TEORIJSKA I PRIMENJENA MEHANIKA 2, p.p. 37–40, 1976.

THE BRACHISTOCHRONIC MOTION OF A GYROSCOPE MOUNTED ON THE GIMBALS

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Introduction

Let us consider a dynamical system which is required to pass from a given configuration A to another given configuration B. It is under the action of a given conservative field of force and its total mechanical energy is assigned. It is required to find a workless constraint which will carry it from A to B in a minimal-stationary time. The corresponding trajectory of the system is known as a brachistochrone, and the motion is called brachistochronic. The workless reactions of the constraint are in fact control forces. These forces protect the optimality of the motion, that is, the minimality of the time taken in passing from one configuration to the other.

John Bernoulli (1696) was the first who studied a problem of this kind. He considered the brachistochronic motion of a particle under the influence of gravity. His results have been generalized, for a mechanical system of n degrees of freedom, by Pennachietti [1], McConnell [2] and Đukic [3]. The brachistochronic motion of the Lagrange's gyroscope was analyzed in the references [2] and [3].

The rigid body dynamics is a well elaborated part of the dynamics (see for an extensive review [5]). The brachistochronic motion of the rigid bodies has received a small attention although the problems of this kind have been of the practical interest in recent years.

Motion of a gyroscope mounted on the gimbals is an interesting and, at the same time practical problem (see for example [6] — [8]). Here, the brachistochronic motion of the gyroscope is analyzed.

In this paper the summation convention will be observed and small italic indices imply a range of values from 1 to n.

Let us consider a holonomic, scleronomic conservative mechanical system with n-degrees of freedom, where the q^i are regarded as the generalized coordinates, whose kinetic energy is

$$T=\frac{1}{2} g_{ij}(q^r)\,\dot{q}^i\,\dot{q}^j, \quad (g_{ij}=g_{ji};\ \cdot \equiv d/dt), \tag{1}$$

where t is time and g_{ij} are function of the q' s only. This system is acted upon by a conservative field of force of potential energy $\Pi(q^r)$ and by the generalized control forces-control variables u_i. We shall assume that throughout any motion under consideration the energy equation

$$T+\Pi=h, \tag{2}$$

where h is a constant, always holds. This means that the control forces u_i are workless, i.e., we must have $u_i\dot{q}^i=0$. According to [2] and [3], brachistohronic motion of the mechanical system under consideration is described by the differential equations

$$\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}^r}-\frac{\partial T_1}{\partial q^r}=0 \tag{3}$$

and realized with the help of the workless optimal control forces

$$u_i=2\frac{\partial \Pi}{\partial q^i}-\frac{1}{h-\Pi}\frac{\partial T}{\partial \dot{q}^i}\frac{\partial \Pi}{\partial q^r}\dot{q}^r, \tag{4}$$

where

$$T_1=\frac{T}{h-\Pi}, \tag{5}$$

and where $T_1=1$ during the motion.

Analysis

Let us consider a gyroscope mounted on the gimbals, where (for more details see [6] — [8]): x_1, y_1, z_1 are rectangular axes fixed in space; ψ is rotation angle of the external ring; θ is rotation angle of the hoop, which is situated in the ring; x, y, z are rectangular axes fixed relatively to the hoop; φ is rotation angle of the gyroscope with respect to the hoop; I is moment of inertia of the ring with respect to the z_1 axis: A°, B°, C° are moments of inertia of the hoop with respect to the x, y, z axes; A, A, C are moments of inertia of the gyroscope with respect to the x, y, z axes; g is acceleration of gravity; m is the mass of the gyroscope, ξ is the distance of the gyroscope's centre of gravity from the fixed point.

In this case the function (5) is of the following form

$$T_1=\{\dot{\psi}^2[I+(A+B^\circ)\sin^2\theta+C^\circ\cos^2\theta]+(A+A^\circ)\dot{\theta}^2+c(\dot{\varphi}+\dot{\psi}\cos\theta)^2\}$$
$$\times[2(h-mg\xi\cos\theta)]^{-1};\quad T_1=1. \tag{6}$$

The equations (3) of the brachistochronic motion have two first integrals (φ and ψ are the cyclic coordinates)

$$\frac{c(\dot{\varphi}+\dot{\psi}\cos\theta)}{h-mg\xi\cos\theta}=r_0, \tag{7}$$

$$\frac{\dot{\psi}[I+(A+B^\circ)\sin^2\theta+C^\circ\cos^2\theta]+c(\dot{\varphi}+\dot{\psi}\cos\theta)\cos\theta}{h-mg\xi\cos\theta}=k, \tag{8}$$

where r_0 and k are constants. Combining (5) — (8) we have

$$\dot{u}^2 = \frac{f(u)}{\varepsilon - eu^2} = \frac{(1-u^2)(\alpha - au)}{\varepsilon - eu^2}\{(\varepsilon - eu^2)[2 - r_0^2\, b(\alpha - au)] - (k - r_0)^2(\alpha - au)\},$$

(10) $$\dot{\psi} = \frac{(k - r_0)(\alpha - au)}{\varepsilon - eu^2}; \qquad \dot{\varphi} = r_0 b(\alpha - au) - u\dot{\psi},$$

where:

$$u = \cos\theta; \quad \alpha = \frac{h}{A + A^\circ}; \quad a = \frac{mg\xi}{A + A^\circ} > 0;$$

(11) $$\varepsilon = \frac{I + A + B^\circ}{A + A^\circ} > 0; \quad \varepsilon = \frac{A + B^\circ - C^\circ}{A + A^\circ}; \quad b = \frac{A + A^\circ}{C} > 0.$$

The equation (9) shows that u is an elliptic function of t. If this equation is solved, so that u is a known function of t, than φ and ψ can be obtained from (10) by quadratures.

Further, let us consider a special case when the moments of inertia are satisfying the following condition

(12) $$A + B^\circ - C^\circ = 0.$$

In this case (9) and (10) reduce into

(13)
$$\frac{du}{dt} = \sqrt{\frac{2a}{\beta}}\sqrt{(u - u_1)(u^2 - 1)(u - u_2)};$$
$$\frac{d\psi}{du} = \frac{r_0 - k}{\varepsilon}\sqrt{\frac{a\beta}{2}}\sqrt{\frac{u - u_1}{(u^2 - 1)(u - u_2)}};$$
$$\frac{d\varphi}{du} = \frac{k - r_0}{\varepsilon}\sqrt{\frac{a\beta}{2}}(u - \nu)\sqrt{\frac{u - u_1}{(u^2 - 1)(u - u_2)}},$$

where

(14)
$$u_1 = \frac{\alpha}{a}; \quad \beta = \frac{2(I + A + B^\circ)}{mg\xi\left[(k - r_0)^2 + \frac{r_0^2}{C}(I + A + B^\circ)\right]} > 0;$$
$$u_2 = u_1 - \beta; \quad \nu = \frac{r_0(I + A + B^\circ)}{C(k - r_0)}.$$

In a special case, when the constant u_1 is equal to one, equations (13) have the following solution

(15)
$$\psi = \frac{r_0 - k}{\varepsilon}\sqrt{\frac{a\beta}{2}}\ln[2R(u) + 2u + \beta] + C_1;$$
$$\varphi = \frac{k - r_0}{\varepsilon}\sqrt{\frac{a\beta}{2}}\left\{R(u) - \left(\nu + \frac{\beta}{2}\right)\ln[2R(u) + 2u + \beta]\right\} + C_2;$$
$$\sqrt{\frac{2a}{\beta}}\,t = C_3 - \frac{1}{\sqrt{2\beta}}\ln\frac{4\beta + (2 + \beta)(u - 1) + 2R(u)\sqrt{2\beta}}{u - 1},$$

where

$$R(u)=\sqrt{u^2+\beta u-(1-\beta)} \tag{16}$$

and C_1, C_2, C_3 are integration constants. Equations (15) are equations of the brachistochronic motion of the gyroscope mounted on the gimbals. The integration constants C_1, C_2, C_3, r_0, k and the minimal time τ, necessary to transfer the gyroscope from a state $\theta(0)=\theta_0$, $\varphi(0)=\varphi_0$; $\psi(0)=\psi_0$ into a desired state $\theta(\tau)=\theta_\tau$, $\varphi(\tau)=\varphi_\tau$; $\psi(\tau)=\psi_\tau$, may be found if (15) is substituted into these equations. It is not an easy task to solve so obtained complicated algebraic problem.

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БРАХИСТОХРОНОЕ ДВИЖЕНИЕ ГИРОСКОПА В КАРДАНОВОМ ПОДВЕСЕ

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Резюме

Анализированно брахистохроное движение гироскопа в Кардановом подвесе при влиянии силы тяжести. В одном спепиальном случае принято решение проблеми.

(Received March 26. 1976)

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