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ON A DISCRETE MODEL OF AN UNSTATIONARY LINEAR SYSTEM

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In this a work a procedure for construction of a discrete model of a linear unstationary system is proposed. In expression

(1) $$\dot{x} = A(t)\,x + B(t)\cdot u, \quad x(t_0) = x_0,$$

$x(t)$ is a globe vector of the column with coordinates x_i, $(i = 1, 2, \ldots, n)$, $A(t)$ is a well-known quadratic matrix of order n, $B(t)$ is a known matrix of the type (nxr), and $u(t)$ is known function with coordinates u_i, $(i = 1, 2, \ldots, r)$.

For construction of a discrete model for an initially continuous system (1) let us start from the solution of an equation (1) given by the Cauchy formula

(2) $$x(t) = X[t, t_0]\,x(t_0) + \int_{t_0}^{t} X[t, s]\,B(s)\cdot u(s)\,ds$$

The direct application of the Cauchy formula (2) is evidently related to the familiarity with the fundamental matrix. The determination of a fundamental matrix for a linear unstationary system is not due in general case in a finite form. That is why we have often to determine a globe vector numerically using a discrete model. We suppose that, the elements $a_{ij}(t)$ of the matrix $A(t)$ are continuous bounded and slow varying time functions, so that the fundamental matrix may be represented in the form [1]

(3) $$X[t, t_0] = \sum_{n=0}^{\infty} A_n(t_0)\,(t - t_0)^n/n!$$

In (3) matrices $A_n(t_0)$ are defined by

$A_0(t_0) = \mathrm{I}$ (unity matrix of order n)

$A_1(t_0) = A(t_0)$, $A_2(t_0) = \dot{A}(t_0) + A^2(t_0), \ldots$

Therefore

(4) $$A_n(t_0) = \sum_{p=0}^{n-1} \binom{n-1}{p} \overset{(p)}{A}(t_0)\,A_{n-1-p}(t_0), \quad n \geqslant 1$$

where

$$\overset{(p)}{A}(t) = \frac{d^p A(t)}{dt^p}, \quad \overset{(0)}{A}(t) = A(t)$$

With this the first few terms of the fundamental matrix become:

$$X[t,\ t_0]=I+A(t_0)(t-t_0)+(t-t_0)^2/2!\,(\dot{A}+A^2)_{t=t_0}+ \tag{5}$$

$$+(t-t_0)^3/3!\,(\ddot{A}+2\dot{A}A+A\dot{A}+A^3)_{t=t_0}+\cdots$$

If a discretization of time is carried out, the step of discretization T being constant and $t_{k+1}-t_k=T$, $(k=0,\ 1,\ 2,\ \ldots,\ s)$, $t_\beta-t_0=sT$, it is important that T is chosen so that

$$u(t)=u(t_k)\ \text{for}\ t\in[t_k,\ t_{k+1}) \tag{6}$$

with the sufficient accuracy.

Now we can calculate $x(t_1)$, $x(t_2)$, ... by means of formula (2), where $x(t_1)$, $x(t_2)$, ... take the role of $x(t_0)$. In such a way we have the recurrence formula:

$$x(t_{k+1})=X[t_{k+1},\ t_k]\,x(t_k)+\Big(\int_{t_k}^{t_{k+1}} X[t_{k+1},\ s]\,B(s)\,ds\Big)\,u(t_k) \tag{7}$$

Using formula (3) expression (7) becomes:

$$x(t_{k+1})=\Big(\sum_{n=0}^{\infty} A_n(t_k)\,T^n/n!\Big)\,x(t_k)+ \tag{8}$$

$$+\Big(\int_{t_k}^{t_{k+1}}\sum_{n=1}^{\infty} A_n(s)\,(t_{k+1}-s)^n/n!\ B(s)\,ds\Big)\,u(t_k)$$

If in expression (8) notation $E(t_k)$ for the first infinite series with $x(t_k)$, and $F(t_k)$ for an integral with $u(t_k)$ are introduced we get:

$$x(t_{k+1})=E(t_k)\ x(t_k)+F(t_k)\ u(t_k) \tag{9}$$

In a stationary case matrices A and B are constant, and from (4) we can see that $A_n=A^n$, so that expression (8) reduces to:

$$x(t_{k+1})=e^{AT}\,x(t_k)+\Big(\int_0^T e^{Az}\,dz\Big)\,Bu(t_k) \tag{10}$$

where, for the sake of simplicity, a substitution in the integral in eq. (8) is made $t_{k+1}-s=z$, $ds=-dz$.

Expression (10) is known [2], as a discrete model of a linear stationary system in the form $\dot{x}=Ax+Bu$, A and B being constant matrices. Comparing (9) and (10) we can see that matrix $E(t_k)=E(T)$ is now a function of T. This matrix is also constant if T is fixed. The same conclusion is valid for matrix $F(t_k)$.

In an unstationary case expression (8) can be used in two ways. In the cases where infinite series in (8) are expansions of known matrices $E(t_k)$ and

$F(t_k)$, i.e. when it is possible to find boundary functions of series the problem is solved in a finite form. Matrices so determined are to be introduced in (9 and the expression can be used as discrete model.

If it is not possible, expression (8) can be used in approximately the same way as the Taylor series is employed in the scalar case, replacing the infinite series by the first few members of the same series. In infinite series (8) we assume n to take the values 0, 1, 2, ... , m, where m is to be selected in the wanted accuracy of the approximation. In this case the discrete model is presented by expression (9), where matrices $E(t_k)$ and $F(t_k)$ are

(11) $$E(t_k)=\sum_{n=0}^{m} A_n(t_k)\, T^n/_{n!}, \quad m\geqslant 1$$

(12) $$F(t_k)=\int_{t_k}^{t_{k+1}} \sum_{n=0}^{m} A_n(s)\,(t_{k+1}-s)^n/_{n!}\, B(s)\, ds$$

and matrices $A_n(t)$ are defined in (4).

We observe that matrices E and F are here the time functions, while in the stationary system they were functions of the discretization step T. The positive magnitude $T<1$ in equations (11) and (12) is to be chosen so that the series converge fast, satisfyng at the same time expression (6). Magnitude T could be changed during the test of discrete model (9) on the computer. This magnitude may be decreased, if the first results show great deviation from the results expected. In the case when, magnitude T could not be further decreased, the aproximation precision could be increased by increasing m, until the satisfactory accuracy is reached.

To check the result, we will solve an example. The motion of a material point with coordinates $y(t)$ is described by equation

(13) $$\ddot{y}+p\dot{y}+qe^{2pt}\,y=u(t), \quad y(t_0)=y_0, \quad \dot{y}(t_0)=\dot{y}_0,$$

where p and q are the well-known constants, and $u(t)$ is Dirac function.

The solution of this equation for the initial conditions $y_0=0$, $\dot{y}_0=0$ is given in [3] as:

(14) $$y(t)=\frac{1}{\sqrt{q}}\, e^{-pt} \sin\left[\frac{\sqrt{q}}{p}\,(e^{pt}-e^{pt_0})\right]$$

The problem could be numerically solved by the discrete model (9) on the computer and then compared to the accurate solution (14). To that end we shall transform differential equation (13) into the Cauchy form (1) by substituting $y=x_1$, $\dot{y}=x_2$. In this case matrices $A(t)$ and $B(t)$ are (for $p=0{,}5$, $q=0{,}2$)

(15) $$A(t)=\begin{pmatrix} 0 & 1 \\ -0{,}25\, e^{0{,}4t} & -0{,}2 \end{pmatrix}, \quad B(t)=B=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To use discrete model (9) we have to calculate matrices $E(t_k)$ and $F(t_k)$ according to (11) and (12). Taking $m=3$ we have:

$$E(t_k)=I+A(t_k)\,T+A_2(t_k)\,T^2/2!+A_3(t_k)\,T^3/3! \tag{16}$$

With notation (4), expression (16) reduces to

$$E(t_k)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\\ -0{,}25\,e^{0{,}4\,t_k} & -0{,}2\end{pmatrix}\cdot T+$$

$$+\begin{pmatrix}-0{,}25\,e^{0{,}4\,t_k} & -0{,}2\\ -0{,}05\,e^{0{,}4\,t_k} & 0{,}04-0{,}25\,e^{0{,}4\,t_k}\end{pmatrix}T^2/2!+$$

$$+\begin{pmatrix}0{,}4\,e^{0{,}4\,t_k} & 0{,}04-0{,}25\,e^{0{,}4\,t_k}\\ -0{,}03\,^{0{,}4\,t_{k+4}-2}\,e^{0{,}8\,t_k} & -5^{-3}-0{,}1\,e^{0{,}4\,t_k}\end{pmatrix}T^3/3!$$

In the same way we can calculate $F(t_k)$, for $m=3$, from (12)

$$F(t_k)=\int_{t_k}^{t_{k+1}}\left(\begin{matrix}(t_{k+1}-s)-0{,}1\,(t_{k+1}-s)^2+\\ 1-0{,}2\,(t_{k+1}-s)+(0{,}02-0{,}125\,e^{0{,}4\,s})\,(t_{k+1}-s)^2+\end{matrix}\right.$$

$$\left.\begin{matrix}+\dfrac{1}{6}(0{,}04-0{,}25\,e^{0{,}4\,s})\,(t_{k+1}-s)^3\\ +\dfrac{1}{6}(-5^{-3}-0{,}1\,e^{0{,}4\,s})\,(t_{k+1}-s)^3\end{matrix}\right|\,ds$$

So calculated matrices $E(t_k)$ and $F(t_k)$ are substituted into eq. (9) and difference equation for recurrence solving of globe vector $x(t_k)$ is obtained. In this example a step of discretization is $T=0{,}05\,s$, $t_0=2\,s$, $t_k=kT+t_0$, $(k=0,$ $1,\,2,\,\ldots,\,120)$, $t_\beta=t_{120}=8\,s$, and the Dirac function is approximately presented by

$$u(t)=\delta(t-t_0)=\begin{cases}20 & \text{for } 2\leqslant t<2{,}05\\ 0 & \text{for } t\geqslant 2{,}05\end{cases} \tag{17}$$

Now, all elements in discrete model (9) necessary for numerical solving are defined. Calculations are done on a computer IBM 1130 of the computer center at the Faculty of Electrical Engineering, Belgrade by two methods:

1. Using the program for a discrete model (9).

2. Using 4th Runge-Kutta method for integration.

The obtained results are presented in Table 1 for sake of the comparison. The table contains only the characteristic data from a large number of data provided by the computer. From this table we can conclude that numerical solution (9) gives us a good aproximation of the exact solution (14). If the curves of the accurate solution (14) and approximate solution (9) are drawn in the same coordinate system. the difference between two curves could hardly be seen in the Figure, because they almost overlap.

t_k	exact solution curve (14)	numerical solution by 9.	numerical solution by *R KGS*
2	0	0	0
2,6	0,544	0,524	0,566
3	0,807	0,793	0,858
3,6	0,960	0,958	1,037
4	0,868	0,872	0,945
4,5	0,537	0,549	0,595
5	0,055	0,069	0,076
5,5	−0,397	−0 387	−0,418
6	−0,596	−0,596	−0,644
6,5	−0,406	−0,414	−0,477
7	0,052	0,052	0,056
7,5	0,415	0,433	0,445
8	0,282	0,288	0,311

The greatest drawback of this method is the error evaluation. In a general case there is no finite expression practically useable. It is sure that the use of (9) for solving a great number of examples will contribute to better understanding of the error problem and perhaps some other shortcomings.

Comparing this method with the Runge-Kutta method we can expect better numerical results for $m>3$ (for linear systems) on the basis of discrete model (9), because the Runge-Kutta method is a general method for solving nonlinear differential equations, so that less results are to be expected with respect to discrete model (9) which is especially established for linear systems. Apart from that, the precision of the numerical calculations in the Runge-Kutta method is increased only by decreasing the discretisation step T, while in a discrete model the precision could be increased by enhancing m up to the desired accuracy.

REFERENCES

[1] Mikičić, J. D., *Jedna približna metoda za određivane fundamentalne matrice sistema linearnih diferencijalnih jednačina*, Matematički vesnik 8 (23), 1971, Beograd.

[2] Stojić, R. M., *Kontinualni sistemi automatskog upravljanja*, Građevinska knjiga, Beograd 1973.

[3] Солодов, А. В., *Линейные системы автоматического управления с переменными параметрами*, Физматгиз, Москва, 1962.

ДИСКРЕТНЫЙ МОДЕЛЬ НЕСТАЦИОНАРНОГО ЛИНЕАРНОГО СИСТЕМА

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Резюме

В работе рассматривается движение системы, состояние которой можно представить уравнениями

$$\dot{x} = A(t)\,x + B(t)\,u$$

Здесь показано, как можно приблизительной метод для определения фундаменталной матрицы использовать для нахождения матрицы $E(t_k)$ и $F(t_k)$ в дискретных моделях

$$x(t_{k+1}) = E(t_k)\,x(t_k) + F(t_k)\,u(t_k)$$

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