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TOPOLOGICAL FORMULATIONS IN ESTABLISHING FLEXIBILITY MATRIX FOR SPACE FRAMES

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The present paper deals with solving space frames under the assumption of rigid joints, as well as of elastic ties of the bars in joints.

The flexibility matrix method consists in expressing the element displacements as function of forces by means of *element flexibility matrix* $[\alpha]$. Besides it, the *transformation matrix* $[B]$ is also involved, and it expresses the forces in the bars as function ox external and unknown forces, on the statically determined basic system.

The *flexibility matrix of the structure* will be

$$[a]=[B]^T\,[\alpha]\,[B] \tag{1}$$

The difficulty to determine the transformation matrix [B] makes this method to be much less used than the one of stiffness matrix, for which advanced sistematizations exist. But the flexibility matrix method is advantageous not only in the case of pin-jointed structures — when the number of unknowns is extremely small — it is also in the case of space structures with stiff joints whose number of contours does not exceed the number of joints, as is the case of tall structures (Fig. 1).

In this paper, the transformation matrix $[B]$ is determined by use of the system topology, namely, of the incidence relations between the joints and the tree on one hand, and between sections and contours on the other hand.

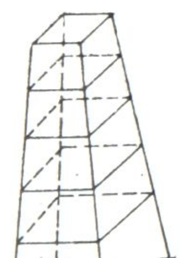

Fig. 1

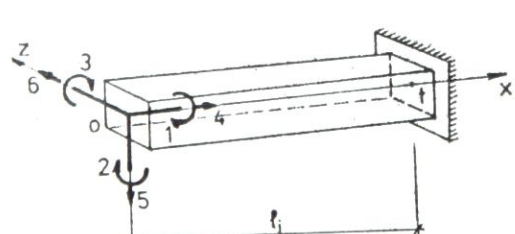

Fig. 2

The procedure may be applied to space structures with stiff joints, or with elastic restrains, for which an invariant geometric tree, fixed in space, can be found. This is the case of space frames without pins or the one of the frames with pins for which a statically determined basic system can be chosen such that complete sectionings are made at the pins.

The coordinate system for a bar "i" is the one of Fig. 2.

The ox axis of the $(oxyz)$ coordinate system coincides with the bar axis and is directed from its initial point to its terminal one.

The coordinates 1, 2, 3 correspond to the moments (notations), and the coordinates 4, 5, 6 to the forces (translations) of the origin.

Let $[\lambda]_i$ be the matrix of the direction cosines of the coordinate system $OXYZ$ of the structure with respect to the bar i coordinate system $(oxyz)_i$:

$$[\lambda]_i = \begin{bmatrix} \lambda_{xX} & \lambda_{xY} & \lambda_{xZ} \\ \lambda_{yX} & \lambda_{yY} & \lambda_{yZ} \\ \lambda_{zX} & \lambda_{zY} & \lambda_{sZ} \end{bmatrix} \tag{2}$$

By means of matrix $[\lambda]_i$ *anx* vector (moment of force) can be projected from the system $OXYZ$ in the system $(oxyz)_i$.

In the coordinate system of the structure two types of forces are acting:

1) The unknowns at the sections k;

2) External forces that can be reduced at the joints j of the structure.

Therefore, the matrix $[B]$ must be divided into two submatrices

$$[B] = [[B]_1\, [B]_2] \tag{3}$$

The submatrix $[B]_1$ expresses the forces in the element coordinates as functions of the unknowns at the sections k, while the submatrix $[B]_2$ expresses the same forces in terms of the outer forces at the joints j of the stcucture. For the structure coordinates the same order of numbering will be kept: first the three moments, then the three forces.

To give an example, let us consider the simple structure of Fig. 3a.

To turn it into a statically determined system, three complete sectionings are made: $k=1, 2, 3$, denoted by encircled numbers in Fig. 3. It is thus obtained the tree of Fig. 3 b. By closing only by one section, a *closed contour* (*circuit*) is obtained. Every section has two faces and by establishing a sense in passing from a face to the other, there results the sense of the contours. In Fig. 3 b, these senses of the contours C_k $(k=1, 2, 3)$ are represented.

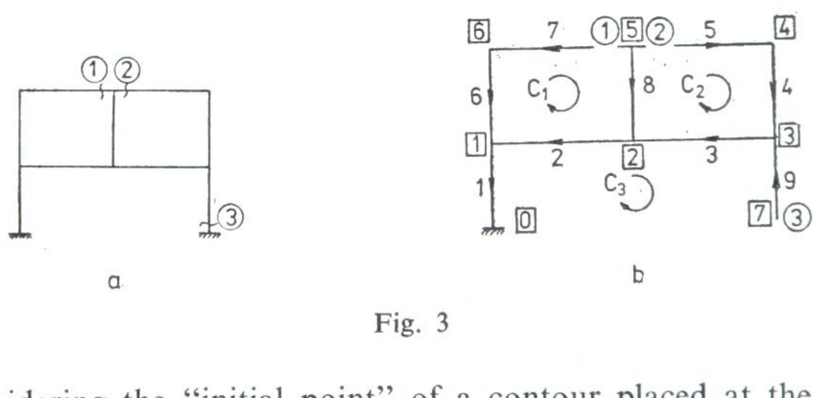

Fig. 3

By considering the "initial point" of a contour placed at the section, each bar of the contour will have got a rank, denoted by e, in the given contour. For example, in the contour C_1 the bar 8 has $e=1$, the bar 2, $e=2$, the bar 6, $e=3$, and the bar 7, $e=4$. In the contour C_2 the bar 5 has $e=1$, the bar 4, $e=2$, the bar 3, $e=3$, the bar 8, $e=4$, and in the contour C_3 the order of bars is 1, 2, 3, 9.

On the tree the direction of the bars is chosen along the paths from the joints j (written in squares in Fig. 3b) to the ground (joint o). Looking at the tree, from each joint j there exists a single path to the ground and each bar on this path can be assigned a rank e. For example, for the joint $j=5$, the rank (order number) of the bar 8 is $e=1$, of the bar 2 is $e=2$, of the bar 1 is $e=3$.

Let us build up a matrix $[B^*]$ as follows:

$$[B^*]=[C \mid T] \tag{4}$$

where C is the contour-bar incidence matrix, and [T] is the tree-bar incidence matrix. The element c_{ik} of the matrix C shows the incidence of the bar i with the contour C_k: if the bar i does not belong to the contour C_k, then $c_{ik}=0$, if it does then $c_{ik}= =+1$ or -1 depending on whether its sense is the same or contrary to the sense of the contour C_k. Since there exists a one-to-one mapping between sections and contours and to every section there corresponds a group of 6 unknowns, it follows that a one-to-one mapping occurs between the groups of unknowns and the contours. A group of unknowns yields forces in merely the bars of the corresponding contour, hence the matrix $[C]$ shows exactly which is the distribution of zero elements in the the matrix $[B]_1$ of equation (3), namely, if a zero exists at place (ik) in C, at the same place in $[B]_1$ will exist a (6×6) null submatrix. If at this palce the elements c_{ik} is 1 or $-$ 1, the corresponding matrix in $[B]_1$ is not null.

The tree-bar incidence matrix $[T]$ has the element t_{ij} equal to 1 or to 0 depending on whether the bar i is or is not on the path from the joint j to the ground (joint 0). Since the external forces are reduced at the joints, the forces in each joint will yield forces namely in the bars situated on the path from that joint to the ground. Hence if the element t_{ij} in the matrix $[T]$ equals zero, the corresponding (6×6) submatrix of the matrix $[B]_2$ of equations (3), is null, and if t_{ij} 0, then the corresponding submatrix of $[B]_2$ is nonnull, too.

Therefore the matrix $[B^*]$ of equation (4) shows exactly the distribution of the coefficients in the transformation matrix $[B]$.

As an instance, consider Fig. 3 b:

$[B^*]=$

bars i	sections k ①	②	③	joints j [1]	[2]	[3]	[4]	[5]	[6]	[7]
1	0	0	-1	+1	+1	+1	+1	+1	+1	+1
2	+1	0	-1	0	+1	+1	+1	+1	0	+1
3	0	+1	-1	0	0	+1	+1	0	0	+1
4	0	+1	0	0	0	0	+1	0	0	0
5	0	+1	0	0	0	0	0	0	0	0
6	-1	0	0	0	0	0	0	0	+1	0
7	-1	0	0	0	0	0	0	0	0	0
8	+1	-1	0	0	0	0	0	+1	0	0
9	0	0	-1	0	0	0	0	0	0	+1

[C] (sections columns) [T] (joints columns)

But the contour-bar and tree-bar incidence relations are useful not only to show the qualitative aspect of the transformation matrix B, but also for its construction.

It is useful to introduce first the new incidence matrices C and T which will have the same distribution of the zero elements, but instead of 1 will have the rank e of the corresponding bar in the contour or in the tree.

Thus, for Fig. 3b, one can obtain:

$[\bar{B}] =$

bars i	sections k ①	②	③	joints j [1]	[2]	[3]	[4]	[5]	[6]	[7]
1	0	0	−1	1	2	3	4	3	2	4
2	+2	0	−2	0	1	2	3	2	0	3
3	0	+3	−3	0	0	1	2	0	0	2
4	0	+2	0	0	0	0	1	0	0	0
5	0	+1	0	0	0	0	0	0	0	0
6	−3	0	0	0	0	0	0	0	1	0
7	−4	0	0	0	0	0	0	0	0	0
8	−1	−4	0	0	0	0	0	1	0	0
9	0	0	−4	0	0	0	0	0	0	1

$[\bar{C}]$ (sections columns) $[\bar{T}]$ (joints columns)

Let us show how can be built up the matrix $[B]$ from the matrix $[\bar{B}]$. Every coefficient of $[\bar{B}]$ is replaced by a (6×6) submatrix. It has been shown that if this coefficient is null, the submatrix will be null, too. The problem still remains for the nonnull coefficients.

Let us first build up $[B]_1$ from $[\bar{C}]$. The matrix $[B]_1$ expresses the forces from the initial points of the bars (Fig. 2) in terms of the unknown forces at sections k. For both of them, the numbering is as follows: first the three moments, then the three forces, in the order ox_i, oy_i, cz_i for bars and OK_k, OY_k, OZ_k for sections (Fig. 4a).

unknowns / internal forces	moments at k	forces at k
moments in i	$[\lambda]_i$	$[d]_{ik}$
forces in i	$[0]$	$[\lambda]_i$

a

external forces / internal forces	moments at j	forces at j
moments in i	$[\lambda]_i$	$[d]_{ij}$
forces in i	$[0]$	$[\lambda]_i$

b

Fig. 4

The (6×6) submatrix has been divided in four (3×3) submatrices. Since the moments do not yield forces, the submatrix in position (2, 1) of Fig. 4a is null. The moments of section k are projected by the direction cosine submatrix $[\lambda]_i$ of the coordinate system $(OXYZ)$ with respect to $ox_iy_iz_i$ on the terminal point of the bar i.

Similarly, the forces of section k are turned into forces for the bar i by the same matrix $[\lambda]_i$.

The more difficult problem occurs regarding the submatrix in position (1, 2), denoted $[d]_{ik}$, which expresses the transition from the forces at the section k to the moments in the bar i. Obviously, when one makes this transition, the distances from the initial point of the bar i to the supports of the forces in section k are involved.

There are presented two ways of determing $[d]_{ik}$. According to the first one there are added the "distances" of the bars situated between section k and the initial point of the bar i, while according to the second one, $[d]_{ik}$ is determined directly, on the basic of global coordinates of the section k and of the initial point of the bar i.

Way a). For a bar f of length l_f, the moments acting on the terminal *joint* which are yielded by the forces from the initial point of the *bar* f are given by

$$|X_f|=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & l_f \\ 0 & -l_f & 0 \end{bmatrix} \tag{7}$$

(X is the sign of the vectorial product).

For the equation (6) both the forces and the moments are expressed in the co-ordinate system $(oxzy)_f$. If they were expressed in the system $(OXYZ)$ of the structure, the transition matrix would be:

$$[d]_f=[\lambda]_f^T[X]_f[\lambda]_f \tag{8}$$

If the bar i has the same sense as the contour C_k, then

$$[d]_{ik}=[\lambda]_i \sum_{f=1}^{e-1} \alpha_f [d]_f \tag{9a}$$

where e is the order number (rank) of the bar i in the contour C_k and α_f is $+1$ or -1 depending on whether the bar f has the same sense or inverse sense as the contour C_k.

If the bar i is directed conversely to the contour C_k, then $[d]_i$ must also occur in the sum (9a), hence:

$$[d]_{ik}=[\lambda]_i \sum_{f=1}^{e} \alpha_f [d]_f \tag{9b}$$

To build up the submatrices $[d]_{ik}$ of equation (9), the matrix $[\bar{C}]$ of equation (6) is quite useful. Matrix $[\bar{C}]$ shows not only which relation of (9a) and (9b) should be used, by means to the sign of c_{ik}, but also the rank e of the bar i, as well as what of the preceding bars are involved in the sum of equation (9) and which is their sign.

Way b). Let (X_i, Y_i, Z_i) be the coordinates of the initial point of the bar i with respect to the system $(OYXZ)$ of the structure, and let (X_k, Y_k, Z_k) be the coordinates of the section k in the same system. Let us denote:

$$\Delta X=X_i-X_k \quad \Delta Y=Y_i-X_k \quad \Delta Z=Z_i-Z_k$$

Then the moments acting on the initial point (origin) of the bar i, expressed in the system $(OXYZ)$ and yielded by the forces in section k will be:

$$\begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}=\begin{bmatrix} 0 & \Delta Z & -\Delta Y \\ -\Delta Z & 0 & +\Delta X \\ +\Delta Y & -\Delta X & 0 \end{bmatrix}_{ik} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} \tag{10}$$

By projecting these moments on the axes $(oxyz)_i$ of the bar i one obtains:

(11)
$$[d]_{ik}=[\lambda]_i\begin{bmatrix}0 & \Delta Z & -\Delta Y\\ -\Delta Z & 0 & +\Delta X\\ +\Delta Y & -\Delta X & 0\end{bmatrix}$$

no matter how is the sense of the bar i and of the other bars f with respect to the sense of the contour C_k.

It still remains to build up the matrix $[B]_2$ by means of the matrix $[\overline{T}]$ of equation (6). In figure (4b) have been represented the corresponding (3×3) matrices. To establish the submatrix $[d]_{ij}$ one can follow similar ways to the ones in the case of $[d]_{ik}$.

Way a). The expression of $[d]_{ij}$ is simpler than in the preceding case since the bars are directed from the nodes j to the ground:

(12)
$$[d]_{ij}=[\lambda]_i\sum_{f=1}^{e-1}[d]_f.$$

The matrix $[\overline{T}]$ shows which bars are involved in the sum (12) and namely the bars with their rank less than the rank e of the bar i.

Way b). Denoting

$$\Delta X=X_i-X_j \quad \Delta Y=Y_j-Y_i \quad \Delta Z=Z_i-Z_j$$

the expression of $[d]_{ij}$ giving the moments of the initial point of the bar i in terms of the forces in the joint j, will be:

(13)
$$[d]_{ij}=[\lambda]_i\begin{bmatrix}0 & \Delta Z & -\Delta Y\\ -\Delta Z & 0 & +\Delta Y\\ +\Delta Y & -\Delta X & 0\end{bmatrix}_{ij}.$$

Note that in the matrix of equation system

(14)
$$[a]_{11}=[B]_1^T[\alpha][B]_1$$

each bar contributes only to the coefficients corresponding to the contours which it belongs to, and the flexibility submatrix of a contour results as the sum of the flexibility submatrices of the bars it consists of.

Therefore it results a perfect analogy between the "joint" in stiffness matrix method and the "contour" in flexibility matrix method.

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VARIANTES TYPOLOGIQUES DANS L'ETABLISSEMENT DE LA MATRICE DE FLEXIBILITE DES PORTIQUES SPATIAUX

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Résumé

Ce travail concerne la résolution des systèmes spatiaux par la méthode de la matricé de flexibilité, dans l'hypothèse que les noeuds sont rigides ou à encastrements élastiques. Pour déterminer la matrice de transformation on utilise les relations d'incidence noeuds-arbre et des sections contours. On fait l'analogie entre „le noeud" de la méthode de la matrice de rigidité et »le contour« de la méthode de la matrice de flexibilité ce qui annulle le décalage existant entre les deux méthodes, de tous les points de vue.

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