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COMPOSITE MATERIALS AND MECHANICS WITH INTERNAL DEGREES OF FREEDOM

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Many continuum theories have been proposed which are generalized in the sense that, besides the mean displacement field, additional degrees of freedom are assumed (micromorphic, micropolar, multipolar theories). The aim is to characterize the internal structure of the body more in detail. Especially, in some cases this approach has been used for describing heterogeneous materials, for instance laminated or fibre-reinforced bodies. Apart from such well established applications, there seems to be a lack concerning a sufficient physical interpretation of the additional degrees of freedom. In the same way, physical reasons for the accepted constitutive relations are desirable.

This state of affairs suggests an attempt of deriving such a generalized continuum theory by means of remodeling and averaging the exact microscopic equations. The notation "microscopic" does not refer to the atomic scale but to the characteristic length of heterogeneity.

We consider inclusions being stochastically distributed within a matrix M. The microscopic equation of motion is written as

$$Lu = f = [L_M + \sum_i L_i]\, u \tag{1}$$

where L is the operator of linear elasticity, which, in the present case, is a stochastic operator, with piecemeal constant coefficients. L_i describes, in a suitable way, how the properties of the i-th inclusion deviate from those of the matrix. Outside the i-th grain L_i acts as null-operator.

Equ. (1) can be rearranged into

$$u = G_M \{ f + \sum_i [L_i (G_i f + \sum_\alpha \varphi_{\alpha i} Q_{\alpha i})]\} \tag{2}$$

together with

$$Q_{\alpha i} = B[f] + \sum_\beta \sum_{j(\neq i)} A_{\alpha\beta\, ij}\, Q_{\beta j}\,. \tag{3}$$

Here, G_M, G_i are Green-operators for the pure matrix and the homogeneous material of the i-th inclusion, respectively. $\{\varphi_{\alpha i}\}$ is a certain system of solutions of

$$L \varphi_{\alpha i} = 0$$

restricted to the interior of the i-th inclusion. $Q_{\alpha i}$ are called the "multipoles" of the i-th inclusion, which are governed by the infinite system (3) where $B_{\alpha i}$,

$A_{\alpha\beta ij}$ are well defined quantities. While Equs. (2) and (3) describe the microscopic problem in a new fashion, we are interested in the mean displacement field $\langle u \rangle$. Performing the ensemble average of Equs. (2) and (3), besides $\langle u \rangle$ we are confronted with conditional averages of the multipoles under the condition that the i-th inclusion is placed at the position R_i. In the case of equal inclusions these averages can be denoted by $Q_\alpha(R_i)$. They are considered as the additional macroscopic degrees of freedom.

From (3) we obtain a hierarchy

$$\begin{aligned} \langle Q \rangle &= \langle B \rangle + \langle AQ \rangle \\ \langle AQ \rangle &= \langle AB \rangle + \langle AAQ \rangle \\ \langle AAQ \rangle &= \langle AAB \rangle + \langle AAAQ \rangle \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{4}$$

where, for the sake of clarity, all indices and sums are omitted. This hierarchy will be closed by putting $Q_{\alpha i} \to Q_\alpha(R_i)$ anywhere. Then successive substitution finally leads to equations of the type

$$\begin{aligned} Q_\alpha(r) &= b_\alpha(r) + \sum_\beta \int d^3r' \, a_{\alpha\beta}(r, r') \, Q_\beta(r') \\ \langle u(r) \rangle &= c(r) + \sum_\alpha \int d^3r' \, d_\alpha(r, r') \, Q_\alpha(r'). \end{aligned} \tag{5}$$

In the case of statistical homogeneity in Fourier-space an algebraic system is achieved:

$$\begin{aligned} Q_\alpha(k) &= \tilde{b}_\alpha(k) + \sum_\beta \tilde{a}_{\alpha\beta}(k) \, Q_\beta(k) \\ \langle u(k) \rangle &= \tilde{c}(k) + \sum_\alpha \tilde{d}(k) \, Q_\alpha(k). \end{aligned} \tag{6}$$

The coefficients depend on the statistical distribution of the grain positions. If the hierarchy (4) is broken off after the n-th equation, the distribution function for n grains is needed. The further exploration has been done under the following restrictions: $f=0$, i.e. free waves, spherical inclusions, and most important, small k, i.e. large wave-lengths in comparison with the correlation-length 1.

The inspection of the coefficients in (6) is a cumbersome task. The main results are as follows: The leading terms of the coefficients are certain powers $(kl)^n$, with $n=n(\alpha)$. Thus deciding for an accuracy up to a certain order in kl it can be shown that most of the multipoles are negligible. Therefore we are left, approximatively, with a finite number of additional degrees of freedom. However, besides powers of kl in $\tilde{a}_{\alpha\beta}(k)$ there appear terms like $(kl)^m \log kl$. This means, that the coefficients in question cannot be expanded into power series with arbitrarily high accuracy. Therefore, in turn, by retransformation into the physical space the kernel $a_{\alpha\beta}(r)$ of (5) can be treated as a differential operator only with a restricted accuracy.

In other words, as a matter of principle, a local theory being described by differential-equations for the additional fields is meaningful only up to a certain order. It turns out that the relative accuracy, which can be achieved by a local theory, is of the order $(kl)^4$.

VERBUNDWERKSTOFFE UND MECHANIK MIT INNEREN FREIHEITSGRADEN

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Die mikroskopische Bewegungsgleichung für einen Verbund mit Einschlüssen in der Matrix wird umgeformt. Durch Mittelungsprozesse wird eine makroskopische Mechanik hergeleitet, die neben dem mittleren Verschiebungsfeld noch "Multipolfelder" enthält. Eine lokale, durch Differentialgleichungen beschriebene Theorie ist prinzipiell auf den langwelligen Bereich beschränkt.

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