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FUNCTIONAL RELATIONS IN NON—LINEAR VISCOELASTJCITY

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1. Introduction

The paper deals with functional constitutive equations for non-linear viscoelastic materials having hereditary characteristics. The author introduces a new algorithm based on tensorial expansions in order to obtain a closer form of functional relationships. In this way, he has derived finite functional formulae in contradistinction of the infinite integral series obtained according to the Weierstrass approximation theorem of functionals. Constitutive functional equations define the stress tensor $\sigma_{ij}(t)$ at the time t as a continuous functional of the strain tensor $\varepsilon_{ij}(\tau)$ in the range $t_0 \leqq \tau \leqq t$ or as that of the strain tensor and of tensors of time derivatives of the strain, respectively, and as an ordinary function of these tensors at the time $\tau = t$.

2. Functional Stress-Strain Relations

The stress tensor for an isotropic material having memory of strain tensors in its past state at N intervals of time $\tau_0 = t_0, \tau_1, \tau_2, \tau_3 \ldots \tau_{N-1}$ prior to its present state at time $\tau_N = t$ may be expressed by

$$\sigma_{ij}(t) = f_{ij}[\varepsilon_{kl}(t_0), \varepsilon_{kl}(\tau_1), \varepsilon_{kl}(\tau_2) \ldots \varepsilon_{kl}(\tau_{N-1}), \varepsilon_{kl}(t)]. \tag{1}$$

If the isotropic tensorial function on the right-hand side of the preceding equation can be expanded into an absolutely convergent tensorial series, the author has obtained, according to rules of the tensorial algebra, the following finite formula

$$\begin{aligned}\sigma_{ij}(t) = \Phi_0 \delta_{ij} + \sum_{K=0}^{N} \Phi_{K+1} \varepsilon_{ij}(\tau_K) + \sum_{K=0}^{N} \Phi_{N+K+2} \varepsilon_{i\lambda}(\tau_K) \varepsilon_{\lambda j}(\tau_K) + \\ + \sum_{K=0}^{N-1} \Phi_{2N+K+2} [\varepsilon_{i\lambda}(t) \varepsilon_{\lambda j}(\tau_K) + \varepsilon_{i\lambda}(\tau_K) \varepsilon_{\lambda j}(t)],\end{aligned} \tag{2}$$

where Φ_K are scalar functions of invariants. Eq. (2) is quite an analogous one to those derived in the author's papers [1] and [2]. An infinite series can be replaced by a finite formula, because all symmetrical tensorial products of the second rank may be expressed in terms of a finite number of fundamental tensors and invariants.

The stress tensor $\sigma_{ij}(t)$ in Eq. (2) may be decomposed into $3N+2$ parts as follows

$$(3)\qquad {}^{1}\sigma_{ij}(t) = {}^{1}\Phi_0\,\delta_{ij} + \Phi_1\,\varepsilon_{ij}(\tau_0),$$

$$(4)\qquad {}^{2}\sigma_{ij}(t) = {}^{2}\Phi_0\,\delta_{ij} + \Phi_2\,\varepsilon_{ij}(\tau_1),$$

$$(5)\qquad {}^{3}\sigma_{ij}(t) = {}^{2}\Phi_0\,\delta_{ij} + \Phi_3\,\varepsilon_{ij}(\tau_2),$$

$$\dots\dots\dots\dots\dots\dots$$

$$(6)\qquad {}^{N}\sigma_{ij}(t) = {}^{N}\Phi_0\,\delta_{ij} + \Phi_N\,\varepsilon_{ij}(\tau_{N-1}),$$

$$(7)\qquad {}^{N+1}\sigma_{ij}(t) = {}^{N+1}\Phi_0\,\delta_{ij} + \Phi_{N+1}\,\varepsilon_{ij}(t),$$

$$(8)\qquad {}^{N+2}\sigma_{ij}(t) = {}^{N+2}\Phi_0\,\delta_{ij} + \Phi_{N+2}\,\varepsilon_{i\lambda}(\tau_0)\,\varepsilon_{\lambda j}(\tau_0),$$

$$(9)\qquad {}^{N+3}\sigma_{ij}(t) = {}^{N+3}\Phi_0\,\delta_{ij} + \Phi_{N+2}\,\varepsilon_{i\lambda}(\tau_1)\,\varepsilon_{\lambda j}(\tau_1),$$

$$(10)\qquad {}^{N+4}\sigma_{ij}(t) = {}^{N+4}\Phi_0\,\delta_{ij} + \Phi_{N+4}\,\varepsilon_{i\lambda}(\tau_2)\,\varepsilon_{\lambda j}(\tau_2),$$

$$(11)\qquad {}^{2N+1}\sigma_{ij}(t) = {}^{2N+1}\Phi_0\,\delta_{ij} + \Phi_{2N+1}\,\varepsilon_{i\lambda}(\tau_{N-1})\,\varepsilon_{\lambda j}(\tau_{N-1}),$$

$$(12)\qquad {}^{2N+2}\sigma_{ij}(t) = {}^{2N+2}\Phi_0\,\delta_{ij} + \Phi_{2N+2}\,\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(t),$$

$$(13)\qquad {}^{2N+3}\sigma_{ij}(t) = {}^{2N+3}\Phi_0\,\delta_{ij} + \Phi_{2N+3}\,[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau_0) + \varepsilon_{i\lambda}(\tau_0)\,\varepsilon_{\lambda j}(t)],$$

$$(14)\qquad {}^{2N+4}\sigma_{ij}(t) = {}^{2N+4}\Phi_0\,\delta_{ij} + \Phi_{2N+4}\,[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau_1) + \varepsilon_{i\lambda}(\tau_1)\,\varepsilon_{\lambda j}(t)],$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(15)\qquad {}^{3N+1}\sigma_{ij}(t) = {}^{3N+1}\Phi_0\,\delta_{ij} + \Phi_{3N+1}\,[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau_{N-1}) + \varepsilon_{i\lambda}(\tau_{N-1})\,\varepsilon_{\lambda j}(t)],$$

$$(16)\qquad {}^{3N+2}\sigma_{ij}(t) = {}^{3N+2}\Phi_0\,\delta_{ij} + \Phi_{3N+2}\,\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(t).$$

Multiplying Egs. (3) up to (7) by δ_{ij} and ${}^{K+1}\sigma_{ij}(t) = {}^{K+1}\Phi_0\,\delta_{ij} + \Phi_{K+1}\,\varepsilon_{ij}(\tau_K)$, successively, we obtain

$$(17)\qquad {}^{1}\Phi_0 = \frac{1}{3}\left(\mathrm{I}_{1\sigma} - \mathrm{I}_{\varepsilon 0}\sqrt{\frac{3\,\mathrm{II}_{1\sigma} - \mathrm{I}_{1\sigma}^2}{3\,\mathrm{II}_{\varepsilon 0} - \mathrm{I}_{\sigma 0}^2}}\right) = {}^{1}\sigma_M(t) - \frac{2\,{}^{1}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_0)}\,\varepsilon_M(\tau_0),$$

$$(18)\qquad \Phi_1 = \sqrt{\frac{3\,\mathrm{II}_{1\sigma} - \mathrm{I}_{1\sigma}^2}{3\,\mathrm{II}_{\varepsilon 0} - \mathrm{I}_{\varepsilon 0}^2}} = \frac{2\,{}^{1}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_0)},$$

$$(19)\qquad {}^{2}\Phi_0 = \frac{1}{3}\left(\mathrm{I}_{2\sigma} - \mathrm{I}_{\varepsilon 1}\sqrt{\frac{3\,\mathrm{II}_{2\sigma} - \mathrm{I}_{2\sigma}^2}{3\,\mathrm{II}_{\varepsilon 1} - \mathrm{I}_{\varepsilon 1}^2}}\right) = {}^{2}\sigma_M(t) - \frac{2\,{}^{2}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_1)}\,\varepsilon_M(\tau_1),$$

$$(20)\qquad \Phi_2 = \sqrt{\frac{3\,\mathrm{II}_{2\sigma} - \mathrm{I}_{2\sigma}^2}{3\,\mathrm{II}_{\varepsilon 1} - \mathrm{I}_{\varepsilon 1}^2}} = \frac{2\,{}^{0}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_1)},$$

$$(21)\qquad {}^{N}\Phi_0 = \frac{1}{3}\left(\mathrm{I}_{N\sigma} - \mathrm{I}_{\varepsilon(N-1)}\sqrt{\frac{3\,\mathrm{II}_{N\sigma} - \mathrm{I}_{N\sigma}^2}{3\,\mathrm{II}_{\varepsilon(N-1)} - \mathrm{I}_{\varepsilon(N-1)}^2}}\right) = {}^{N}\sigma_M(t) - \frac{2\,{}^{N}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_{N-1})}\,\varepsilon_M(\tau_{N-1}),$$

(22) $$\Phi_N = \sqrt{\frac{3\,\mathrm{II}_{N\sigma} - \mathrm{I}^2_{N\sigma}}{3\,\mathrm{II}_{\varepsilon(N-1)} - \mathrm{I}^2_{\varepsilon(N-1)}}} = \frac{2\,{}^N\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_{N-1})},$$

(23) $${}^{N+1}\Phi_0 = \frac{1}{3}\left(\mathrm{I}_{(N+1)\sigma} - \mathrm{I}_{\varepsilon N}\sqrt{\frac{3\,\mathrm{II}_{(N+1)\sigma} - \mathrm{I}^2_{(N+1)\sigma}}{3\,\mathrm{II}_{\varepsilon N} - {}^2_{\varepsilon N}}}\right) = {}^{N+1}\sigma_M(t) - \frac{2\,{}^{N+1}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(t)}\,\varepsilon_M(t),$$

(24) $$\Phi_{N+1} = \sqrt{\frac{3\,\mathrm{II}_{(N+1)\sigma} - \mathrm{I}^2_{(N+1)\sigma}}{3\,\mathrm{II}_{\varepsilon N} - \mathrm{I}^2_{\varepsilon N}}} = \frac{2\,{}^{N+1}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(t)},$$

where

(25) $$\mathrm{I}_{K\sigma} = {}^K\sigma_{ij}(t)\,\delta_{ij},\quad \mathrm{II}_{K\sigma} = {}^K\sigma_{ij}(t)\,{}^K\sigma_{ij}(t)$$

are invariants of partial stress tensors,

(26) $$\mathrm{I}_{\varepsilon K} = \varepsilon_{ij}(\tau_K)\,\sigma_{ij},\quad \mathrm{II}_{\varepsilon K} = \varepsilon_{ij}(\tau_K)\,\varepsilon_{ij}(\tau_K)$$

are invariants of the strain tensor at the time τ_K,

(27) $${}^K\sigma_M(t) = \frac{1}{3}\left[{}^K\sigma_{11}(t) + {}^K\sigma_{22}(t) + {}^K\sigma_{33}(t)\right]$$

is the partial mean stress,

(28) $$\varepsilon_M(\tau_K) = \frac{1}{3}\left[\varepsilon_{11}(\tau_K) + \varepsilon_{22}(\tau_K) + \varepsilon_{33}(\tau_K)\right]$$

is the mean strain at the time τ_K,

(29) $${}^K\sigma_{\mathrm{I}}(t) = \sqrt{\sigma^2_{11}(t) + \sigma^2_{22}(t) + \sigma^2_{33}(t) + 3\left[\sigma^2_{12}(t) + \sigma^2_{23}(t) + \sigma^2_{31}(t)\right]}$$

is the intensity of the partial stress tensor

(30) $$\varepsilon_{\mathrm{I}}(\tau_K) = \frac{2}{3}\sqrt{\varepsilon^2_{11}(\tau_K) + \varepsilon^2_{22}(\tau_K) + \varepsilon^2_{33}(\tau_K) + 3\left[\varepsilon^2_{12}(\tau_K) + \varepsilon^2_{23}(\tau_K) + \varepsilon^2_{31}(\tau_K)\right]}$$

is the intensity of the strain tensor at the time τ_K.

Multiplying Eqs. (8) up to (12) by δ_{ij} and ${}^{N+K+2}\sigma_{ij}(t) = {}^{N+K+2}\Phi^0\,\delta_{ij} + \Phi_{N+K+2}\,\varepsilon_{i\lambda}(\tau_K)$ successively, we find

(31) $$\begin{aligned} {}^{N+2}\Phi_0 &= \frac{1}{3}\left[\mathrm{I}_{(N+2)\sigma} - \mathrm{II}_{\varepsilon 0}\sqrt{\frac{3\,\mathrm{II}_{(N+2)\sigma} - \mathrm{I}^2_{(N+2)\sigma}}{3\,\mathrm{IV}_{\varepsilon 0} - \mathrm{II}^2_{\varepsilon 0}}}\right] = \\ &= {}^{N+2}\sigma_M(t) - \frac{4\,{}^{N+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_0)}\,\varepsilon^2_S(\tau_0) \end{aligned}$$

$$\Phi_{N+2} = \sqrt{\frac{3\,\mathrm{II}_{(N+2)\sigma} - \mathrm{I}^2_{(N+2)\sigma}}{3\,\mathrm{IV}_{\varepsilon 0} - \mathrm{II}^2_{\varepsilon 0}}} = \frac{4\,{}^{N+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_0)} \tag{32}$$

$$\begin{aligned} {}^{N+3}\Phi_0 &= \frac{1}{3}\left[\mathrm{I}_{(N+3)\sigma} - \mathrm{II}_{\varepsilon 1}\sqrt{\frac{3\,\mathrm{II}_{(N+3)\sigma} - \mathrm{I}^2_{(N+3)\sigma}}{3\,\mathrm{IV}_{\varepsilon 1} - \mathrm{II}^2_{\varepsilon 1}}}\right. = \\ &= {}^{N+3}\sigma_M(t) - \frac{4\,{}^{N+3}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_1)}\,\varepsilon^2_S(\tau_1), \end{aligned} \tag{33}$$

$$\Phi_{N+3} = \sqrt{\frac{3\,\mathrm{II}_{(N+3)\sigma} - \mathrm{I}^2_{(N+3)\sigma}}{3\,\mathrm{IV}_{\varepsilon 1} - \mathrm{II}^2_{\varepsilon 1}}} = \frac{4\,{}^{N+3}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_1)}, \tag{34}$$

$$\begin{aligned} {}^{2N+1}\Phi_0 &= \frac{1}{3}\left[\mathrm{I}_{(2N+1)\sigma} - \mathrm{II}_{\varepsilon(N-1)}\sqrt{\frac{3\,\mathrm{II}_{(2N+1)\sigma} - \mathrm{I}^2_{(2N+1)}}{3\,\mathrm{IV}_{\varepsilon(N-1)} - \mathrm{II}^2_{\varepsilon(N-1)}}}\,\sigma\right. = \\ &= {}^{2N+1}\sigma_M(t) - \frac{4\,{}^{2N+1}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_{N-1})}\,\varepsilon^2_S(\tau_{N-1}), \end{aligned} \tag{35}$$

$$\Phi_{2N+1} = \sqrt{\frac{3\,\mathrm{II}_{(2N+1)\sigma} - \mathrm{I}^2_{(2N+1)\sigma}}{3\,\mathrm{IV}_{\varepsilon(N-1)} - \mathrm{II}^2_{\varepsilon(N-1)}}} = \frac{4\,{}^{2N+1}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_{N-1})}, \tag{36}$$

$$\begin{aligned} {}^{2N+2}\Phi_0 &= \frac{1}{3}\left[\mathrm{I}_{(2N+2)\sigma} - \mathrm{II}_{\varepsilon N}\sqrt{\frac{3\,\mathrm{II}_{(2N+2)\sigma} - \mathrm{I}^2_{(2N+2)\sigma}}{3\,\mathrm{IV}_{\varepsilon N} - \mathrm{II}^2_{\varepsilon N}}}\right] = \\ &= {}^{2N+2}\sigma_M(t) - \frac{4\,{}^{2N+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_N)}\,\varepsilon^2_S(\tau_N) \end{aligned} \tag{37}$$

$$\Phi_{2N+2} = \sqrt{\frac{3\,\mathrm{II}_{(2N+2)\sigma} - \mathrm{I}^2_{(2N+2)\sigma}}{3\,\mathrm{IV}_{\varepsilon N} - \mathrm{II}^2_{\varepsilon N}}} = \frac{4\,{}^{2N+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_N)}, \tag{38}$$

where

$$\mathrm{IV}_{\varepsilon K} = \varepsilon_{ij}(\tau_K)\,\varepsilon_{j\lambda}(\tau_K)\,\varepsilon_{\lambda\mu}(\tau_K)\,\varepsilon_{\mu i}(\tau_K), \tag{39}$$

is the fourth invariant of the strain tensor at the time τ_K,

$$\varepsilon_S(\tau_K) = \frac{1}{\sqrt{3}}\sqrt{\varepsilon^2_{11}(\tau_K) + \varepsilon^2_{22}(\tau_K) + \varepsilon^2_{33}(\tau_K) + 2\,[\varepsilon^2_{12}(\tau_K) + \varepsilon^2_{23}(\tau_K) + \varepsilon^2_{31}(\tau_K)]} \tag{40}$$

is the second-order mean strain and

$$\varepsilon_{\mathrm{II}}(\tau_K) = \frac{2}{3\sqrt[4]{2}}\sqrt[4]{3\,\mathrm{IV}_{\varepsilon K} - \mathrm{II}^2_{\varepsilon K}} \tag{41}$$

is the second-order intensity of the strain tensor.

Multiplying Eqs. (13) up to (15) by δ_{ij} and ${}^{K}\sigma_{ij}(t)$ successively, we find

(42)
$$ {}^{2N+3}\Phi_0 = \frac{1}{3}\left[\mathrm{I}_{(2N+3)\sigma} - \Pi_{\varepsilon N\varepsilon 0} \sqrt{\frac{3\,\mathrm{II}_{(2N+3)\sigma} - \mathrm{I}^2_{(2N+3)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon 0\varepsilon 0} + \Pi_{\varepsilon N\varepsilon 0\varepsilon N\varepsilon 0}\right) - \Pi^2_{\varepsilon N\varepsilon 0}}} \right] = $$
$$ = {}^{2N+3}\sigma_M(t) - \frac{4\,{}^{2N+3}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_0)}\,\varepsilon^2_j(t,\tau_0), $$

(43)
$$ \Phi_{2N+3} = \sqrt{\frac{3\,\mathrm{II}_{(2N+3)\sigma} - \mathrm{I}^2_{(2N+3)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon 0\varepsilon 0} + \Pi_{\varepsilon N\varepsilon 0\varepsilon N\varepsilon 0}\right) - \Pi^2_{\varepsilon N\varepsilon 0}}} = \frac{4\,{}^{2N+3}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_0)}, $$

(44)
$$ {}^{2N+4}\Phi_0 = \frac{1}{3}\left[\mathrm{I}_{(2N+4)\sigma} - \Pi_{\varepsilon N\varepsilon 1} \sqrt{\frac{3\,\mathrm{II}_{(2N+4)\sigma} - \mathrm{I}^2_{(2N+4)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon 1\varepsilon 1} + \Pi_{\varepsilon N\varepsilon 1\varepsilon N\varepsilon 1}\right) - \Pi^2_{\varepsilon N\varepsilon 1}}} \right] = $$
$$ = {}^{2N+4}\sigma_M(t) - \frac{4\,{}^{2N+4}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_1)}\,\varepsilon^2_j(t,\tau_1), $$

(45)
$$ \Phi_{2N+4} = \sqrt{\frac{3\,\mathrm{II}_{(2N+4)\sigma} - \mathrm{I}^2_{(2N+4)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon 1\varepsilon 1} + \Pi_{\varepsilon N\varepsilon 1\varepsilon N\varepsilon 1}\right) - \Pi^2_{\varepsilon N\varepsilon 1}}} = \frac{4\,{}^{2N+4}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_1)}, $$

. . . .

(46)
$$ {}^{3N+1}\Phi_0 = \frac{1}{3}\left[\mathrm{I}_{(3N+2)\sigma} - \right. $$
$$ \left. - \Pi_{\varepsilon N\varepsilon(N-1)} \sqrt{\frac{3\,\mathrm{II}_{(3N+1)\sigma} - \mathrm{I}^2_{(3N+1)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon(N-1)\varepsilon(N-1)} + \Pi_{\varepsilon N\varepsilon(N-1)\varepsilon N\varepsilon(N-1)} - \Pi^2_{\varepsilon N\varepsilon(N-1)}\right.}} \right] = $$
$$ = {}^{3N+1}\sigma_M(t) - \frac{4\,{}^{3N+1}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_{N-1})}\,\varepsilon^2_j(t,\tau_{N-1}), $$

(47)
$$ \Phi_{3N+1} = \sqrt{\frac{3\,\mathrm{II}_{(3N+1)\sigma} - \mathrm{I}^2_{(3N+1)\sigma}}{\frac{3}{2}\left(\Pi_{\varepsilon N\varepsilon N\varepsilon(N-1)\varepsilon(N-1)} + \Pi_{\varepsilon N\varepsilon(N-1)\varepsilon N\varepsilon(N-1)} - \Pi^2_{\varepsilon N\varepsilon(N-1)}\right.}} = $$
$$ = \frac{4\,{}^{3N+1}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t,\tau_{N-1})}\,\varepsilon^2_j(t,\tau_{N-1}) $$

where

(48)
$$ \Pi_{\varepsilon N\varepsilon K} = \varepsilon_{ij}(t)\,\varepsilon_{ij}(\tau_K), $$

$$\Pi_{\varepsilon N \varepsilon N \varepsilon K \varepsilon K} = \varepsilon_{ij}(t)\, \varepsilon_{j\lambda}(t)\, \varepsilon_{\lambda\mu}(\tau_K)\, \varepsilon_{\mu i}(\tau_K), \qquad \Pi_{\varepsilon N \varepsilon K \varepsilon N \varepsilon K} = \varepsilon_{ij}(t)\, \varepsilon_{j\lambda}(\tau_K)\, \varepsilon_{\lambda\mu}(t)\, \varepsilon_{\mu i}(\tau_K)$$

are joint invariants of strain tensors,

$$\varepsilon_j(t, \tau_K) = \frac{1}{\sqrt{3}} \sqrt{\{\varepsilon_{11}(t)\varepsilon_{11}(\tau_K) + \varepsilon_{22}(t)\varepsilon_{22}(\tau_K) + \varepsilon_{33}(t)\varepsilon_{33}(\tau_K) + 2[\varepsilon_{12}(t)\varepsilon_{12}(\tau_K) + \varepsilon_{23}(t)\,\varepsilon_{23}(\tau_K) + \varepsilon_{31}(t)\,\varepsilon_{31}(\tau_k)]\}}, \tag{50}$$

is the joint second-order mean strain and

$$\varepsilon_{jj}(t, \tau_K) = \frac{2}{3\sqrt[4]{2}} \sqrt[4]{\frac{3}{2}(\Pi_{\varepsilon N \varepsilon N \varepsilon K \varepsilon K} + \Pi_{\varepsilon N \varepsilon K \varepsilon N \varepsilon K}) - \Pi^2_{\varepsilon N \varepsilon K}} \tag{51}$$

is the joint second-order intensity of strain tensors.

Introducing the expressions (17) to (24), (33) to (38) and (42) to (47) into Eqs. (3) up to (16) and adding them, we obtain

$$\begin{aligned} \sigma_{ij}(t) - \sigma_M(t)\delta_{ij} &= \frac{2}{3}\sum_{K=0}^{N} \frac{{}^{K+1}\sigma_{\mathrm{I}}(t)}{\varepsilon_{\mathrm{I}}(\tau_K)} [\varepsilon_{ij}(\tau_K) - \varepsilon_M(\tau_K)\,\delta_{ij}] + \\ &+ \frac{4}{9}\sum_{K=0}^{N} \frac{{}^{N+K+2}\sigma_{\mathrm{I}}(t)}{\varepsilon^2_{\mathrm{II}}(\tau_K)} [\varepsilon_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(\tau_K) - \varepsilon^2_S(\tau_K)\,\delta_{ij}] + \\ &+ \frac{4}{9}\sum_{K=0}^{N-1} \frac{{}^{2N+K+2}\sigma_{\mathrm{I}}(t)}{\varepsilon^2_{jj}(t, \tau_K)} \left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau_K) + \varepsilon_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(t)] - \varepsilon^2_j(t, \tau_K)\,\delta_{ij}\right\}. \end{aligned} \tag{52}$$

If N tends to infinity, the functions

$$\widetilde{G}[\varepsilon_{\mathrm{I}}(\tau_K), t] = \frac{{}^{K+1}\sigma_{\mathrm{I}}(t)}{3\,\varepsilon_{\mathrm{I}}(\tau_K)}, \quad \widetilde{M}[\varepsilon_{\mathrm{II}}(\tau_K), t] = \frac{{}^{N+K+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{\mathrm{II}}(\tau_K)}, \tag{53}$$

$$\widetilde{H}[\varepsilon_{jj}(t, \tau_K), t] = \frac{{}^{2N+K+2}\sigma_{\mathrm{I}}(t)}{9\,\varepsilon^2_{jj}(t, \tau_K)} \tag{54}$$

tend to $d\widetilde{G}[\varepsilon_{\mathrm{I}}(\tau), t]$, $d\widetilde{M}[\varepsilon_{\mathrm{II}}(\tau), t]$, $d\widetilde{H}[\varepsilon_{ij}(t, \tau), t]$ and the three sums on the right-hand side of Eq. (52) become Stieltjes integrals.

Then, we get

$$\begin{aligned} \sigma_{ij}(t) - \sigma_M(t)\,\delta_{ij} &= 2\int_{t_0}^{t} [\varepsilon_{ij}(\tau) - \varepsilon_M(\tau)\,\delta_{ij}]\, d\widetilde{G}[\varepsilon_{\mathrm{I}}(t), t] + \\ &+ 4\int_{t_0}^{t} [\varepsilon_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(\tau) - \varepsilon^2_s(\tau)\,\delta_{ij}]\, d\widetilde{M}[\varepsilon_{\mathrm{II}}(\tau), t] + \\ &+ 4\int_{t_0}^{t_0} \left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau) + \varepsilon_{i\lambda}(\tau)\varepsilon_{\lambda j}(t)] - \varepsilon^2_j(t, \tau)\,\delta_{ij}\right\} d\widetilde{H}[\varepsilon_{jj}(t, \tau), t]. \end{aligned} \tag{55}$$

Under certain conditions the Stieltjes integrals may be transformed into those of Riemann and we obtain finally

$$\sigma_{ij}(t) - \sigma_M(t)\,\delta_{ij} = 2\int_{t_0}^{t} [\varepsilon_{ij}(\tau) - \varepsilon_M(\tau)\,\delta_{ij}] \frac{d\tilde{G}[\varepsilon_{\mathrm{I}}(\tau), t]}{d\varepsilon_{\mathrm{I}}(\tau)} \frac{d\varepsilon_{\mathrm{I}}(\tau)}{d\tau} d\tau +$$

$$(56)\qquad + 4\int_{t_0}^{t} [\varepsilon_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(\tau) - \varepsilon_s^2(\tau)\,\delta_{ij}] \frac{d\widetilde{M}[\varepsilon_{\mathrm{II}}(\tau), t]}{d\varepsilon_{\mathrm{II}}(\tau)} \frac{d\varepsilon_{\mathrm{II}}(\tau)}{d\tau} d\tau +$$

$$+ 4\int_{t_0}^{t} \left\{ \frac{1}{2} [\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau) + \varepsilon_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(t)] - \varepsilon_j^2(t, \tau)\,\delta_{ij} \right\} \frac{d\widetilde{H}[\varepsilon_{jj}(t, \tau), t]}{d\varepsilon_{jj}(t, \tau)} \frac{d\varepsilon_{jj}(t, \tau)}{d\tau} d\tau.$$

3. Functional Relations between Stress, Strain and Strain Derivatives

The author has obtained more general functional relations of non-linear rheology on expressing the stress tensor for an isotropic material having memory of strain tensors and strainderivatives tensors in its past state at N intervals of time by the following function

$$(57)\qquad \begin{aligned} \sigma_{ij}(t) = f_{ij}[&\varepsilon_{kl}(t_0), \varepsilon_{kl}(\tau_1), \varepsilon_{kl}(\tau_2), \cdots \varepsilon_{kl}(\tau_{N-1}), \varepsilon_{kl}(t), \\ &\dot{\varepsilon}_{mn}(t_0), \dot{\varepsilon}_{mn}(\tau_1), \dot{\varepsilon}_{mn}(\tau_2), \cdots \dot{\varepsilon}_{mn}(\tau_{N-1}), \dot{\varepsilon}_{mn}(t), \\ &\ddot{\varepsilon}_{mn}(t_0), \ddot{\varepsilon}_{mn}(\tau_1), \ddot{\varepsilon}_{mn}(\tau_2), \cdots \ddot{\varepsilon}_{mn}(\tau_{N-1}), \ddot{\varepsilon}_{mn}(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots]. \end{aligned}$$

Proceeding in quite an analogous way as it has been shown in the Section 2, the author has developped, according to the rules of the tensorial algebra, the function (57) into the following relation

$$(58)\qquad \begin{aligned} \sigma_{ij}(t) = {} & \Phi_0\,\delta_{ij} + \sum_{K=0}^{N} \Phi_{K+1}\,\varepsilon_{ij}(\tau_K) + \sum_{K=0}^{N} \Phi_{N+K+2}\,\varepsilon_{i\lambda}(\tau_K)\,\varepsilon_{\lambda j}(\tau_K) + \\ & + \sum_{K=0}^{N-1} \Phi_{2N+K+2}[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau_K) + \varepsilon_{i\lambda}(\tau_K)\,\varepsilon_{\lambda j}(t)] + \\ & + \sum_{K=0}^{N} \Phi_{3N+K+3}\,\dot{\varepsilon}_{ij}(\tau_K) + \sum_{K=0}^{N} \Phi_{4N+K+4}\,\dot{\varepsilon}_{i\lambda}(\tau_K)\,\dot{\varepsilon}_{\lambda j}(\tau_K) + \\ & + \sum_{K=0}^{N-1} \Phi_{5N+K+4}[\varepsilon_{i\lambda}(t)\,\dot{\varepsilon}_{\lambda j}(\tau_K) + \dot{\varepsilon}_{i\lambda}(\tau_K)\,\varepsilon_{\lambda j}(t)] + \\ & + \sum_{K=0}^{N} \Phi_{6N+K+4}\,\ddot{\varepsilon}_{ij}(\tau_K) + \sum_{K=0}^{N} \Phi_{7N+K+5}\,\ddot{\varepsilon}_{i\lambda}(\tau_K)\,\ddot{\varepsilon}_{\lambda j}(\tau_K) + \\ & + \sum_{K=0}^{N-1} \Phi_{8N+K+5}[\varepsilon_{i\lambda}(t)\,\ddot{\varepsilon}_{\lambda j}(\tau_K) + \ddot{\varepsilon}_{i\lambda}(\tau_K)\,\varepsilon_{\lambda j}(t)] + \end{aligned}$$

Introducing the expressions (53), (54) and

(59) $$\tilde{\eta}_1[\dot{\varepsilon}_1(\tau_K), t] = \frac{{}^{3N+K+3}\sigma_1(t)}{3\dot{\varepsilon}_1(\tau_K)}, \quad \tilde{\xi}_1[\dot{\varepsilon}_{11}(\tau_K), t] = \frac{{}^{4N+K+4}\sigma_1(t)}{9\,\dot{\varepsilon}_{11}^2(\tau_K)},$$

(60) $$\tilde{\varphi}_1[\dot{\varepsilon}_{jj}(t, \tau_K), t] = \frac{{}^{5N+K+4}\sigma_1(t)}{9\dot{\varepsilon}_{jj}(\tau_K)},$$

(61) $$\tilde{\eta}_2[\ddot{\varepsilon}_1(\tau_K), t] = \frac{{}^{6N+K+4}\sigma_1(t)}{3\ddot{\varepsilon}_1(\tau_K)}, \quad \tilde{\xi}_2[\ddot{\varepsilon}_{11}(\tau_K), t] = \frac{{}^{7N+K+5}\sigma_1(t)}{9\ddot{\varepsilon}_{11}^2(\tau_K)},$$

(62) $$\tilde{\varphi}_2[\dddot{\varepsilon}_{jj}(t, \tau_K), t] = \frac{{}^{8N+K+5}\sigma_1(t)}{9\dddot{\varepsilon}_{jj}(t, \tau_K)},$$

where

(63) $$\ddot{\varepsilon}_{jj}(t, \tau_K) = \frac{2}{3\sqrt[4]{2}}\sqrt[4]{\frac{3}{2}(\Pi_{\varepsilon N \varepsilon N \dot{\varepsilon} K \dot{\varepsilon} K} + \Pi_{\varepsilon N \dot{\varepsilon} K \varepsilon N \dot{\varepsilon} K}) - \Pi^2_{\varepsilon N \dot{\varepsilon} K}}$$

is the joint second-order intensity of the strain and strain-rate tensor and

(64) $$\dddot{\varepsilon}_{jj}(t, \tau_K) = \frac{2}{3\sqrt[4]{2}}\sqrt[4]{\frac{3}{2}(\Pi_{\varepsilon N \varepsilon N \ddot{\varepsilon} K \ddot{\varepsilon} K} + \Pi_{\varepsilon N \ddot{\varepsilon} K \varepsilon N \ddot{\varepsilon} K}) - \Pi^2_{\varepsilon N \ddot{\varepsilon} K}}$$

is the joint second-order intensity of the strain and strain-acceleration tensor, we obtain

(65) $$\begin{aligned}
\sigma_{ij}(t) - \sigma_M(t)\delta_{ij} &= 2\sum_{K=0}^{N} \tilde{G}[\varepsilon_1(\tau_K), t]\,[\varepsilon_{ij}(\tau_K) - \varepsilon_M(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{K} \widetilde{M}[\varepsilon_{11}(\tau_K), t]\,[\varepsilon_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(\tau_K) - \varepsilon_S^2(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{N} \widetilde{H}[\varepsilon_{jj}(\tau_K, t), t]\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\varepsilon_{\lambda j}(\tau_K) + \varepsilon_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(t)] - \varepsilon_j^2(t, \tau_K)\delta_{ij}\right\} + \\
&+ 2\sum_{K=0}^{N} \tilde{\eta}_1[\dot{\varepsilon}_1(\tau_K), t]\,[\dot{\varepsilon}_{ij}(\tau_K) - \dot{\varepsilon}_M(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{N} \tilde{\xi}_1[\dot{\varepsilon}_{11}(\tau_K), t]\,[\dot{\varepsilon}_{i\lambda}(\tau_K)\dot{\varepsilon}_{\lambda j}(\tau_K) - \dot{\varepsilon}_S^2(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{N} \tilde{\varphi}_1[\ddot{\varepsilon}_{jj}(\tau_K, t), t]\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\dot{\varepsilon}_{\lambda j}(\tau_K) + \dot{\varepsilon}_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(t)] - \ddot{\varepsilon}_j^2(t, \tau_K)\delta_{ij}\right\} + \\
&+ 2\sum_{K=0}^{N} \tilde{\eta}_2[\ddot{\varepsilon}_1(\tau_K), t]\,[\ddot{\varepsilon}_{ij}(\tau_K) - \ddot{\varepsilon}_M(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{N} \tilde{\xi}_2[\ddot{\varepsilon}_{11}(\tau_K), t]\,[\ddot{\varepsilon}_{i\lambda}(\tau_K)\ddot{\varepsilon}_{\lambda j}(\tau_K) - \ddot{\varepsilon}_S^2(\tau_K)\delta_{ij}] + \\
&+ 4\sum_{K=0}^{N} \tilde{\varphi}_2[\dddot{\varepsilon}_{jj}(t, \tau_K), t]\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\ddot{\varepsilon}_{\lambda j}(\tau_K) + \ddot{\varepsilon}_{i\lambda}(\tau_K)\varepsilon_{\lambda j}(t)] - \dddot{\varepsilon}_j^2(t, \tau_K)\delta_{ij}\right\}
\end{aligned}$$

If N tends to infinity, we find a relation with Stieltjes integrals, which has an analogous form to Eq. (55):

$$
\begin{aligned}
\sigma_{ij}(t)-\sigma_M(t)\,\delta_{ij} = {} & 2\int_{t_0}^{t}[\varepsilon_{ij}(\tau)-\varepsilon_M(\tau)\,\delta_{ij}]\,dG[\varepsilon_{\mathrm{I}}(\tau),\,t]+\\
& +4\int_{t_0}^{t}[\varepsilon_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(\tau)-\varepsilon_S^2(\tau)\,\delta_{ij}]\,d\widetilde{M}[\varepsilon_{\mathrm{II}}(\tau),\,t]+\\
& +4\int_{t_0}^{t}\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\,\varepsilon_{\lambda j}(\tau)+\varepsilon_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(t)]-\varepsilon_j^2(t,\tau)\,\delta_{ij}\right\}d\widetilde{H}[\varepsilon_{jj}(t,\tau),\,t]+\\
& +2\int_{t_0}^{t}[\dot{\varepsilon}_{ij}(\tau)-\dot{\varepsilon}_M(\tau)\,\delta_{ij}]\,d\,\widetilde{\eta}_1[\dot{\varepsilon}_{\mathrm{I}}(\tau),t]+\\
& +4\int_{t_0}^{t}[\dot{\varepsilon}_{i\lambda}(\tau)\,\dot{\varepsilon}_{\lambda j}(\tau)-\dot{\varepsilon}_S^2(\tau)\,\delta_{ij}]\,d\,\widetilde{\xi}_1[\dot{\varepsilon}_{\mathrm{II}}(\tau),t]+\\
& +4\int_{t_0}^{t}\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\,\dot{\varepsilon}_{\lambda j}(\tau)+\dot{\varepsilon}_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(t)]-\ddot{\varepsilon}_j^2(t,\tau)\,\delta_{ij}\right\}d\,\widetilde{\varphi}_1[\ddot{\varepsilon}_{jj}(t,\tau),\,t]+\\
& +2\int_{t_0}^{t}[\ddot{\varepsilon}_{ij}(\tau)-\ddot{\varepsilon}_M(\tau)\,\delta_{ij}]\,d\,\widetilde{\eta}_2[\ddot{\varepsilon}_{\mathrm{I}}(\tau),t]+\\
& +4\int_{t_0}^{t}[\ddot{\varepsilon}_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(\tau)-\ddot{\varepsilon}_S^2(\tau)\,\delta_{ij}]\,d\,\widetilde{\xi}_2[\ddot{\varepsilon}_{\mathrm{II}}(\tau),\,t]+\\
& +4\int_{t_0}^{t}\left\{\frac{1}{2}[\varepsilon_{i\lambda}(t)\,\ddot{\varepsilon}_{\lambda j}(\tau)+\ddot{\varepsilon}_{i\lambda}(\tau)\,\varepsilon_{\lambda j}(t)-\dddot{\varepsilon}_{jj}^2(t,\tau)\,\delta_{ij}\right\}d\,\widetilde{\varphi}_2[\dddot{\varepsilon}_{jj}(t,\tau)\,t]+\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{66}
$$

The Stieltjes integrals may be transformed, under certain conditions, into those of Riemann and we obtain a functional relation analogous to Eq. (56).

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RELATIONS FONCTIONNELLES DANS LA VISCOÉLASTICITÉ NON—LINÉAIRE

Zdenek Sobotka

Résumé

L'auteur a dérivé les équations fonctionnelles pour les matériaux viscoélastiques non-linéaires qui ont de caractères héréditaires. Il fait usage d'un algorithme nouveau qui a pour la base les expansions tensorielles. On peut obtenir de cette manière les relations fonctionnelles de formes finies qui sont différentes de séries infinies d'intégrales multiples, obtenues suivant le théorème de Weierstrass.

La présente contribution contient les relations fonctionnelles contraintes-déformations et les relations entre les tenseurs de contraintes, de déformations et de dérivées de déformations.

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