TEORIJSKA I PRIMENJENA MEHANIKA 2, pp. 57—66, 1976.

ON SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS OF GENERALIZED PLANE ELASTOSTATICS

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I. Introductory remarks

In the last thirty years, a number of generalizations concerning the classical theory of thin plates has been elaborated, e.g. [7]. While solving boundary value problems for model [7], M. Muršič applies Mushelišvili's mapping methods and those of Cauchy's integrals. Yet he mentions neither the theorem of uniqueness of solution nor the existence of the solution of the problem. The boundary value problem treated in [8] is expressed by equation

$$-\varkappa\,\varphi(z)+z\,\overline{\varphi'(z)}+\overline{\psi(z)}+n\,\overline{\varphi''(z)}=f(z),\qquad z\in C \tag{1—1}$$

$$n=\text{const.}$$

where C is the boundary curve of the region in which the boundary value problem is being solved. In the present article, Eqn. (1—1) will be further generalized and the theorems, missing in [8], will be proved. The method used is taken from J. D. Sherman [4], [5], 1940. In solving elastostatic problems, this method has shown a great deal of vitality and universality [6], and besides, it allows practical treatment. It is applicable in various forms also to more complicated problems and, with smaller adaptations, to problems of thin plate bending.

II. Definition of Boundary Value Problems

Let the finite region D be finitely muliply connected. The boundary curve is designated by

$$C=C_0\cup C_1\cup C_2\cup\cdots\cup C_N \tag{2—1}$$

where C_0 is the exterior boundary curve of the region. Further, the following task is to be accomplished:

Define holomorphic function $\varphi_0(z)$ and $\psi_0(z)$ in D with the expressions

$$\begin{aligned}\varphi(z)&=\sum_{k=1}^{k=N}A_k\ln(z-z_k)+\varphi_0(z)=\varphi_1(z)+\varphi_0(z)\\ \psi(z)&=-\varkappa\sum_{k=1}^{k=N}\overline{A}_k\ln(z-z_k)+\psi_0(z)=\psi_1(z)+\psi_0(z)\end{aligned} \tag{2—2}$$

with constants A_k, $1 \leqslant k \leqslant N$ so that the following might be fulfilled:

1.

$$\tilde{G}(\varkappa) = -\varkappa\varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} + \sum_{k=1}^{k=M} n_k \overline{\varphi^{(k)}(z)} = f(z) \tag{2—3}$$

$$z \in C, \qquad n_k = \text{const}, \qquad k = 1, 2, \ldots, M$$

where $f(z)$ is a given function on C and

2. $\varphi^{(M)}(z)$ and $\psi(z)$ are continuous functions on C.

With regard to the properties of the boundary curve C, we require that continuous derivatives of $M+2$ — order exist for $z(s)$; it must anywhere answer the condition $H\mu$ on C, while $f(z)$ should be a M-times continuously derivable function on C. Here, constants n_k are given and $\varkappa > 1$.

In the case of an infinite region outside C, the following requirements are added:

1.

$$\sum_{k=1}^{k=N} A_k = 0 \tag{2—4}$$

2. $\varphi_0(z)$ and $\psi_0(z)$ are holomorphic functions at $z = \infty$:

$$\begin{aligned} \varphi_0(z) &= a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \\ \psi_0(z) &= b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \end{aligned} \tag{2—5}$$

At $\varkappa = -1$, Eqn. (2—3) is substituted by the following equation for finite regions

$$\tilde{G}(-1) = \varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} + \sum_{k=2}^{k=M} n_k \overline{\varphi^{(k)}(z)} = f(z) + \beta_k \qquad z \in C_k \tag{2—6}$$

Instead of constants A_k, established before constants β_k must be determined.

C_k being encircled in the positive direction, the subsequent expression follows from (2—6)

$$-2\pi i(1+\varkappa) A_k = f(s_{k+1} - 0) - f(s_k + 0) \tag{2—7}$$

if the parameter s runs along C_k within the limits from s_k to s_{k+1}.

If the expressions for $\tilde{G}(-1)$ and $f(z)$ are designated by $\tilde{G}_1(-1)$ and $f_1(z)$, where $\varphi(z)$ and $\psi(z)$ are represented by functions $\varphi_1(z)$ and $\psi_1(z)$ from (2—2), we can write

$$\begin{aligned} \tilde{G}_0(-1) &= G_0(-1) + \sum_{k=2}^{k=M} n_k \overline{\varphi_0^{(k)}(z)} = f(z) - f_1(z) + \beta_k \\ G_0(-1) &= \varphi_0(z) + \overline{z\varphi_0'(z)} + \overline{\psi_0(z)} \end{aligned} \tag{2—8}$$

where $\tilde{G}_0(-1)$ represents the expression for $\tilde{G}(-1)$ for functions $\varphi_0(z)$ and $\psi_0(z)$. Knowing function $f_1(z)$ we know also

$$f_0(z)=f(z)-f_1(z), \qquad z\in C \tag{2—10}$$

The following requirement is made for this function

$$\mathrm{Re}\Big[\int_C f_0(z)\,d\bar{z}\Big]=0 \tag{2—11}$$

which is a momentous balance condition in the case of elastostatic problem and at the same time also a condition necessary to solve the present boundary value problem.

Eqn. (2—11) need not be fulfilled for the infinite region, which however does not go for (2—4) and (2—5).

III. Theorem of Uniqueness of Solution

Let $f(z)=0$ in (2—3). Let us examine the expression

$$I=\int_C\big[\tilde{G}(\varkappa)\overline{\varphi'(z)}\,d\bar{z}-\overline{\tilde{G}(\varkappa)}\,\varphi'(z)\,dz\big] \tag{3—1}$$

If we write

$$G(\varkappa)=-\varkappa\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)} \tag{3—2}$$

we obtain

$$I=\int_C\big[G(\varkappa)\overline{\varphi'(z)}\,d\bar{z}-\overline{G(\varkappa)}\,\varphi'(z)\,dz\big]+\sum_{k=1}^{k=M}n_k\int_C\big[\overline{\varphi'(z)}\,\overline{\varphi^{(k)}(z)}\,d\bar{z}-\varphi'(z)\,\varphi^{(k)}(z)\,dz\big]$$

Since

$$\int_C\varphi'(z)\,\varphi^{(k)}(z)\,dz=0$$

we have

$$I=\int_C\big[G(\varkappa)\overline{\varphi'(z)}\,d\bar{z}-\overline{G(\varkappa)}\varphi'(z)\,dz\big] \tag{3—3}$$

Following from the application of Stokes's expression [1] is

$$I=-4i\iint_D\big[(\varkappa-1)\,\mathrm{Re}^2[\varphi'(z)]+(\varkappa+1)\,I_m^2[\varphi'(z)]\big]\,dx\,dy \tag{3—4}$$

from (3—3) is $I=0$, from (3—4)

$$\begin{aligned}&\varphi'(z)=0, \qquad z\in D\cup C\\ &\varphi(z)=c_1=\text{const.}, \qquad z\in D\cup C\end{aligned} \tag{3—5}$$

With regard to (2—3)

$$\psi(z) = \varkappa \overline{c}_1, \qquad z \in D \cup C \tag{3—6}$$

where c_1 is an arbitrary complex constant.

With regard to (2—2) we have finally also

$$A_1 = A_2 = \cdots = A_N = 0 \tag{3—7}$$

Following from [1], (2—5), and (2—6) is an analogous result in the infinite region.

In the case of $\varkappa = -1$ and for finite regions we take

$$I = \int_C [\tilde{G}_0(-1)\overline{\varphi_0'(z)}\, d\overline{z} - \overline{\tilde{G}_0(-1)}\varphi_0'(z)\, dz] \tag{3—8}$$

By analogous deduction we obtain [1]:

$$I = \int_C [G_0(-1)\varphi_0'(z)\, d\overline{z} - \overline{G_0(-1}\, \varphi_0'(z)\, dz] = \tag{3—9}$$

$$= -2i \iint_D [\varphi_0'(z) + \overline{\varphi_0'(z)}]\, dxdy$$

wherefrom, in the presence of arbitrary constants β_k, follows at $f_0(z) = 0$:

$$\operatorname{Re}[\varphi_0'(z)] = 0, \qquad z \in D \cup C$$

and from the above

$$\varphi_0(z) = icz + c_1, \qquad z \in D \cup C \tag{3—10}$$

where c is an arbitrary real constant and c_1 an arbitrary complex one. From (2—9) we obtain at $f_0(z) = 0$

$$\psi_0(z) = -\overline{c}_1 + \overline{\beta}_k, \quad z \in C \tag{3—11}$$

and therefrom, we have first

$$\beta_0 = \beta_1 = \beta_2 = \cdots = \beta_N \tag{3—12}$$

If $\beta_0 = 0$, however, we obtain

$$\begin{aligned} \psi_0(z) &= -\overline{c}_1 \qquad z \in D \cup C \\ \beta_0 &= \beta_1 = \beta_2 = \cdots = \beta_N = 0 \end{aligned} \tag{3—13}$$

In a similar way, this is valid also for infinite regions, the only difference being due to requrement (2—5)

$$c = 0 \tag{3—14}$$

IV. Sherman's Method of Proving the Existence of Boundary Value Problem Solution

Let us first examine the case $\varkappa > 1$ and set up

$$\varphi_0(z) = \frac{1}{2\pi i} \int_C \frac{\omega(t)\, dt}{t - z}$$

$$\psi_0(z) = \frac{1}{2\pi i} \int_C \frac{-\varkappa \overline{\omega(t)} - \bar{t}\, \omega'(t)}{t - z}\, dt - \sum_{k=1}^{k=N} \frac{1}{2\pi i} \int_C \frac{k!\, \bar{n}_k\, \omega(t)\, dt}{(t - z)^{k+1}} \qquad z \in D$$

(4—1)
$$A_k = \int_{C_k} \omega(t)\, ds$$

For the time being, $\omega(t)$ should be a function defined on G and having a sufficient number of continuous derivations in any point on C. Following from (2—3) and (4—1) is

(4—2)
$$\tilde{G}_0(x) = G_0(x)$$

which means that in our case Sherman's equation [1] which is of the Fredholm type of second order equals that in the classical instance of boundary value problem of displacements.

(4—3)
$$\varkappa\, \omega(t_0) + \frac{\varkappa}{\pi} \int_C \omega(t)\, \vartheta'(t_0, t)\, dt + \frac{1}{\pi} \int_C \overline{\omega(t)}\, e^{2i\vartheta(t_0, t)} \cdot \vartheta'(t_0, t)\, dt =$$
$$= -f(t_0) - \varkappa \sum_{k=1}^{k=N} \left[\ln(t_0 - z_k) + \overline{\ln(t_0 - z_k)}\right] \int_{C_k} \omega(t)\, ds +$$
$$+ \sum_{k=1}^{k=N} \frac{t_0}{\bar{t}_0 - \bar{z}_k} \int_{C_k} \overline{\omega(t)}\, ds$$

If this equation is solvable, while $z(t)$, $t \in C$ is a $p+1$ — times continously derivable function where $z^{(p+1)}(t)$ fulfills the condition $H\mu$, then $\vartheta(t_0, t) = \arg(t - t_0)$ is p — times continously derivable in every possible way [2]. Following from (4—3) and the existence of the solution of the aforementioned equation is that $\omega(t)$ has p continuous derivations anywhere on C. Let us have

(4—4)
$$p - 1 = M$$

which is to be applied in the continuation of our study.

From the equation

$$(k)!\, \frac{1}{2\pi i} \int_C \frac{\omega(t)\, dt}{(t - z)^{k+1}} = \frac{1}{2\pi i} \int \frac{\omega^{(k)}(t)\, dt}{t - z}$$

follows

(4—5)
$${}_{J}(z) = \frac{1}{2\pi i} \int_C \frac{-\varkappa \overline{\omega(t)} - \bar{t}\, \omega'(t) - \sum_{k=1}^{k=M} \bar{n}_k\, \omega^{(k)}(t)}{t - z}\, dt$$

In order to prove the existence of the solution of Eqn. (4—3) the corresponding homogeneous one, $f(t_0)=0$ namely, has to be considered. Let it have solution $\omega_0(t)$.

From (4—1), (3—5) and (3—6) we have

$$0=\frac{1}{2\pi i}\int_C \frac{\omega_0(t)-c_1}{t-z}\,dt$$

$$0=\frac{1}{2\pi i}\int_C \frac{-\varkappa\,\overline{\omega_0(t)}-\bar{t}\,\omega_0'(t)-\sum_{k=1}^{k=M}\bar{n}_k\,\omega_0^{(k)}(t)-\varkappa\bar{c}_1}{t-z}\,dt \qquad z\in D \tag{4—6}$$

In connection with the function in square brackets, Plemelj's formulas [3] may be used. Hence it follows that two functions $\varphi^*(z)$ and $\psi^*(z)$, holomorphic outside region D, exist so that

$$\varphi^*(t)=i\,[\omega_0(t)-c_1]$$

$$\psi^*(t)=-i\left[\varkappa\,\overline{\omega_0(t)}+\bar{t}\,\omega_0'(t)+\sum_{k=1}^{k=M}\bar{n}_k\,\omega_0^{(k)}(t)+\varkappa\bar{c}_1\right] \qquad t\in C \tag{4—7}$$

and

$$\tilde{G}^*(x)=2\,i\,\varkappa\,c_1$$

$$\varphi^*(\infty)=\psi^*(\infty)=0 \tag{4—8}$$

Following from (2—6), (3—5), (3—6) and (3—7) is

$$\varphi^*(z)=\overset{*}{c}_k$$

$$\psi^*(z)=\varkappa\,\overline{\overset{*}{c}_k}, \qquad z\in C_k \tag{4—9}$$

where $\overset{*}{c}_k$, $k=0, 1\,\ldots, N$ are constants.

With regard to (4—8) $\overset{*}{c}_0=0$ and $c_1=0$ with regard to (4—7) $\omega_0(t)=0$, $t\in C_0$.

Following from (4—7) is also

$$\omega_0(t)=\text{const}, \; t\in C_k, \; 1\leqslant k\leqslant N.$$

and finally from (3—7) and (4—1)

$$\omega_0(t)=0, \qquad t\in C \tag{4—10}$$

Thus, the existence of solution of Eqn. (4—3) is proved.

In the problem for $\varkappa=-1$, expression (4—1) needs adapting. We take

$$\varphi_0(z)=\frac{1}{2\pi i}\int_C \frac{\omega(t)\,dt}{t-z}+\sum_{k=1}^{k=N}\frac{b_k}{z-z_k}$$

$$\psi_0(z)=\frac{1}{2\pi i}\int\limits_C \frac{\overline{\omega(t)}-\bar{t}\,\omega'(t)}{t-z}\,dt+\sum_{k=1}^{k=N}\frac{b_k}{z-z_k}-$$

$$-\sum_{k=2}^{k=M}\frac{1}{2\pi i}\int\limits_C \frac{k!\,\bar{n}_k\,\omega(t)\,dt}{(t-z)^{k+1}}-\sum_{l=1}^{l=N}\sum_{k=2}^{k=M}(-1)^k\frac{k!\,\bar{n}_k\,b_l}{(z-z_l)^{k+1}},\qquad z\in D$$

(4—11)
$$\beta_0=0,\qquad \beta_k=\int\limits_{C_k}\omega(t)\,ds,\quad 1\leqslant k\leqslant N$$

$$b_k=\frac{1}{2\pi i}\int\limits_{C_k}\left[\overline{\omega(t)}\,dt-\omega(t)\,d\bar{t}\right]$$

Due to the requirement in (2—11), Sherman's equation (4—3), too, has to be corrected. For the sake of a more appropriate form of expression, the corrected equation is written simply in the form of (2—8).

(4—12)
$$G_0(-1)+\sum_{k=2}^{k=M} n_k\,\overline{\varphi_0^{(k)}(z)}+izK=f_0(z)+\beta_k,\qquad z\in C_k$$

where

(4—13)
$$K=\frac{1}{2\pi}\int\limits_C\left[\frac{\omega(t)\,dt}{t^2}+\frac{\overline{\omega(t)}\,d\bar{t}}{\bar{t}^2}\right]-i\sum_{k=1}^{k=N} b_k\left[z_k^{-2}-\bar{z}_k^{-2}\right]$$

$(t=0\in D)$

From (4—12) we obtain

(4—14)
$$K\cdot i\int\limits_C z\,d\bar{z}=\mathrm{Re}\Big[\int\limits_C f_0(z)\,d\bar{z}\Big]$$

and

(4—15)
$$K=0\Leftrightarrow \mathrm{Re}\Big[\int\limits_C f_0(z)\,d\bar{z}\Big]=0$$

Let us examine the conditions at $f_0=0$. In that case, (4—15) is fulfilled. Let the solution of Sherman's equation be $\omega_0(t)$. Following from (4—11), (3—10) and (3—13) is

(4—16)
$$icz=\frac{1}{2\pi i}\int\limits_C\frac{\omega_0(t)\,dt}{t-z}+\sum_{k=1}^{k=N}\frac{b_k}{z-z_k}-c_1$$

(4—17)
$$0=\frac{1}{2\pi i}\int\limits_C\left[\frac{\overline{\omega_0(t)}-\bar{t}\,\omega_0'(t)+\sum_{k=1}^{k=N}\frac{b_k}{t-z_k}-\sum_{k=2}^{k=M}\bar{n}_k\,\omega_0^{(k)}(t)+\bar{c}_1}{t-z}-\right.$$

$$\left.-\frac{1}{t-z}\sum_{l=1}^{l=N}\sum_{k=2}^{k=M}(-1)^k\frac{k!\,\bar{n}_k\,b_l}{(t-z_l)^{k+1}}\right]dt,\qquad z\in D$$

Deriving Eqn. (4—16) with respect to z, for $z=0$, we have

$$ic = \frac{1}{2\pi i}\int_C \frac{\omega_0(t)}{t^2}\,dt - \sum_{k=1}^{k=N} b_k z_k^{-2}$$

Considering (4—13) and (4—15) we obtain

(4—18) $$c=0$$

Thus, (4—16) can be written in a more lucid form

(4—19) $$0 = \frac{1}{2\pi i}\int_C \frac{\omega_0(t) - \sum_{k=1}^{k=N} \frac{b_k}{t-z_k} - c_1}{t-z}\,dt$$

Following from (4—17) and (4—19) is that $\varphi^*(z)$ and $\psi^*(z)$ for which can be written

$$\varphi^*(t) = t\left[\omega_0(t) + \sum_{k=1}^{k=N}\frac{b_k}{t-z_k} - c_1\right]$$

(4—20) $$\psi^*(t) = i\left[\overline{\omega_0(t)} - \bar{t}\,\omega_0'(t) + \sum_{k=1}^{k=N}\frac{\bar{b}_k}{t-z_k} - \sum_{k=2}^{k=M}\bar{n}_k\,\omega_0^{(k)}(t) - \right.$$
$$\left. - \sum_{l=1}^{l=N}\sum_{k=2}^{k=M}(-1)^k\frac{\bar{n}_k b_l}{(t-z_l)^{k+1}} + \bar{c}_1\right], \qquad t\in C$$

are holomorphic functions outside D and

$$\varphi^*(\infty) = \psi^*(\infty) = 0$$

(4—21) $$\tilde{G}^*(-1) = -2ic_1 + i\left[\sum_{k=1}^{k=N}\frac{b_k}{t-z_k} - \overline{\sum_{k=1}^{k=N}\frac{b_k}{t-z_k}} - \sum_{k=1}^{k=N}\left(\frac{b_k}{t-z_k}\right)'\right]$$

By integration of the above equation over C_k, $1\leqslant k\leqslant N$ on the variable $\bar{z}$ follows

(4—22) $$b_k = 0, \qquad 1\leqslant k\leqslant N$$

whereupon the last equation of (4—21) reads like this:

(4—23) $$\tilde{G}^*(-1) = -2ic_1$$

Following from Eqns. (3—10) — (3—14) is

$$\varphi^*(z) = i\overset{*}{c}_k z + \overset{*}{c}_{1k}$$

(4—24) $$\psi^*(z) = \overset{*}{c}_{2k}$$

$$\overset{*}{c}_{2k} + \overline{\overset{*}{c}_{2k}} = -2ic_1$$

from (4—21)

(4—25) $$\overset{*}{c}_0 = 0$$

and from (4—22) also

(4—26) $$\overset{*}{c}_k = 0, \qquad 1 \leqslant k \leqslant N$$

and

(4—27) $$\omega_0(t) = \text{const.}, \quad t \in C_k, \quad 0 \leqslant k \leqslant N$$

From (4—24) and (4—21) we obtain also

(4—28) $$\overset{*}{c}_{10} = \overset{*}{c}_{20} = 0$$

and from the last equation of (4—24)

(4—29) $$c_1 = 0$$

Thus we have

(4—30) $$\omega_0(t) = 0, \qquad t \in C_0$$

and from (3—13), (4—11) and (4—27) also

(4—31) $$\omega_0(t) = 0, \quad t \in C_k \quad 1 \leqslant k \leqslant N$$

and together

(4—32) $$\omega_0(t) = 0, \qquad t \in C$$

The above proves the existence of solution of the treated problem at $\varkappa = -1$ and under the aforementioned conditions.

By insignificant alterations, it is possible to prove the existence of the problem solution for infinite regions as well. Taking the region D to be simply connected and $M = 2$, then our study proves the existence of solution of the first and the second boundary value problem in moderately thick plate bending. In the case of multiply connected regions, it is necessary to use the somewhat complete expressions (2—2), (4—11), yet no essential qualitative additions are needed.

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LÖSUNGEN EINIGER RANDWERTPROBLEME IN DER GENERALIZIERTEN ELASTOSTATIK

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Zusammenfassung

Ein Randwertproblem, das in der Anwendung als ein Randwertproblem bei der Biegung einer mässig dicken Platte ausgelegt werden kann, wird durch die Theorie der Funktionen einer komplexen Veränderlichen erörtert. Dabei ist die Lösungsexistenz nach der Methode von Šerman bewiesen, wobei das Lösungsverfahren auf die Fredholm'sche Integralgleichung zweiter Art übersetzt wird.

Received January 10, 1976.

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