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TRANSONIC CASCADE FLOWS AS A WEAK SOLUTION OF BOUNDARY VALUE PROBLEM (SMALL DISTURBANCE THEORY)

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Next contribution deals with mathematical model of two-dimensional cascade transonic flow in the frame of perturbation theory. The weak solution is defined and some properties of the solution are derived.

1. Consider transonic cascade flow with the governing equation for small disturbance transonic perturbation potential

(1) $$[1-M_\infty^2-(\varkappa+1)M_\infty^2\varphi_x]\varphi_{xx}+\varphi_{yy}=0^*$$

or in conservation form

(1 a) $$\left[(1-M_\infty^2)\varphi_x-\frac{1}{2}(\varkappa+1)M_\infty^2\varphi_x\right]_x+[\varphi_y]_y=0,$$

M_∞ is the upstream Mach number, $\varkappa$ is Poisson's constant. In small disturbance theory the flow tangency conditions at the body surface are linearised

(2) $$\varphi_y=f'_{h,d}(x),\ x\in\langle-1+nl_1,\ 1+nl_1\rangle,\ n=\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots;$$

(3) $$f_h(-1+nl_1)=f_d(-1+nl_1),\ f_h(1+nl_1)=f_d(1+nl_1).$$

$\bar{l}=(l_1,\ l_2)$ — vector period, $\operatorname{tg}\beta=l_2/l_1$, $\left(\frac{\pi}{2}-\beta\right)$ — stagger angle.

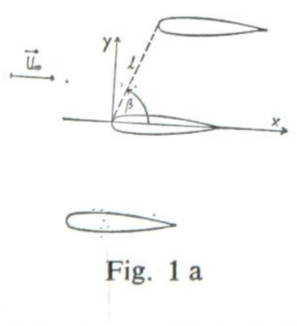

Fig. 1 a

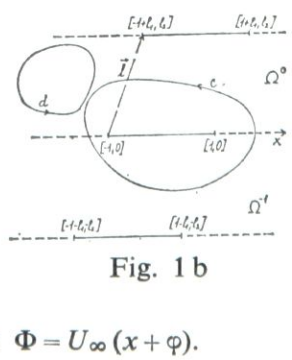

Fig. 1 b

* The Velocity potential is considered in the form $\Phi=U_\infty(x+\varphi)$.

We will define the classical solution of the problem which is the mathematical model of transonic cascade flows described by equation (1). l — periodicity of a velocity vector allows to formulate the problem only in the domain

$$x \in (-\infty, \infty),\ y \in \langle 0, l_2 \rangle, \tag{4}$$

respectively in the bounded domain $\Omega_G^0 \equiv G_1 G_2 G_3 G_4$,

$$G_1[-\bar{x}, 0],\ G_2[\bar{x}, 0],\ G_3[\bar{x}+l_1, l_2],\ G_4[-\bar{x}+l_1, l_2],\ \bar{x} \geqq 7, \tag{5}$$

because we know (from practical experience) that conditions at infinity are aproximitely realized in the finite distance ([1]). Periodicity conditions are fulfilled for φ_x, φ_y

$$\begin{aligned} &\varphi_x(x, 0) = \varphi_x(x+l_1, l_2), \\ &\varphi_y(x, 0) = \varphi_y(x+l_1, l_2),\ x \in (-\infty, -1) \cup (1, \infty). \end{aligned} \tag{6}$$

Conditions at $-\infty$ (at $\overline{G_1 G_2}$):

$$\varphi_x = \varphi_y = 0. \tag{7}$$

Conditions at $+\infty$ are unknown in advance, the conditions are considered as a part of the solution. As the flow at $+\infty$ is homogeneous and paralell, we consider

$$\varphi_x = K_1,\ \varphi_y = K_2,\ x \to +\infty\big((x, y) \in \overline{G_2 G_3}\big) \tag{8}$$

K_1, K_2 are unknown constants. Green's formula for conservation form (1 a) leads to a relation

$$0 = \iint_{\Omega_G^0} \left\{ \left[(1 - M_\infty^2)\varphi_x - \frac{1}{2}(\varkappa+1) M_\infty^2 \varphi_x^2 \right]_x + [\varphi_y]_y \right\} dx\, dy =$$

$$= \oint_{\partial \Omega_G^0} \left[(1 - M_\infty^2)\varphi_x - \frac{1}{2}(\varkappa+1) M_\infty^2 \varphi_x^2 \right] dy - \varphi_y\, dx =$$

$$= \int_{G_1 G^2} + \int_{G_2 G_3} + \int_{G_3 G_4} + \int_{G_4 G_1}.$$

But

$$\int_{G_1 G_4} = 0\ (\varphi_x = \varphi_y = 0,\ (x, y) \in \overline{G_1 G_4}),$$

$$\int_{G1\, G_2} + \int_{G_3 G_4} = 0 \text{ (periodicity conditions (6), property (3)).}$$

It means that

$$\int_{G_2 G_3} \left[(1 - M_\infty^2)\varphi_x - \frac{1}{2}(\varkappa+1) M_\infty^2 \varphi_x^2 \right] dy - \varphi_y\, dx = 0. \tag{9}$$

As $\varphi_x = K_1$, $\varphi_y = K_2$ at $\overline{G_2 G_3}$, (9) can be fulfilled only for

$$\left[(1-M_\infty^2)K_1 - \frac{1}{2}(\varkappa+1)M_\infty^2 K_1^2\right]\operatorname{tg}\beta - K_2 = 0,\ dy = \operatorname{tg}\beta\cdot dx. \tag{10}$$

Next condition for K_1, K_2 will be relation for circulation γ of perturbation velocity vector

$$K_2\sin\beta + K_1\cos\beta = -\gamma/l,\ l = \sqrt{l_1{}^2 + l_2{}^2}, \tag{11}$$

$$\gamma = -\oint_c \varphi_x\,dx + \varphi_y\,dy,$$

c is any smooth curve, enclosing the slit $y=0$, $|x|<1$, $c\in\Omega^0\cup\Omega^{-1}$ (see fig 1 b). The system (10), (11) gives two solutions, but we want the solution which allows for $\gamma=0$ only $K_1 = K_2 = 0$. Then

$$\begin{aligned} K_1 &= [(\varkappa+1)M_\infty^2]^{-1}\cdot\{[1-M_\infty^2+\operatorname{cotg}^2\beta] - \\ &-\sqrt{[1-M_\infty^2+\operatorname{cotg}^2\beta]^2 + 2(\varkappa+1)M_\infty^2\gamma\cos\beta/l\sin^2\beta} \\ K_2 &= -\left(\frac{\gamma}{l} + K_1\cos\beta\right)(\sin\beta)^{-1} \\ \gamma &\geqslant -\frac{1}{2}l\sin^2\beta\,[1-M_\infty^2+\operatorname{cotg}^2\beta]\ \ [(\varkappa+1)M_\infty^2\cos\beta]^{-1} \end{aligned} \tag{12}$$

As we consider a sharp trailing edge we must satisfy the Joukowski condition. Within the framework of small disturbance theory, this reduces to a requirement

$$\begin{aligned} \varphi_x(x, 0+) &= \varphi_x(x+l_1, l_2-0) \\ \varphi_y(x, 0+) &= \varphi_y(x+l_1, l_2-0),\ x>1. \end{aligned} \tag{13}$$

Now we can define the classical solution of our problem.

Definition 1: *Let $\varphi\in C^2(\Omega^0)$ satisfies equation (1) in Ω^0, conditions* (2), (3), (6), (7), (8) *and the Joukowski condition. Then we say that φ is the classical solution of our problem in Ω^0.*

Definition 2: *Let $\varphi\in C^2(\Omega_G^0)$ satisfies equation* (1) *in Ω_G^0, conditions* (2), (3), (6), (7), (8) *and the Joukowski condition. Then we say that φ is a classical solution of our problem in Ω_G^0.*

It is evident that next theorem is valid.

Theorem 1: *Let φ be the solution given by D1 (D2). Then $\varphi_x = K_1$, $\varphi_y = K_2$ satisfy conditions* (12) *at $+\infty$ $(\overline{G_2 G_3})$.*

2. The equation (1) is nonlinear equation of mixed elliptic-hyperbolic type then the solution of our problem will not be necessary continuous. We have to expect the possibility of the existence disconinuity in φ_x, φ_y along some curve (shock wave). Then our classical definition would not be suitable. We must find a generalized definition, definition of a weak solution. We use in this definition conservation law (1 a) in integral form

$$\oint_d \left[(1 - M_\infty^2)\varphi_x - \frac{1}{2}(\varkappa + 1) M_\infty^2 \varphi_x^2 \right] dy - \varphi_y \, dx = 0 \tag{14}$$

d is any closed curve in Ω^0, Ω^0_G. From the condition of vortexfree motion, i.e. $[\varphi_x]_y - [\varphi_y]_x = 0$, we consider

$$\oint_d \varphi_x \, dx + \varphi_y \, dy = 0 \tag{15}$$

Let φ_x, φ_y be discontinuous along some curve r. Then relations (14), (15) give

$$\left\langle (1 - M_\infty^2)\varphi_x - \frac{1}{2}(\varkappa + 1) M_\infty^2 \varphi_x^2 \right\rangle dy_r - \langle \varphi_y \rangle dx_r = 0, \tag{16}$$

$$\langle \varphi_x \rangle \, dx_r + \langle \varphi_y \rangle \, dy_r = 0, \tag{17}$$

where $\langle \varphi_x \rangle = \varphi_{x/1} - \varphi_{x/2}$ — a jump in the value across the shock wave. Function φ is continuous along that curve r, as we can see from (15), (17) and φ_x, φ_y satisfy relation (16) along curve r.

Now we define class of functions $K(\Omega^0)$:

a) $\varphi \in C(\Omega^0)$

b) φ_x, φ_y are discontinuous only along a curves r or in some isolated points Ω^0. The set of the curves and the points has finite number of elements localized in a bounded domain. Functions φ_x, φ_y have bounded limits from different sides of a curve r. Functions φ_x, φ_y, φ_{xx}, φ_{yy}, φ_{xy} are continuous in other points of domain Ω^0.

c) $|\varphi_x| \leqq L_1$, $|\varphi_y| \leqq L_2$; L_1, L_2 — positive real numbers.

The function $\varphi \in K(\Omega^0)$ is element $K_l(\Omega^0)$ when relation (6) is fulfilled.

Definition 3: *Consider $\varphi \in K_l(\Omega^0)$. Let φ satisfies relation* (14), *conditions* (2), (3), (6), (7), (8) *and the Joukowski condition. Then we say that φ is the weak solution of our problem in Ω^0.*

Definition 4: *Consider $\varphi \in K_l(\Omega^0_G)$ Let φ satisfies relation* (14), *conditions* (2), (3), (6), (7), (8) *and the Joukowski condition. Then we say that φ is the weak solution of our problem in Ω^0_G.*

Remark 1: We can see that any classical solution given by $D\,1$ satisfies $D\,3$, i.e. any classical solution is the weak solution. But there exists a weak solution given by $D\,3$ which does not satisfy $D\,1$.

3. Now we would like to find more suitable form of defined weak solution. Consider a function φ given by $D\,3$ ($D\,4$). That function satisfies the equation (1) in all points of continuity Ω^0, and relation (16) along a curve r. Green's formula used in the domain $\Omega^0\,(\Omega^0_G)$ gives (with consideration of the described discontinuity along r)

$$0=\iint\limits_{\Omega^0_G}\left\{\left[(1-M^2_\infty)\,\varphi_x-\frac{1}{2}(\varkappa+1)\,M^2_\infty\,\varphi^2_x\right]_x+[\varphi_y]_y\right\}dx\,dy=$$

$$=\oint\limits_{\partial\Omega^0_G}\left\{\left[(1-M^2_\infty)\,\varphi_x-\frac{1}{2}(\varkappa+1)\,M^2_\infty\,\varphi^2_x\right]dy+-\varphi_y\,dx\right\}\psi\,(x,\,y)+$$

$$+\int\limits_r\left[\left\{(1-M^2_\infty)\,\varphi_x-\frac{1}{2}(\varkappa+1)\,M^2_\infty\,\varphi^2_x\right]dy-\varphi_y\,dx\right\}\psi\,(x,\,y)+$$

$$+\iint\limits_{\Omega^0_G}\left\{\left[1-M^2_\infty-\frac{1}{2}(\varkappa+1)\,M^2_\infty\,\varphi_x\right]\varphi_x\psi_x+\varphi_y\,\psi_y\right\}dx\,dy.$$

By using (2), (3), (6), (7), (8) and (16) we derive

$$\iint\limits_{\Omega^0_G}\left\{\left[1-M^2_\infty-\frac{1}{2}(\varkappa+1)\,M^2_\infty\,\varphi_x\right]\varphi_x\psi_x+\varphi_y\,\psi_y\right\}dx\,dy=$$

(18)

$$=\int\limits_{-1}^{1}f_h\cdot\psi_x\,(x,\,0+)\,dx-\int\limits_{-1}^{1}f_d\cdot\psi_x\,(x+l_1,\,l_2-0)\,dx$$

for every l — periodical* function $\psi\in K(\Omega^0)$ resp. $K(\Omega^0_G)$ that the double integral in (18) is convergent.**

On the contrary, every $\varphi\in K_l(\Omega^0)$ resp. $\varphi\in K_l(\Omega^0_G)$ satisfying (18), (7), (8) and the Joukowski condition satisfies our definition of a weak solution given by $D\,3$ ($D\,4$).

Definiton 5: *Consider $\varphi\in K_l(\Omega^0)$. Let relation* (18) *be valid for every suitable l — periodical $\psi\in K(\Omega^0)$. Let* (7), (8) *with the Joukowski condition be satisfied. Then φ is the weak solution of our problem in Ω^0.*

Definition 6: *Consider $\varphi\in K_l(\Omega^0_G)$. Let relation* (18) *be valid for every l — periodical $\psi\in K(\Omega^0_G)$. Let* (7), (8) *with the Joukowski condition be satisfied. Then φ is the weak solution of our problem in Ω^0_G.*

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* $\psi\,(x,\,0)=\psi\,(x+l_1,\,l_2),\ |\,x\,|>1$.

** ... that double integral in (18) converges when $(\Omega^0_G\to\Omega^0)$.

Remark 2: Theorem 1 is valid for function φ given by $D\,5$ ($D\,6$).

Theorem 2: *The definitions $D\,5$ and $D\,3$ are equaivalent, the same assertion is true for $D\,6$ and $D\,4$.*

4. *Basic properties of the weak solution*

Lemma: *Let $\varphi_1(x, y)$, $\varphi_2(x, y)$ be functions fulfilling conditions b), c) of the definition $K(\Omega^0)$. Then*

$$\iint\limits_{\Omega_G^0} (\varphi_1 \psi_x + \varphi_2 \psi_y)\, dxdy = 0$$

for every l — periodical $\psi \in K(\Omega^0)$ if and only if $\varphi_1 = \varphi_2 = 0$.

Theorem 3: *Consider $f_h = f_d = 0$. Then our weak solution is $\varphi =$const., i.e. $\varphi_x = \varphi_y = 0$.*

Proof: From $f_h = f_d = 0$ and (18) we get

$$\iint\limits_{\Omega_G^0} \left\{ \left[1 - M_\infty^2 - \frac{1}{2} (\varkappa + 1) M_\infty^2 \varphi_x \right] \varphi_x \psi_x + \varphi_y \psi_y \right\} dx\, dy = 0 \tag{19}$$

for every l — periodical $\psi \in K(\Omega^0)$. The lemma and relation (19) give

$$\left[1 - M_\infty^2 - \frac{1}{2} (\varkappa + 1) M_\infty^2 \varphi_x \right] \varphi_x = 0, \quad \varphi_y = 0.$$

$\varphi_x = \varphi_y = 0$ along $\overline{G_1 G_4}$ and in the small vicinity of $\overline{G_1 G_4}$. As $\varphi_y = 0$, then $\varphi_x = \varphi_y = K_1 = K_2 = 0$ along $\overline{G_2 G_3}$ and in the small vicinity of $\overline{G_2 G_3}$. But lemma and (19) give $\varphi_x = 0$ or $\varphi_x = 2(1 - M_\infty^2)/(\varkappa + 1) M_\infty^2$ only. If we would consider $\varphi_x = 2(1 - M_\infty^2)/(\varkappa + 1) M_\infty^2$ in a part of the domain Ω_G^0, we would consider two jumps in the value of φ_x, from $\varphi_{x_1} = 0$ to $\varphi_{x_2} = 2(1 - M_\infty^2)/(\varkappa + 1) M_\infty^2$ and then from φ_{x_2} to φ_{x_1}, because $\varphi_x = 0$ along $\overline{G_2 G_3}$. But only one of these jumps is allowed (see [2]) as we know from the theory of differential equations (see [2]). Then $\varphi_x = \varphi_y = 0$ in $\Omega_G^0(\Omega^0)$.

Remark 3: Consider $f_h = f_d = 0$, then the upstream parallel flow field does not change.

The next theorem implies from lemma and $D\,5$.

Theorem 4: *Consider $f_h \neq 0$ or $f_d \neq 0$. Then only one of the relations $\varphi_x = 0$, $\varphi_y = 0$ can be fulfilled in every subdomain of the domain Ω^0 (Ω_G^0).*

Theorem 5: *Consider two weak solutions $\varphi(x, y)$, $\tilde{\varphi}(x, y)$ for given f_h, f_d. Then $\varphi - \tilde{\varphi} =$const., i.e. $\varphi_x = \tilde{\varphi}_x$, $\varphi_y = \tilde{\varphi}_y$ in $\Omega_G^0(\Omega^0)$.*

Proof: Let φ and $\tilde{\varphi}$ fulfill $D\,5$. Then relation

$$(20)\qquad \iint\limits_{\Omega_G^0} \left\{\left(1 - M_\infty^2 - \frac{\varkappa+1}{2} M_\infty^2 [\varphi_x + \tilde{\varphi}_x]\right)(\varphi_x - \tilde{\varphi}_x)\,\psi_x + (\varphi_y - \tilde{\varphi}_y)\,\psi_y\right\} dxdy = 0$$

is satisfied for every l — periodical $\psi \in K(\Omega^0)$. The relation (20) and the lemma give

$$\operatorname{lub} |\varphi_y - \tilde{\varphi}_y| = 0$$

$$(21)\qquad \operatorname{lub} |\varphi_x - \tilde{\varphi}_x| \left| 1 - M_\infty^2 - \frac{\varkappa+1}{2} M_\infty^2 (\varphi_x + \tilde{\varphi}_x) \right| = 0.$$

The upstream conditions: $\varphi_x = \tilde{\varphi}_x = 0$, $\varphi_y = \tilde{\varphi}_y = 0$. From (21) and (12) we get $K_2 = \tilde{K}$, $K_1 = \tilde{K}_1$. The second condition (21) will be satisfied for

$$\operatorname{lub} |\varphi_x - \tilde{\varphi}_x| = 0 \quad \text{or} \quad 0 = \operatorname{lub} \left| 1 - M_\infty^2 - \frac{\varkappa+1}{2} M_\infty^2 (\varphi_x + \tilde{\varphi}_x) \right| \quad \text{in } D \subset \Omega_G^0.$$

The condition $\operatorname{lub} 1 - M_\infty^2 - \frac{\varkappa+1}{2} M_\infty^2 (\varphi_x + \tilde{\varphi}_x) | = 0$ can be eliminated by the way as condition $\varphi_x = \frac{2}{\varkappa+1} \cdot \frac{1 - M_\infty^2}{M_\infty^2}$ in the proof of theorem 3.

The similar way is successfull in proving next three theorems:

Theorem 6: *A small change in the boundary conditions* ($|f_h - \tilde{f}_h| + |f_d - \tilde{f}_d| < \varepsilon$) *implies a small change in the weak solution*

$$\left\{ \iint\limits_{\Omega_G^0} [(\varphi_x - \tilde{\varphi}_x)^2 + (\varphi_y - \tilde{\varphi}_y)^2]\, dxdy \right\}^{1/2} < k_1 \cdot \varepsilon$$

Theorem 7: *A small change in* M_∞^2 ($|M_\infty^2 - \widetilde{M}_\infty^2| < \varepsilon$) *implies a small change in the weak solution for* $M_2^2 \leq \lambda < 1$ (M_∞ — *Mach number at* $-\infty$, $\overline{G_1 G_4}$, M_2 — *Mach number at* $+\infty$, $\overline{G_2 G_3}$).

Theorem 8: *Consider a weak solution* φ *in* Ω^0 *and* $\tilde{\varphi}$ *in* Ω_G^0 *for the same* f_h, f_d. *Then for every* $\varepsilon > 0$ *a domain* Ω_G^0 *exists that*

$$\left\{ \iint\limits_{\Omega_G^0} [(\varphi_x - \tilde{\varphi}_x)^2 + (\varphi_y - \tilde{\varphi}_y)^2]\, dxdy \right\}^{1/2} < k_2 \cdot \varepsilon.$$

Theorem 8 is very important for numerical solution two-dimensional transonic cascade flows.

Conclusion

We can say that our transonic cascade problem as a weak solution is uniqueness and continuously dependent on the shape of blades and upstream conditions given by upstream Mach number. We do not prove the existence of our problem in the given class of functions. But we can hope that a solution exists as we can presume from the first numerical results achieved in the last time.

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ОБОБЩЕНОЕ РЕШЕНИЕ КРАЕВОЙ ЗАДАЧИ ОКОЛОЗВУКОГО ОБТЕКАНИЯ ДВУМЕРНОЙ РЕШЕТКИ ТОНКИХ ПРОФИЛЕЙ

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Резюме

Равота занимается формулировкой проблемы двумерного околозвукового обтекания решетки тонких профилей как краевой задачи для уравнения смешаного эллиптическо-гиперболического типа. Задача есть однозначно разрешима и решение непреривно зависит от формы обтекаемых профилей в решетке и от входного числа Маха $M_1 < 1$.

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