad \eta=-\frac{\partial\psi}{\partial\theta},\quad \eta^i=-\frac{\partial\psi}{\partial\theta_i}$$

and the free energy is a function of the form

(7.13) $$\psi=\psi(\varepsilon_{ij},\, f_{ij},\, d_{ijk},\, T,\, \theta_i)$$

where

$$T=\theta-\theta,\qquad \frac{T}{\theta_0}\ll 1.$$

If we suppose that initial stresses do not exist, and that is a very small quantity, the free energy is a qudratic polynomial form

(7.14) $$\rho_0\psi=\frac{1}{2}A^{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}B^{ijkl}f_{ij}f_{kl}+\frac{1}{2}C^{ijklmn}d_{ijk}d_{lmn}-$$
$$-\frac{1}{2}d\cdot T^2-\frac{1}{2}F^{ij}\theta_i\theta_j+G^{ijkl}\varepsilon_{ij}f_{kl}+E^{ij}\varepsilon_{ij}T+H^{ij}f_{ij}T+K^{ijkl}d_{ijk}\theta_l.$$

A^{ijkl}, B^{ijkl}, C^{ijklmn}, F^{ij}, G^{ijkl}, E^{ij}, H^{ij} and K^{ijkl} are isotropic material tensors.

Making use of (7.12) and (7.14) the linear constitutive relations take the form

(7.15)
$$t^{ij}=\alpha_1 g^{ij}\varepsilon_l+\alpha_2\varepsilon^{ij}+\alpha_3\varepsilon^{ji}+\gamma_1 g^{ij}f_l+2\gamma_2 f^{ij}+e_1 g^{ij}T,$$
$$\tau^{ij}=\beta_1 g^{ij}f_l+2\beta_2 f^{ij}+\gamma_1 g^{ij}\varepsilon_l+\gamma_2(\varepsilon^{ij}+\varepsilon^{ji})+\varkappa_1 g^{ij}T,$$
$$h^{ijk}=\delta_1(g^{ij}d^{kn}_{\cdot\cdot n}+g^{jk}d^{n\cdot i}_{\cdot n})+\delta_2(g^{ij}d^{nk}_{\cdot\cdot n}+g^{ik}d^{n\cdot j}_{\cdot n})+$$
$$+\delta_3(g^{ik}d^{jn}_{\cdot\cdot n}+g^{jk}d^{ni}_{\cdot\cdot n})+\delta_4(d^{kij}+d^{jki})+\delta_5 g^{ij}d^{n\cdot k}_{\cdot n}+$$
$$+\delta_6 g^{ik}d^{nj}_{\cdot\cdot n}+\delta_7 g^{jk}d^{in}_{\cdot\cdot n}+\delta_8 d^{ijk}+\delta_9 d^{ikj}+\delta_{10}d^{jik}+\delta_{11}d^{kji}+$$
$$+k_1 g^{ij}\theta^k+k_2 g^{ik}\theta^j+k_3 g^{jk}\theta^i,$$
$$\rho_0\eta=d\cdot T+e_1\varepsilon_l+\varkappa_1 f_l,\qquad \rho_0\eta^i=f_1\theta^i-k_1 d^{j\cdot i}_{\cdot j}-k_2 d^{ki}_{\cdot\cdot k}-k_3 d^{ik}_{\cdot\cdot k}.$$

In the case of linear constitutive equations for determined irreversible thermodynamic forces, the function Φ must be a homogeneous quadratic form of its variables, in fact

(7.16) $$\rho_0\Phi=\frac{1}{2}M^{ij}\frac{T_{,i}}{\theta}\frac{T_{,j}}{\theta}+N^{ij}\frac{T_{,i}}{\theta}\frac{\theta_j}{\theta}+\frac{1}{2}P^{ij}\frac{\theta_i}{\theta}\frac{\theta_j}{\theta}+\frac{1}{2}Q^{ijmn}\left(\frac{\theta_i}{\theta}\right)_{,j}\left(\frac{\theta_m}{\theta}\right)_{,n}$$

where M^{ij}, N^{ij}, P^{ij} and Q^{ijmn} are isotropic tensors.

The linear constitutive equations for the heat fluxes take the form

$$
\begin{aligned}
q^k &= m_0 g^{ik} T,_i + m_1 g^{ik} \theta_i \\
\delta^k &= m_1 g^{ik} T,_i + m_2 g^{ik} \theta_i \\
q^{kl} &= q_1 g^{kl} \theta,^i_i + q_2 (\theta^l)'^k + q_3 (\theta^k)'^l.
\end{aligned}
\tag{7.17}
$$

Coefficients m_0, m_1, m_2, q_1, q_2 and q_3 must satisfy following conditions

$$
\begin{gathered}
m_0 \geqslant 0, \quad m_2 \geqslant 0, \quad m_0 m_2 - m_1^2 = 0, \\
q_3 - q_2 \geqslant 0, \quad q_3 + q_2 \geqslant 0 \quad 3 q_1 + q_2 + q_3 \geqslant 0.
\end{gathered}
\tag{7.18}
$$

Six mutually independent material constans are in the equations (7.17), while in the corresponding constitutive equations in [2] there ere seven mutually independent constans. However, it can be shown easily, that between two of them there must be some relationship, in fact, six of them are mutually independent.

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ТЕРМОПРУГИЕ МАЕРИАЛЫ С МИКРОСТРУКТУРОЙ

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Резюме

В настоящей работе изучается модель термоупругого материала с микроструктурой, прєдполагая неоднородное распределение температуры внутри макроэлемента. Анализируя такую модель получены нелинейные определяющие уравнения для механических и термодинамических величин. Из общей формы уравнений выведены линеаризованные определяющие уравнения для изотропного термоупругого материјала.

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