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UNDAMPED VIBRATIONS OF ELASTIC THIN-WALLED BEAMS OF OPEN DEFORMABLE CROSS SECTION

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Introduction. Basic Assumptions

Undamped vibrations of straight single span thin-walled beams of open deformable cross sections are analyzed; this is actually the vibration analysis of long prismatic folded plates. The paper is based on the treatise proposed by Kollbrunner and Hajdin (1) which represents the statical analysis of the mentioned thin-walled beams.

Let us consider a single span straight thin-walled beam of an arbitrary open cross section which is constant along the span (Fig. 1). All vector and tensor quantities are defined with respect to the coordinate systems n, s, z and/or x, y, z. Both systems are orthogonal righthanded and fixed to the undeformed configuration of the beam. The axes x and y are selected to be the principal centroidal axes of the cross section, and z is the axis of the beam. The coordinate s is measured along the center line of the walls from the previously defined starting point 0, while n is the normal to the center line.

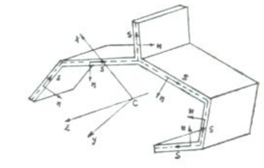

Fig. 1

The displacement vector of the points of the middle surface has components u, v and w with reference to the system n, s, z, respectivelly, and components ξ and η in x and y directions. The same notation, only with the asterisk as the subscript, is being used for displacement components of the points outside the middle surface (i. e. at the distances e along the normal n).

Besides of the assumptions of linearized theory of elasticity as well as of the neglection of thermal influences, the following assumptions have been used:

1. the normal strains of the points of the center line in the direction of the center line, as well as the shear strains in the middle surface are neglected;

2. the displacements perpendicular to the middle surface due to displacements and rotations of the nodal lines are represented by the third order polynomial in coordinate s along the center line.

Differential Equations of Motion. Boundary and Initial Conditions

According to the used assumption, the displacement vector of an arbitrary point of the beam may be expressed in terms of the N generalized coordinates dependent on the position of the cross sections and of the time. To any given thin-walled cross section it corresponds the kinematical mechanism formed by the rigid straight rods whose axes are coinciding with the center line and which are pin-jointed at the nodal lines. If the number of the degrees of freedom of the plane motion of the kinematical mechanism is denoted by n and the number of the nodal points by m, then the number N of the generalized coordinates is equal to $N=n+m$. There are n coordinates $V_i\ (z,\ t)$, which are called the displacement parameters, and m coordinates $\Phi_k\ (z,\ t)$ which represent the angles of rotations of the nodal points.

The translation of the cross section taken as a rigid plane in the directions of the axes x and y, and its rotation around an arbitrary point in the cross section plane are taken as the first three motion parameters V_i. Thatway, the theory of the thin-walled beams of undeformable cross sections may be included as the special case of the theory of the thin-walled beams of deformable cross section. Therefore, by the generalized coordinates $V_i\ (z,\ t)$ and $\Phi_k\ (z,\ t)$ the displacements and the rotations of the nodal lines are defined. The displacement components u, v and w are expressed in terms of the generalized coordinates in the form:

$$
(1) \qquad
\begin{aligned}
u_* &= \sum_{i=0}^{n} V_i\, u_v^{(i)} = \sum_{k=1}^{m} \Phi_k\, u_\Phi^{(k)} + u_p \\
v_* &= \sum_{i=1}^{n} V_i \left(v^{(i)} - \frac{d u_v^{(i)}}{ds}\, e \right) - \sum_{k=1}^{m} \Phi_k \frac{d u_\Phi^{(k)}}{ds}\, e - \frac{\partial u_p}{\partial s}\, e \\
w_* &= - \sum_{i=0}^{n} \frac{\partial V_i}{\partial z} \left(u^{(i)} + u_v^{(i)} e \right) - \sum_{k=1}^{m} \frac{\partial \Phi_k}{\partial z}\, u_\Phi^{(k)}\, e - \frac{\partial u_p}{\partial z}\, e
\end{aligned}
$$

The functions beside the generalized coordinates $u_v^{(i)}$, $u_\Phi^{(k)}$, $v^{(i)}$, $\omega^{(i)}$ are the third order polynomials in coordinate s, and they are obtained from the assumptions being used. The function $u_k\ (z,\ s,\ t)$ represents the set of deflections of all rectangular plates the thin-walled beam is composed of due to external loading perpendicular to the plates if assuming that each plate is fixed along the nodal line.

The strain components are expressed in terms of displacements:

$$
(2) \qquad \varepsilon_{z_*} = \frac{\partial w_*}{\partial z} \qquad \varepsilon_{s_*} = \frac{\partial v_*}{\partial s} \qquad \gamma_{zs_*} = \frac{\partial v_*}{\partial z} + \frac{\partial w_*}{\partial s} \qquad \varepsilon_{n_*} = \gamma_{nz_*} = \gamma_{ns_*} = 0
$$

and, then, the stress components:

$$
(3) \qquad \sigma_z = E' \left(\varepsilon_{z_*} + \nu \varepsilon_{s_*} \right) \qquad \sigma_s = E' \left(\varepsilon_{s_*} + \nu \varepsilon_{z_*} \right) \qquad \tau_{zs} = E' \frac{1-\nu}{2} \gamma_{zs_*}
$$

where $E'=E/(1-\nu^2)$, E is the Young's modulus, ν the Poisson's ratio. The normal stresses σ_n are, like in the theory of thin plates, neglected, and the shearing stresses τ_{zn} and τ_{sn}, due to the assumptions being used, cannot be expressed in terms of displacements.

The cross sectional internal forces (stress resultants) are defined like in the theory of thin plates; all other quantities are expressed by displacements, or, by generalized coordinates V_i and Φ_k.

The differential equations of motion are derived by applying the principle of the virtual work due to a variation of displacements: the increment of the work of the surface forces, body forces and inertia forces due to a variation of displacements is equal to the increment of the work of the internal forces due to the corresponding variation of the state of deformation. Therefore, the following work equation is valid over the unit length of the beam:

$$\int_F \left(\frac{\partial \vec{\sigma}_z}{\partial z} \delta \vec{R}_* + \vec{\sigma}_z \frac{\partial \delta \vec{R}_*}{\partial z} \right) dF_* + \int_s \overline{p} \delta \vec{R} ds - \rho \int_F \vec{a}_* \delta \vec{R}_* dF_* = \tag{4}$$

$$= \int_F (\sigma_z \delta \varepsilon_{z_*} + \tau_{zs} \delta \gamma_{zs_*} + \sigma_s \delta \varepsilon_{s_*}) \, dF_*$$

The vector of virtual displacement $\delta \vec{R}$ is assumed in the same form as the vector of actual displacement, Eqs. (1), the only difference is that the virtual generalized coordinates exist and that the terms u_p, dependent on actual loading, do not exist.

After many transformations, Eq. (4) leads to the equation describing the longitudinal vibrations

$$E'F \frac{\partial^2 W_0}{\partial z^2} - \rho F \frac{\partial^2 W_0}{\partial t^2} = -p_z \tag{5}$$

where W_0 (z, t) represents the translation of the cross sections in the longitudinal direction of the beam, F the cross section area, ρ the material density, p_z the longitudinal loading, as well as to the system of linear partial differential equations of the fourth order in unknown generalized coordinates. This system can be written in matrix form*

$$K_1 \vec{\psi}_{,zzzz} - K_2 \vec{\psi}_{,zz} + K_4 \vec{\psi} - \frac{\rho}{E'} K_1 \vec{\psi}_{,ttzz} + \frac{\rho}{E'} K_3 \vec{\psi}_{,tt} = \frac{1}{E'} \vec{K}_0 \tag{6}$$

where the notations are introduced:

$$\vec{\psi} = \begin{Bmatrix} V \\ \Phi \end{Bmatrix}, \; \vec{V} = \begin{Bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{Bmatrix}, \; \vec{\Phi} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_k \end{Bmatrix} \tag{7}$$

The quantities $\vec{K}_l$ $(l=1, \ldots, 4)$ are symmetric square matrices of the order $m+n$ with constant elements dependent on the cross section geometry as well as on the selected displacement parameters V_i. The absolute term $\vec{K}_0$ (z, t) is the vector whose elements are dependent on loading. With the comma and index z or t, it is denoted the derivative with respect to the corresponding coordinate.

Eq. (5) is the well-known equation of the longitudinal vibrations of beams and it will not be considered in the following text.

The boundary conditions due to displacements can be written in matrix form:

$$z=0, \quad z=l: \quad \vec{\psi} = \vec{\psi}^* \quad \vec{\psi}_{,z} = \vec{\psi}^*_{,z} \tag{8}$$

where the column vectors on the right sides represent the known generalized coordinates and their derivatives.

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* In the following, all letters with vector symbols „→" denote matrix quantities.

The boundary conditions due to forces can be derived by the principle of virtual displacements, and after being expressed in terms of displacements, they have the form:

$$(9) \qquad z=0,\ l: \quad \vec{K}_1\vec{\psi}_{,zzz}-\vec{L}_1\vec{\psi}_{,z}=-\frac{1}{E'}(\vec{Q}^*-\vec{Q}^0);\quad \vec{K}_1\vec{\psi}_{,zz}+y\vec{L}_2\vec{\psi}=-\frac{1}{E'}(\vec{M}^*-\vec{M}^0)$$

The homogeneous boundary conditions at some characteristic support conditions have the form:

for the built-in end

$$(10\,a) \qquad \vec{\psi}=0 \quad \vec{\psi}_{,z}=0$$

for the simply supported end

$$(10\,b) \qquad \vec{\psi}=0 \quad \vec{\psi}_{,zz}=0$$

for the free end

$$(10\,c) \qquad \vec{K}_1\vec{\psi}_{,zzz}-\vec{L}_1\vec{\psi}_{,z}=0 \quad \vec{K}_1\vec{\psi}_{,zz}+\nu\vec{L}_2\vec{\psi}=0$$

The initial conditions are defined by known positions and by known velocities of all points of the beam at the time t=0:

$$(11) \qquad \vec{\psi}=\vec{F}^0 \quad \vec{\psi}_{,t}=\vec{G}^0$$

Therefore, the elastic undamped vibrations of the straight thin-walled beams of open deformable cross section constant over the span are defined by the system of linear partial differential equations of the fourth order with constant symmetrical coefficients, Eq. (6), as well as by corresponding boundary and initial conditions. The number of equations depends on the form of the cross section of the beam and, in generally, it is very high, f. i., at the "I" cross section this number is equal to 13.

Solution of Differential Equation of Motion

1. Free Harmonic Vibrations

The free vibrations are defined by the homogeneous equations (6) and by the corresponding boundary conditions (10). The solution of the equation (6) is assumed in the form

$$(12) \qquad \vec{\psi}(z,t)=\vec{\bar{\psi}}(z)\,e^{i\omega t}$$

where $i_2=-1$. Thatway, the following system of ordinary differential equations is obtained:

$$(13) \qquad \vec{K}_1\vec{\bar{\psi}}''''-\vec{K}_2\vec{\bar{\psi}}''+\vec{K}_4\vec{\bar{\psi}}-\frac{\rho}{E'}\omega^2(-\vec{K}_1\vec{\bar{\psi}}''+\vec{K}_3\vec{\bar{\psi}}=0$$

Thus, the problem of the free harmonic vibrations is the eigenvalue problem of the system of linear ordinary differential equations of the fourth order with constant coefficients. In general case, the solution of such an eigenvalue problem is not possible to be obtained for an arbitrary type of boundary conditions.

Only, when both ends of the beam are simply supported, the boundary conditions (10b) are satisfied by the particular solution

$$\vec{\psi}(z)=\vec{C}_s \sin\lambda_s z \quad (s=1,\ 2,\ \ldots) \tag{14}$$

where $\lambda_s=S\,\pi/l$ $(s=1,\ 2,\ \ldots.)$, and l being the length of the span, while $\vec{C}_s$ is the column vector of $n+m$ unknown constant elements. Substituting (14) into Eq. (13) the homogeneous system of algebraic equations in unknown elements of vector $\vec{C}_s$ is obtained:

$$\left[\lambda_s^4\vec{K}_1+\lambda_s^2\vec{K}_2+\vec{K}_4-\frac{\rho}{E'}\omega^2(\lambda_s^2\vec{K}_1+\vec{K}_3)\right]\vec{C}_s=\vec{O} \quad (s=1,\ 2,\ \ldots) \tag{15}$$

The conditions to the existence of the nontrivial solution yields the frequency equation

$$\det\left[\lambda_s^4\vec{K}_1+\lambda_s^2\vec{K}_2+\vec{K}_4-\frac{\rho}{E'}(\lambda_s^2\vec{K}_1+\vec{K}_3)\right]=\vec{O} \quad (s=1,\ 2,\ \ldots) \tag{16}$$

Eq. (16) is the algebraic equation of the degree $n+m$ in the unknown ω^2, $(q=1,\ 2,\ \ldots,\ n+m)$, and, also, after substituting them into (15) there are $n+m$ constant column vectors determined to the multiplicative constant factor. It means that for each (sinusoidal) mode of vibration of the axis of the beam as a whole, which may be referred to as the "external" mode, there are $n+m$ vibration modes within cross section, the "internal" modes of vibrations.

By using the notations

$$\vec{R}_1=\vec{K}_1D^4-\vec{K}_2D^2+\vec{K}_4 \quad \vec{R}_2=-\vec{K}_1D^2+\vec{K}_3 \quad D^\mu=\frac{d^\mu}{dz^\mu}\ (\mu=1,\ 2,\ \ldots) \tag{17}$$

Eqs. (13) may be written in the operator form:

$$\vec{R}_1\vec{\psi}=\frac{\rho}{E'}\omega^2\vec{R}_2\vec{\psi} \tag{18}$$

If assumed that the external modes $\vec{\psi}_r$ and $\vec{\psi}_s$ are performed in only on internal mode, then, it can be shown that the natural modes are orthonormalized:

$$\int_0^l \vec{\psi}_r^T R_2\vec{\psi}_s dz=\begin{cases}1 \text{ if } r=s\\ 0 \text{ if } r\neq s\end{cases} \tag{19}$$

where T denotes the transpose of vector column. The orthogonality condition (19) is not valid if one end of the beam is free, therefore, the cantilever beams will not be considered in the following text.

2. Forced Harmonic Vibrations

We shall consider the beam under the influence of the harmonic loading; the absolute term is given by

$$\vec{K}_0(z, t = \vec{\bar{K}}_0(z)\, e^{i\Omega t} \tag{20}$$

The unknown generalized coordinates are assumed in the form

$$\vec{\psi}(z, t) = \vec{\bar{\psi}}(z)\, e^{i\Omega t} \tag{21}$$

The partial differential equations are deduced thatway to ordinary differential equations

$$\vec{R}_1 \vec{\bar{\psi}} - \frac{\rho}{E'}\Omega^2 \vec{R}_2 \vec{\bar{\psi}} = \frac{1}{E'} \vec{\bar{K}}_0 \tag{22}$$

Considering the orthogonality relations (19) it is possible to expand the unknown function $\vec{\bar{\psi}}(z)$ into series in terms of natural modes of the free harmonic vibrations $\vec{\bar{\psi}}_m(z)$:

$$\vec{\bar{\psi}}(z) = \sum_{m=1}^{\infty} a_m \vec{\bar{\psi}}_m(z) \tag{23}$$

as well as to determine the unknown coefficients a_m. Therefore, the solution of the Eq. (22), i. e., of the Eqs. (6), is obtained in the form

$$\vec{\psi}(z, t) = \frac{e^{i\Omega t}}{\rho} \sum_{n-1}^{\infty} \frac{1}{\omega_n^2\left(1 - \frac{\Omega^2}{\omega_n^2}\right)} \vec{\bar{\psi}}_n(z) \int_0^l \vec{\bar{\psi}}_n^T(\xi) \vec{\bar{K}}_0(\xi)\, d\xi \tag{24}$$

For simply supported beam the solution (34) has the form)

$$\vec{\psi}(z, t) = \frac{e^{i\Omega t}}{\rho} \sum_{n=1}^{\infty} \frac{\sin\lambda_n z}{\omega_n^2\left(1 - \frac{\Omega^2}{\omega_n^2}\right)} \vec{C}_n \vec{C}_n^T \int_0^l \vec{\bar{K}}_0(\xi) \sin\lambda_n \xi\, d\xi \tag{25}$$

It can be seen that the resonance occurs when $\Omega \to \omega_n$. Thus, the resonance may occur when the angular frequency of external loading Ω is close to any of $n+m$ natural frequencies of all external modes.

3 Nonperiodic Forced and Free Vibrations

Let us consider Eqs. (5), the initial condition (11), and the boundary conditions which make the orthogonality relations (19) valid.

After applying the Laplace transform to Eqs. (6) and introducing the relations

$$\vec{\bar{\psi}}(z, p) = \int_0^{\infty} \vec{\psi}(z, p)\, e^{-pt}\, dt \qquad \vec{\bar{K}}_0(z, p) = \int_0^{\infty} \vec{K}_0(z, t)\, e^{-pt}\, dt \tag{26}$$

one becomes the system of ordinary differential equations:

(27) $$\vec{R}_1\vec{\psi}+\frac{\rho}{E'}p^2\vec{R}_2\vec{\psi}=\frac{1}{E'}\vec{K}_0-\frac{\rho}{E'}(\vec{K}_1-\vec{K}_3)(p\vec{F}^0+\vec{G}^0)$$

The unknown function $\vec{\psi}(z, p)$, i. e., the Laplace transform of the function $\vec{\psi}(z, t)$, is expanded into series in terms of the natural modes of the free harmonic vibrations $\vec{\psi}(z)$:

(28) $$\vec{\psi}(z, p)=\sum_{m=1}^{\infty} a_m(p)\vec{\psi}_m(z)$$

Due to the orthogonality conditions (19), the coefficients $a_m(p)$ of the series (28), as well as the function $\vec{\psi}(z, p)$ can be found. After carrying out the inverse Laplace transform, the final solution is obtained in the form

(29) $$\vec{\psi}(z, t)=\frac{1}{\rho}\sum_{m=1}^{\infty}\frac{1}{\omega_n}\vec{\psi}_n(z)\int_0^l\vec{\psi}_n^T(\xi)\int_0^t\vec{K}_0(\xi,\tau)\sin\omega_n(t-\tau)\,d\tau\,d\xi-$$
$$-\sum_{N=1}^{\infty}\vec{\psi}_n(z)\int_0^l\vec{\psi}_n^T(\xi)[\vec{K}_1-\vec{K}_3]\left[\vec{F}^0(\xi)\cos\omega_n t+\frac{1}{\omega_n}\vec{G}^0(\xi)\sin\omega_n t\right]d\xi$$

For $\vec{K}_0(z, t)=0$, the solution (29) is related to nonperiodic free vibrations due to the different initial conditions. However, the more often case is when the initial conditions are homogeneous, $\vec{F}^\circ(z)=\vec{G}^\circ(z)=0$, and when $\vec{K}^\circ(z, t)=0$.

Thin-walled Beams of Nondeformable Cross Sections

The theory of the thin-walled beams of open nondeformable cross sections is included as the special case of the presented treatise of open deformable sections. When the cross sections are nondeformable in their planes, only the first three displacement parameters V_1, V_2, and V_3, which describe the rigid plane motion of the cross sections, are different from zero. All other generalized coordinates, as well as the function u_p are equal to zero. The Eqs. (6) are deduced then to three differential equations of motion.

After giving the corresponding geometrical interpretation to the coefficients existing in these equations, and after introducing the concept of the shear centre D and concentrating the element of the cross section area along the center line, the following three differential equations can be written:

(30) $$\begin{bmatrix} I_{xx} & 0 & 0\\ 0 & I_{yy} & 0\\ 0 & 0 & I_{\omega\omega}\end{bmatrix}\begin{Bmatrix}\xi_D\\ \eta_D\\ \varphi_D\end{Bmatrix}''''-\frac{GK}{E'}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\xi_D\\ \eta_D\\ \varphi_D\end{Bmatrix}''-$$
$$-\frac{\rho}{E'}\begin{bmatrix} I_{xx} & 0 & 0\\ 0 & I_{yy} & 0\\ 0 & 0 & I_{\omega\omega}\end{bmatrix}\begin{Bmatrix}\xi_D\\ \eta_D\\ \varphi_D\end{Bmatrix}''+\frac{\rho}{E'}F\begin{bmatrix}1&0&y_D\\0&1&-x_D\\y_D&-x_D&r^2\end{bmatrix}\begin{Bmatrix}\xi_D\\ \eta_D\\ \varphi_D\end{Bmatrix}=\frac{1}{E'}\begin{Bmatrix}p_x+m_x'\\ p_y+m_y'\\ m_D+m_\omega'\end{Bmatrix}$$

Here, the comma denotes the derivatives with respect to z, and the point, with respect to t. All other quantities are common in the theory of the thin-walled beams of open nondeformable cross section. The eqs. (30) were proposed by Vlasov (3).

It can be seen, when the cross section has one axis of symmetry, and the shear center is located at this axis (i. e., x or y is equal to zero), the system (30) is decomposed to an independent equation of transverse vibrations in the direction of the axis of symmetry, and to a system of two equations of the coupled transverse-torsional vibrations. In the case when the shear center and the centroid of the cross section are coinciding, most often at the cross section having two axes of symmetry, i. e., when $x_D=y_D=0$, the system (30) is decomposed to three independent equations.

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UNGEDÄMPFTE SCHWINGUNGEN VON ELLASTISCHEN DÜNNWANDIGEN STABE MIT OFFENEN UNDERFORMIERBAREN QUERSCHNITTEN

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Zusammenfassung

Die Schwingungen der dünnwandigen Stabe mit in ihren Ebenen deformierbaren Querschnitten sind analysiert. Die Analyse ist begrenzt auf die elastischen geraden einspannigen Stäbe mit polygonalen Profilmittellinien. Die Schwingungen der dünnwandigen Stäbe mit underformierbaren Querschnitten sind als Sonderfall der allgemeinen Analyse eingeschlossen.

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