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ELASTIC DIELECTRIC WITH MICROSTRUCTURE

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Abstract

In this paper a dielectric continuum with microstructure is considered as an elastic generalized Cosserat continuum in an electromagnetic field. Thermal effects have been neglected. Both the differential equations of motion, and the equation of the deformation energy balance have been derived by a formulation of the virtual work principle and Piola's theorem.

1. Model of the dielectric continuum.

Let the dielectric body of the volume V, bounded by a closed surface S, be located at an instant t_o in an initial non–deformed configuration B_o and surrounded by an empty space – a vaccum having a volume V_v and an outside surface S_v. In a deformed configuration B, which corresponds to an instant $t > t_o$, the body will have a volume v bounded by a closed surface s. A macroelement dV of the body in the non–deformed configuration B_o will become the element dv in the deformed configuration [1].

We shall assume that there is not a source of the mass, so that the mass of the macroelement remains constant during the deformation, i.e. that

$$dm = \rho_o dV = \rho dv = \text{const.}, \tag{1.1}$$

where

$$\rho_o = \frac{dm}{dV}, \quad \rho = \frac{dm}{dv} \tag{1.2}$$

are the mean mass densities of the macroelements dV and dv, respectively.

Now we observe the mass center $C(X^K)$ of the macroelement. Then, the position of an arbitrary point of the macroelement can be determined by

$$X'^K = X^K + D^K, \tag{1.3}$$

with respect to an arbitrary system of material curvilinear coordinates X^K. If at the point X'^K the density is ρ'_o, then

$$\int_{dV} \rho'_o \, dV' = \rho_o dV = dm, \qquad \int_{dV} \rho'_o D^K dV' = 0, \tag{1.4}$$

since $C(X^K)$ is the mass center of the macroelement.

In deformation, the macroelement dV becomes dv, the point $C(X^K)$ moves to $C(x^k)$, the vector D^K becomes d^k, and the polarization vector P^K becomes p^k. Then, we have

$$x'^k = x^k + d^k, \tag{1.5}$$

where

$$d^k = d^k (X^K, D^K, t). \tag{1.6}$$

We assume for the mapping functions (1.6) that they are continuous and differentiable, and taking into account the fact that D^K is an infinitesimal vector, we can write d^k as a power series

$$d^k = \chi^k_{\cdot K} D^K + \chi^k_{\cdot KL} D^K D^L + \ldots \tag{1.7}$$

If we remain at the first approximation $d^k = \chi^k_{\cdot K} D^K$, then we have the so—called simple materials for which the equation (1.5) can be written as

$$x'^k = x^k + \chi^k_{\cdot K} D^K . \tag{1.8}$$

The microdeformation gradients

$$\chi^k_{\cdot K} = \left(\frac{\partial d^k}{\partial D^K} \right)_{\underset{\sim}{D}=0} = \left(\frac{\partial x'^k}{\partial x'^K} \right)_{\underset{\sim}{X}=0} \tag{1.9}$$

are independent of motion of the point X^K, whence it follows that the deformation is described by the equations

$$x^k = x^k (X^K , t) , \; \chi^k_{\cdot K} = \chi^k_{\cdot K} (X^L , t). \tag{1.10}$$

If at the point x'^k the density is ρ', then

$$\int_{dv} \rho' dv' = \rho dv = dm , \tag{1.11}$$

and, from $(1.4)_2$,

$$\int_{dv} \rho' d^k dv' = \int_{dv} \rho' \chi^k_{\cdot K} D^K dv' = \chi^k_{\cdot K} \int_{dV} \rho'_o D^K dV' = 0, \tag{1.12}$$

where we assume that the mass of the microelement is conserved. From (1.12) we may conclude that $C(x^k)$ is the mass center of the macroelement dv.

At the center $C(X^K)$ we shall observe three non–coplanar vectors $D^k_{(\alpha)}$, $\alpha=$ 1,2,3 $|D^K{}_{(\alpha)}| \neq 0$, which are attached to macroelement dV. Then, according to (1.7), we have

$$d^k_{.(\alpha)} = \chi^k_{.K} \, D^K_{.(\alpha)} , \tag{1.13}$$

wherefrom

$$\chi^k_{.K} = d^k_{.(\alpha)} \, D^{(\alpha)}_{.K} \tag{1.14}$$

where $D^K_{.(\alpha)}$ and $D^{(\alpha)}_{.K}$ are mutually reciprocal triads, i.e.

$$D^K_{.(\alpha)} \, D^{(\alpha)}_{.L} = \delta^K_L , \qquad D^K_{.(\alpha)} \, D^{(\beta)}_{.K} = \delta^\beta_\alpha . \tag{1.15}$$

The microdeformation gradients $\chi^k{}_K$ are completely determined by the deformation of the three non–coplanar vectors $D^K{}_{(\alpha)}$. Using (1.14), the equation (1.7) can be written as

$$d^k = d^k_{.(\alpha)} \, D^{(\alpha)}_{.K} \, D^K . \tag{1.16}$$

Vectors $D^K{}_{(\alpha)}$ and $d^k{}_{(\alpha)}$ can be considered as *directors*, so that the motion is determined by the equations

$$x^k = x^k \, (X^K, t), \qquad d^k_{.(\alpha)} = d^k_{.(\alpha)} (D^M_{.(\beta)} \, (X^2), t) . \tag{1.17}$$

If $d^k{}_{(\alpha)}$ and $d^{(\alpha)}{}_k$ are mutually reciprocal triads, then from (1.16) we have

$$D^K = D^K_{.(\alpha)} \, d^{(\alpha)}_{.k} \, d^k = \chi^K_{.k} \, d^k , \tag{1.18}$$

and the equation (1.5) can be written in the form

$$x'^k = x^k + d^k_{.(\alpha)} \, D^{(\alpha)}_{.K} \, D^K . \tag{1.19}$$

By differentiation, from (1.19) we obtain

$$v'^k = v^k + \nu^k_{\cdot j} \, d^j , \tag{1.20}$$

where we have made use of the expression (1.18), and where

$$\nu^{k}_{\cdot j} = \dot{d}^{k}_{\cdot(\alpha)}\, d^{(\alpha)}_{\cdot j} = \dot{\chi}^{k}_{\cdot K}\, \chi^{K}_{\cdot j} \tag{1.21}$$

Using (1.20) and (1.12), the kinetic energy of a portion v of the body is

$$2T = \int_{v} \rho\, (v^{k}\, v_{k} + I^{KL}\, \dot{d}^{k}_{\cdot(\alpha)}\, \dot{d}_{k(\beta)}\, D^{(\alpha)}_{\cdot K}\, D^{(\beta)}_{\cdot L})\, dv, \tag{1.22}$$

where

$$\rho dv\, I^{KL} = \int_{dv} \rho' D^{K}\, D^{L}\, dv' = \int_{dV} \rho'_{o}\, D^{K}\, D^{L}\, dV'. \tag{1.23}$$

If we introduce the ,,director coefficients of inertia''

$$I^{\alpha\beta} = I^{KL}\, D^{(\alpha)}_{\cdot K}\, D^{(\beta)}_{\cdot L}\,, \tag{1.24}$$

and take into the consideration the polarization kinetic energy [2]

$$2T_{D} = \int_{v} \rho\nu g^{kj}\, \dot{p}_{k}\, \dot{p}_{j}\, dv, \tag{1.25}$$

then, finally, we obtain for the kinetic energy the following expression

$$2T = \int_{v} \rho\, (v^{k}\, v_{k} + I^{\alpha\beta}\, \dot{d}^{k}_{\cdot(\alpha)}\, \dot{d}_{k(\beta)} + \nu g^{kj}\, \dot{p}_{k}\, \dot{p}_{j}\,)\, dv, \tag{1.26}$$

where I^{KL} are the coefficients of inertia of the macroelement with respect to its center of mass, and υ is the inertia coefficient of the polarization.

By differentitation (1.26), and taking into account the mass conservation law (1.1),we obtain

$$\dot{T} = \int_{v} \rho\, (\dot{v}^{k}\, v_{k} + \Gamma^{i(\alpha)}\, \dot{d}_{i(\alpha)} + \Gamma^{i}\, \dot{p}_{i})\, dv, \tag{1.27}$$

where the inertial spins are defined by the expressions

$$\Gamma^{i(\alpha)} = \Gamma^{ij}\, d^{(\alpha)}_{\cdot j} = I^{\alpha\beta}\, \ddot{d}^{i}_{\cdot(\beta)}, \qquad \Gamma^{ij} = I^{\alpha\beta}\, \ddot{d}^{i}_{\cdot(\alpha)}\, d^{j}_{\cdot(\beta)}, \qquad \Gamma^{i} = \nu\ddot{p}^{i}\,. \tag{1.28}$$

The virtual work of the inertial forces, using (1.27), can be written in the form

$$\delta T = \int_V \rho \, (\dot{v}^i \, \delta x_i + \Gamma^{i(\alpha)} \, \delta d_{i(\alpha)} + \Gamma^i \delta p_i) \, dv. \tag{1.29}$$

2. The principle of virtual work.

Let us assume that the surface forces T^i and H^{ij}, as well as the body forces f^i and l^{ij}, act on the dielectric body B, so that their time rate of working is

$$\dot{A}_M = \oint_s (T^i v_i + H^{ij} \nu_{ij}) \; ds + \int_V \rho \, (f^i \, v_i + l^{ij} \, \nu_{ij}) \, dv. \tag{2.1}$$

Using (1.21), this equation can be written as follows

$$\dot{A}_M = \oint_s (T^i \, v_i + H^{i(\alpha)} \, \dot{d}_{i(\alpha)}) \, ds + \int_V \rho \, (f^i \, v_i + l^{i(\alpha)} \, \dot{d}_{i(\alpha)}) \, dv, \tag{2.2}$$

where

$$H^{i(\alpha)} = H^{ij} \, d^{(\alpha)}_{\cdot j} \qquad\qquad l^{i(\alpha)} = l^{ij} \, d^{(\alpha)}_{\cdot j} \; . \tag{2.3}$$

The virtual work of the mechanical forces is

$$\delta A_M = \oint_s (T^i \, \delta x_i + H^{i(\alpha)} \, \delta d_{i(\alpha)}) \, ds + \int_V \rho (f^i \delta x_i + l^{i(\alpha)} \, \delta d_{i(\alpha)}) \, dv. \tag{2.4}$$

Let us assume also that the electromagnetic surface forces S^i and R^i, as well as the electromagnetic body forces F^i and ϵ^i, act on the dielectric body. Their time rate of working is

$$\dot{A}_E = \oint_s (S^i v_i + R^i \dot{p}_i) \, ds + \int_V \rho \, (F^i \, v_i + \epsilon^i \, \dot{p}_i) \, dv, \tag{2.5}$$

wherefrom we obtain the corresponding virtual work

$$\delta A_E = \oint_s (S^i \, \delta x_i + R^i \delta p_i) \, ds + \int_V \rho \, (F^i \delta x_i + \epsilon^i \delta p_i) dv. \tag{2.6}$$

The total virtual work of both the mechanical and electromagnetic forces is obtained from (2.4) and (2.6), using the well known transformations, in the form of

$$\delta A = \oint_s \{ (T^i \, ds - t^{ik} ds_k)\delta x_i + (H^{i(\alpha)} ds - h^{i(\alpha)k} \, ds_k) \, \delta d_{i(\alpha)} +$$

$$+ (R^i ds - r^{ik} \, ds_k) \, \delta p_i + \| s^{ik} + \frac{1}{c} v^k g^i \| \, ds_k \, \delta x_i \} + \tag{2.7}$$

$$+ \int_V \rho \, \{ f^i \delta x_i + l^{i(\alpha)} \delta d_{i(\alpha)} + (s^{ik}{}_{,k} - \frac{1}{c} \frac{\partial}{\partial t} g^i) \delta x_i + \in^i \delta p_i \} dv +$$

$$+ \int_V \{ t^{ik}{}_{,k} \delta x_i + t^{ik} \, \delta x_{i,k} + h^{i(\alpha)k}{}_{,k} \delta d_{i(\alpha)} + h^{i(\alpha)k} \, \delta d_{i(\alpha),k} + r^{ik}{}_{,k} \delta p_i + r^{ik} \, \delta p_{i,k} \} dv,$$

where we have used the well known expressions for the electromagnetic forces.

We now assume the principle of virtual work in the form

$$\delta T + \delta W = \delta A, \tag{2.8}$$

where δW is a variation of strain energy.

$$\delta W = \int_V \rho \delta \, w \, dv, \tag{2.9}$$

and where w is the specific strain energy.

Making use of (2.7), (2.9) and (1.29), (2.8) can be written in the form

$$\int_V \rho \, \{ \dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \delta d_{i(\alpha)} + \Gamma^i \delta p_i \} dv + \int_V \rho \delta \, w \, dv =$$

$$= \Phi_s \{ (T^i ds - t^{ik} ds_k) \, \delta x_i + (H^{i(\alpha)} ds - h^{i(\alpha)k} ds_k) \, \delta d_{i(\alpha)} +$$

$$+ (R^i ds - r^{ik} ds_k) \, \delta p_i + \| s^{ik} + \frac{1}{c} v^k g^i \| \, ds_k \, \delta x_i \} +$$

$$+ \int_V \rho \, (f^i \delta x_i + l^{i(\alpha)} \, \delta d_{i(\alpha)} + (s^{ik}{}_{,k} - \frac{1}{c} \frac{\partial}{\partial t} g^i) \, \delta x_i + \epsilon^i \delta p_i \} dv +$$

$$+ \int_V \{ t^{ik}{}_{,k} \, \delta x_i + t^{ik} \, \delta x_{i,k} + h^{i(\alpha)k}{}_{,k} \, \delta d_{i(\alpha)} + h^{i(\alpha)k} \, \delta d_{i(\alpha),k} +$$

$$+ r^{ik}{}_{,k} \, \delta p_i + r^{ik} \, \delta p_{i,k} \} dv. \tag{2.10}$$

If we apply (2.10) to the tetrahedron and let the tetrahedron shrink to zero while preserving the orientation of its faces, we obtain the boundary conditions

$$T^i = \{t^{ik} - [\![s^{ik} + \frac{1}{c} v^k g^i]\!] \} n_k \, , \tag{2.11}$$

$$H^{i(\alpha)} = h^{i(\alpha)k} n_k \qquad\qquad H^{ij} = h^{ijk} \, n_k \, , \tag{2.12}$$

$$R^i = r^{ik} \, n_k \, , \tag{2.13}$$

where

$$h^{ijk} = h^{i(\alpha)k} \, d^j_{.(\alpha)} \, . \tag{2.14}$$

Now, the equation (2.10) can be written in the form

$$\int_V \rho \, (\dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \, \delta d_{i(\alpha)} + \Gamma^i \delta p_i) \, dv + \int_V \rho \delta \;\; w \, dv =$$

$$= \int_V [(t^{ik}_{,k} + s^{ik}_{,k} - \frac{1}{c} \frac{\partial}{\partial t} g^i + \rho f^i) \, \delta x_i + (h^{i(\alpha)k}_{,k} + l^{i(\alpha)}) \delta d_{i(\alpha)} +$$

$$+ t^{ik} \, \delta x_{i,k} + h^{i(\alpha)k} \, \delta d_{i(\alpha),k} + (r^{ik}_{,k} + \epsilon^i) \, \delta p_i + r^{ik} \, \delta p_{i,k}] dv. \tag{2.15}$$

We shall obtain the differential equations of motion by applying Piola's theorem in the form presented by TRUESDELL and TOUPIN [3]: the equation (2.15) for virtual translations is equivalent to Cauchy's first law of motion, and for rigid displacements, to Cauchy's second law.

For virtual translations we have

$$\delta w = 0, \qquad \delta x_i = \text{const.}, \qquad \delta d_{i(\alpha)} = 0, \qquad \delta d_{i(\alpha),k} = 0,$$

$$\delta p_i = 0, \qquad \delta p_{i,k} = 0, \tag{2.16}$$

and equation (2.15) becomes

$$\int_V \rho \dot{v}^i \delta x_i dv = \int_V (t^{ik}_{,k} + s^{ik}_{,k} - \frac{1}{c} \frac{\partial}{\partial t} g^i + \rho f^i) \, \delta x_i \, dv, \tag{2.17}$$

wherefrom

$$\rho \dot{v}^i = t^{ik}_{,k} + s^{ik}_{,k} - \frac{1}{c} \frac{\partial}{\partial t} g^i + \rho f^i . \tag{2.18}$$

This is Cauchy's first law of motion, i.e. necessary and sufficient condition for the balance of momentum, which represents three differential equations of motion.

If we assume that equation (2.18) is valid, equation (2.15) becomes

$$\int_v \rho\, (\Gamma^{i(\alpha)}\, \delta d_{i(\alpha)} + \Gamma^i \delta p_i)\, dv + \int_v \rho \delta\, w dv =$$

$$\int_v [t^{ik}\, \delta x_{i,k} + (h^{i(\alpha)k}{}_{,k} + l^{i(\alpha)})\, \delta d_{i(\alpha)} + h^{i(\alpha)k}\, \delta d_{i(\alpha),k} +$$

$$+ (r^{ik}{}_{,k} + \epsilon^i)\, \delta p_i + r^{ik} \delta p_{i,k}]\, dv. \qquad (2.19)$$

For virtual rigid displacements we have $\delta w = 0$ and

$$\delta x_{(i,k)} = 0, \qquad \delta x_{[i,k]} = \text{const}, \qquad \delta d_{i(\alpha)} = \delta x_{[i,k]}\, d^k_{.(\alpha)}\, , \qquad (2.20)$$

$$\delta d_{i(\alpha),k} = \delta x_{[i,j]}\, d^j_{.(\alpha),k}\, , \qquad \delta p_i = \delta x_{[i,k]}\, p^k\, , \qquad \delta p_{i,k} = \delta x_{[i,j]}\, p^j_{,k}\, ,$$

and equation (2.19) becomes

$$\int_v [t^{ij} + h^{ijk}{}_{,k} + \rho\, (l^{ij} - \Gamma^{ij}) + (r^{ik}{}_{,k} + \epsilon^i - \Gamma^i)\, p^j +$$

$$+ r^{ik}\, p^j_{,k}]\, \delta x_{[i,j]}\, dv = 0. \qquad (2.21)$$

If we write

$$\tau^{ij} = t^{ij} + h^{ijk}{}_{,k} + \rho(l^{ij} - \Gamma^{ij}) + L^i p^j + r^{ik}\, p^j_{,k}\, , \qquad (2.22)$$

and

$$L^i = r^{ik}{}_{,k} + \epsilon^i - \Gamma^i, \qquad (2.23)$$

hen the equation (2.21) can be written in the form

$$\int_v \tau^{ij}\, \delta x_{[i,j]}\, dv = 0. \qquad (2.24)$$

Since this equation is valid for any $\delta x_{[i,j]}$, we obtain

$$\tau^{[ij]} = 0 \quad , \text{ or } (t^{ij} + h^{ijk}{}_{,k} + \rho(l^{ij} - \Gamma^{ij}) + L^i p^j + r^{ik} p^j_{,k})_{[ij]} = 0. \qquad (2.25)$$

This is Cauchy's second law of motion, i.e. necessary and sufficient condition for the balance of the moment of momentum.

From (2.25) it follows that τ^{ij} is a symmetric tensor. Then, (2.22) and (2.23) represent the system of twelve differential equations of motion, where l^{ij} and ϵ^i are prescribed, while t^{ij}, τ^{ij}, h^{ijk}, r^{ij} and L^i have to be determined from the constitutive equations.

Equation (2.22) can be written in the form

$$\tau^{ij} = t^{ij} + h^{i(\alpha)k}{}_{,k}\, d^j_{.(\alpha)} + h^{i(\alpha)k} d^j_{.(\alpha),k} + r^{ik} p^j_{,k} +$$

$$+ \rho\,(l^{i(\alpha)} - \Gamma^{i(\alpha)})\, d^j_{.(\alpha)} + (r^{ik}{}_{,k} + \epsilon^i - \Gamma^i)\, p^j\,, \tag{2.26}$$

wherefrom it follows

$$h^{i(\alpha)k}{}_{,k} + \rho^{i(\alpha)} = (\tau^{ij} - t^{ij})\, d^{(\alpha)}{}_{.j} - h^{i(\alpha)k}\, d^j_{.(\alpha),k} +$$

$$+ \rho\Gamma^{i(\alpha)} - L^i p^j d^{(\alpha)}{}_{.j} - r^{ik}\, p^j_{,k}\, d^{(\alpha)}{}_{.j}\,. \tag{2.27}$$

Using this equation, (2.19) can be written in the form

$$\int_V \rho\delta\, w dv = \int_V [t^{ik}\,\delta x_{i,k} + (\tau^{ij} - t^{ij} - L^i p^j - r^{ik} p^j_{,k} - h^{i(\alpha)k}\, d^j_{.(\alpha),k})\, x$$

$$x d^{(\alpha)}{}_{.j}\,\delta d_{i(\alpha)} + h^{i(\alpha)k}\,\delta d_{i(\alpha),k} + L^i\,\delta p_i + r^{ik}\,\delta p_{i,k}]\, dv, \tag{2.28}$$

wherefrom

$$\rho\delta w = t^{ik}\,\delta x_{i,k} + (\tau^{ij} - t^{ij} - L^i p^j - r^{ik} p^j_{,k} - h^{i(\alpha)k}\, d^j_{.(\alpha),k})\, x$$

$$x d^{(\alpha)}{}_{.j}\,\delta d_{i(\alpha)} + h^{i(\alpha)k}\delta d_{i(\alpha),k} + L^i\delta p_i + r^{ik}\,\delta p_{i,k}. \tag{2.29}$$

This is the expression for the variation of the specific strain energy, which is form–invariant with respect to the superposed rigid motion. This expression we can write in the form

$$\rho\delta w = t^{ik}\, X^K_{;k}\,\delta x_{i;K} + (\tau^{ij} - t^{ij} - L^i p^j - r^{ik} p^j_{,k} - h^{i(\alpha)k} d^j_{.(\alpha),k})\, X$$

$$xd^{(\alpha)}{}_{.j}\,\delta d_{i(\alpha)} + h^{i(\alpha)k}\,X^{K}_{;k}\delta d_{i(\alpha);K} + L^{i}\delta p_{i} + r^{ik}\,X^{K}_{;k}\,\delta p_{i;K}, \tag{2.30}$$

and we see that the specific strain energy is a function of the form

$$w = w\,(x^{k}_{;K},\, d^{k}_{.(\alpha)},\, d^{k}_{.(\alpha);K},\, p^{k}\ ,\, p^{k}_{;K}). \tag{2.31}$$

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УПРУГИЙ ДИЭЛЕКТРИК С МИКРОСТРУКТУРОЙ

Содержание:

В работе рассматривается сплошная среда с микроструктурой обладающая качествами диэлектрика представлнющая из себя упругий обобщенний континуум Косера в электромагнитном поле. Тепловыми эффектами пренебрегается. Дифференциальные уравнения движений и уравнение баланса энергии деформации выводятся из сформулированного принципа виртуальной работы и теоремы Пиоли.

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