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OSCILLATIONS OF A DISK IN NONHOMOGENEOUS FLUIDS AT A SPECIFIC LAW OF STRATIFICATION

Vladan Djordjević

1. Introduction

It is well known that the flow phenomena arising at small forced oscillations of a body in nonhomogeneous fluids mutually differ very much in those regions of the fluid where the frequency of the body is greater or less than the Brunt-Väisälä frequency which depends on the law of stratification [1]. In the first case the equations describing the steady–state solution are namely of elliptic, and in the second case of hyperbolic type. The Brunt–Väisälä frequency is constant for an exponential law of stratification, so that the flow phenomena in that case possess characteristic properties of elliptic or hyperbolic type of differential equations throughout the fluid. For this law of stratification Görtler has carried out experiments with a half–infinite vertical plate which oscillated in the horizontal direction [1]. In the hyperbolic case the characteristics starting from the edge of the plate were clearly visual within wholly fluid and were also reflected on its boundaries. It is also known that there is a certain relationship between these problems and the problems of small forced oscillations of a body in homogeneous but rotating fluids, whereby the constant double angular velocity in rotating fluids [2] takes over the role of the Brunt–Väisälä frequency in nonhomogeneous fluids. While some concrete problems of oscillations of bodies in rotating fluids have been discussed in detail [3], [4], whereby some new phenomena have been discovered, which could not be forseen by simple analysis of the type of differential equations, as for example the phenomenon of resonance in an enclosed rotating fluid [5], [6], [7], so far solutions of some concrete problems of oscillations of bodies in nonhomogeneous fluids fail, because of certain difficulties of mathematical nature, even in the case of an exponential law of stratification, in which the governing differential equations are reduced to the equations with constant coefficients.

It has been shown in this paper that the governing differential equations can be by means of a convenient transform of coordinates reduced to the equations with constant coefficients at a specific law of stratification, different from exponential one! The solutions of the cases of elliptic and hyperbolic oscilations of a disk at this law of stratification have been obtained by means of the method of integral transforms. It has been shown that the phenomenon of resonance appears in case of hyperbolic oscillations of an enclosed nonhomogeneous fluid, with what the known relationship between these problems and the problems of flows in rotating fluids has been still more deepened. The complex problem of oscillations in a rotating nonhomogeneous fluid has been analysed, too. It has been pointed out at some interesting conclusions following from the mutual influence of the rotation and the stratification of the fluid.

2. The Governing Differential Equations

The governing differential equations in this paper are in the usual notations. The Euler equation:

$$\rho \frac{D\vec{v}}{Dt} = -g\rho\vec{k} - \nabla p,$$

the conditions of incompressibility of individual particles of fluid:

$$\frac{D\rho}{Dt} = 0, \text{or } \frac{\partial\rho}{\partial t} + (\vec{v}, \nabla)\rho = 0$$

and the equation of continuity: $(\nabla, \vec{v}) = 0.$

It is supposed that the fluid is in the field of the gravitational force and that the stratification is statically stable. If $\tilde{\rho}(z)$ and $\tilde{p}(z)$ denote the density and the pressure in the state of equilibrium, they will be bound with the basic equation of hydrostatic:

$$\frac{d\tilde{p}}{dz} = -g\tilde{\rho}(z), \frac{d\tilde{\rho}}{dz} < 0.$$

If the small perturbations appearing at forced oscillations of bodies in such a fluid will be donoted by:

$$\rho = \tilde{\rho}(z) + \rho'(t,\vec{r}), \; p = \tilde{p}(z) + p'(t,\vec{r}), \; \vec{v} = \vec{v}'(t,\vec{r}),$$

and if the products of the perturbation quantities will be neglected, that is if a linearization of the governing equations will be carried out, it will be obtained:

$$\tilde{\rho}\,\frac{\partial \vec{v}'}{\partial t} = - g\rho'\,\vec{k} - \nabla p'$$

$$\frac{\partial \rho'}{\partial t} + \frac{d\tilde{\rho}}{dz}\, w' = 0$$

$$(\nabla, \vec{v}') = 0,$$

or in cylindrical coordinates for the case of axisymmetry:

$$\tilde{\rho}\,\frac{\partial u'}{\partial t} = - \frac{\partial p'}{\partial r}$$

$$\tilde{\rho}\,\frac{\partial w'}{\partial t} = - g\rho' - \frac{\partial p'}{\partial z}$$

$$\frac{\partial \rho'}{\partial t} + \frac{d\tilde{\rho}}{dz}\, w' = 0$$

$$\frac{\partial (ru')}{\partial r} + \frac{\partial (rw')}{\partial z} = 0.$$

We will introduce the following scales for length, time, velocity, density and pressure: r_o, $t_o = \sqrt{r_o/g}$, u_o, ρ_o, $p_o = \rho_o\, u_o \sqrt{g r_o}$, and we will denote the corresponding nondimensional quantities with capitals: $T = t/t_o$, $R = r/r_o$, $Z = z/r_o$, $U = u'/u_o$, $W = w'/u_o$, $Q = \rho'/\rho_o$, $Q_o(Z) = \tilde{\rho}(z)/\rho_o$, $P = p'/p_o$. We will obtain:

$$Q_o\,\frac{\partial U}{\partial T} = - \frac{\partial P}{\partial R}$$

$$Q_o\,\frac{\partial W}{\partial T} = - \frac{\sqrt{g r_o}}{u_o}\, Q - \frac{\partial P}{\partial Z}$$

$$\frac{\partial Q}{\partial T} + \frac{u_o}{\sqrt{g r_o}}\, Q_o' W = 0$$

$$\frac{\partial(RU)}{\partial R} + \frac{\partial(RW)}{\partial Z} = 0,$$

where Q_o' denotes the corresponding derivation. If we introduce now on the usual way the stream function $\psi(T,R,Z)$ and the function $\psi_1 = \psi/R$ we will obtain by eliminating:

$$\frac{\partial^2}{\partial T^2}\left(\frac{\partial^2 \psi_1}{\partial Z^2} + \frac{Q_o'}{Q_o}\frac{\partial \psi_1}{\partial Z}\right) + \left(\frac{\partial^2 \psi_1}{\partial R^2} + \frac{1}{R}\frac{\partial \psi_1}{\partial R} - \frac{\psi_1}{R^2}\right)\left(\frac{\partial^2}{\partial T^2} - \frac{Q_o'}{Q_o}\right) = 0.$$

If the steady–state solution of this equation is denoted by $\tilde{\psi}_1(R,Z)$, it will be: $\tilde{\psi}_1 = \psi_1 \exp(i\beta T)$, where $\beta = \beta_1 \sqrt{r_o/g}$, and β_1 is the frequency of forced oscillations. We thus obtain:

$$\frac{\partial^2 \tilde{\psi}_1}{\partial Z^2} + \frac{Q_o'}{Q_o}\frac{\partial \tilde{\psi}_1}{\partial Z} + \left(1 + \frac{1}{\beta^2}\frac{Q_o'}{Q_o}\right)\left(\frac{\partial^2 \tilde{\psi}_1}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{\psi}_1}{\partial R} - \frac{\tilde{\psi}_1}{R^2}\right) = 0 \quad ..(1)$$

This equation represents the starting point in investigations of small forced oscillations of bodies in nonhomogeneous fluids. The type of the equation obviously depends on the ratio of the nondimensional frequency of forced oscillations and the Brunt–Väisälä frequency $N(Z) = \sqrt{-Q_o'/Q_o}$. In the case of an exponential law of stratification, for example $Q_o = \exp(-Z)$, the equation (1) becomes an equation with constant coefficients. Nevertheless, some concrete solutions of this equation fail even in that, at the first sight, simple case, as it has been remarked in the introduction.

3. Transform of the Governing Equations and Boundary Conditions

In order to obtain some solutions of the equation (1) for any law of stratification, we will carry out the following transform of the coordinate Z:

$$\zeta = \int_o^Z \frac{dZ}{Q_o(Z)}, \quad Z > 0$$

and we will obtain:

$$\frac{\partial^2 \tilde{\psi}_1}{\partial \zeta^2} + \left(Q_o^2 + \frac{Q_o Q_o'}{\beta^2}\right)\left(\frac{\partial^2 \tilde{\psi}_1}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{\psi}_1}{\partial R} - \frac{\tilde{\psi}_1}{R^2}\right) = 0. \quad ...(2)$$

This equation can also become an equation with constant coefficients, if:

$$Q_o^2 + \frac{Q_o Q_o'}{\beta^2} = \pm a^2 , \qquad \ldots (3)$$

where a is an arbitrary constant. It will be of elliptic type in case of the upper sign, and of hyperbolic type in case of the Jower sign. With the initial condition $Q_o(O)=1$, the equation (3) has the solution:

$$Q_o^2 = (1 \mp a^2) \exp(-2\beta^2 Z) \pm a^2 , \qquad \ldots (4)$$

or in dimensional form:

$$\tilde{\rho}^2 (z) = \rho_o^2 \left[(1 \mp a^2) \exp\left(-2 \frac{\beta_1^2}{g} z\right) \pm a^2 \right].$$

Such a law of stratification is very specific and artifical, because it contains the frequency β_1 from which naturally it does not depend. Nevertheless, it could be easily achieved in an eventual experiment. Furthermore, the flow phenomena appearing in a nonhomogeneous fluid probably do not depend qualitatively upon the law of stratification! Consequently, we will accept it in the further work. It must be $a < 1$ in the elliptic case, because the stability condition: $Q_o' < 0$ would not be fulfilled otherwise. In the hyperbolic case the density becomes zero at a height $H_o > 0$. Consequently, the fluid must be bound in the direction of the Z axis. The coordinate ζ can be now calculated. It will be in the elliptic case:

$$\zeta = \frac{1}{2a\beta^2} \ln \frac{(Q_o + a)(1 - a)}{(Q_o - a)(1 + a)} , \quad a < 1$$

what gives: for $Z = 0, \zeta = 0$ and for $Z \to \infty, \zeta \to \infty$. It will be in the hyperbolic case:

$$\zeta = \frac{1}{a\beta^2} \operatorname{arctg} \frac{a(1 - Q_o)}{a^2 + Q_o} ,$$

what gives: for $Z = 0$, $\zeta = 0$ and for $Z = H_o$, $\zeta = \zeta_o = \dfrac{\operatorname{arctd} 1/a}{a\beta^2}$.

We will assume that the flow has been caused by oscillations in the direction of z of a disk $z = 0$, $r < r_o$, which is placed in the aperture of an infinite plate $z = 0$, $r > r_o$. The boundary conditions will be:

$$\zeta = 0 : W = \begin{matrix} \exp(i\beta T), & R < 1 \\ 0 \quad , & R > 1 \end{matrix}$$

or:

$$\zeta = 0 : \frac{\partial \tilde{\psi}_1}{\partial R} + \frac{\tilde{\psi}_1}{R} = \begin{matrix} -1, R < 1 \\ 0, R > 1 \end{matrix} \qquad \ldots . (5)$$

and for $\zeta \to \infty$ in the elliptic case, and $\zeta = \zeta_o$ in the hyperbolic case:

$$\frac{\partial \tilde{\psi}_1}{\partial R} + \frac{\tilde{\psi}_1}{R} = 0, \text{ for all R.} \qquad .. (6)$$

4. Solutions of the Governing Equations

We will find the solutions of the equation (2) at the specific law of stratification (4) and with the boundary conditions (5) and (6) by the method of integral transforms [8]. If we denote the Hankel integral transform of the function $\tilde{\psi}_1$ by $\tilde{\tilde{\psi}}_1$:

$$\tilde{\tilde{\psi}}_1 = \int_o^\infty R J_1 (KR)\, \tilde{\psi}_1 dR,$$

we will obtain in the elliptic case:

$$\tilde{\tilde{\psi}}_1'' - a^2 K^2\, \tilde{\tilde{\psi}}_1 = 0.$$

The solution of this equation, which is finite for $\zeta \to \infty$ is:

$$\tilde{\tilde{\psi}}_1 = A \exp(-aK\zeta),$$

so that the inversion formula leads to:

$$\tilde{\psi}_1 = \int_o^\infty KAJ_1 (RK) \exp(-aK\zeta)\, dK.$$

Applying the boundary condition (5) the constant of integration is obtained as: $A = -J_1(K)/K^2$ so that the function ψ is finally:

$$\psi = -R \exp(i\beta T) \int_0^\infty \frac{J_1(K)\, J_1(RK)}{K} \exp(-aK\zeta)\, dK.$$

Following [9], this integral can be expressed by a series in terms of hypergeometrical functions, if necessary.

It is obtained in the hyperbolic case:

$$\tilde{\tilde{\psi}}_1'' + a^2 K^2 \tilde{\tilde{\psi}}_1 = 0,$$

with the solution:

$$\tilde{\tilde{\psi}}_1 = A \sin aK\zeta + B \cos aK\zeta.$$

The applying of the boundary conditions (6) and (5) leads to:

$$A \sin aK\zeta_o + B \cos aK\zeta_o = 0, \qquad B = -J_1(K)/K^2.$$

This system of equations for calculation of the constants A and B has a finite solution only if $K \neq n\pi/a\zeta_o$ (n–integer). Since K in the inversion formula passes through all values from zero to ∞, this condition will not be satisfied. Consequently, in the hyperbolic case at the specific law of stratification (4), which has been accepted here, the amplitudes of oscillations will become infinite, that is the resonance will appear. As it was remarked in the introduction, the phenomenon of resonance, which appears in rotating homogeneous fluids under similar circumstances, was known still earlier [5], [6], [7]. We assume that the known relationship between these problems has been in such a way, that is by discovering of the phenomenon of resonance in nonhomogeneous fluids, still more deepened.

The corresponding solutions in the region $Z < 0$ can be obtained in a similar way. By the transform:

$$\zeta = \int_Z^o \frac{dZ}{Q_o(Z)}, \quad Z < 0$$

we obtain the same equation (2), as earlier. It will be in the elliptic case:

$$\zeta = \frac{1}{2a\beta^2} \ln \frac{(Q_o - a)(1 + a)}{(Q_o + a)(1 - a)} \quad , a < 1$$

what gives: for Z = 0, $\zeta = 0$ and for $Z \to -\infty$, $\zeta = \zeta_o = \frac{1}{2a\beta^2} \ln \frac{1 + a}{1 - a}$.

It will be in the hyperbolic case:

$$\zeta = \frac{1}{a\beta^2} \operatorname{arctg} \frac{a(Q_o - 1)}{Q_o + a^2} ,$$

what gives: for Z = 0, $\zeta = 0$ and for $Z \to -\infty$, $\zeta = \zeta_o = \frac{\operatorname{arctg} a}{a\beta^2}$

The solution in the elliptic case is:

$$\psi = - R \exp(i\beta T) \int_o^\infty \frac{J_1(K) J_1(RK)}{K} \frac{\operatorname{sh} aK(\zeta_o - \zeta)}{\operatorname{sh} aK\zeta_o} dK$$

while in the hyperbolic case the resonance appears again, as in the region $Z > 0$.

5. Case of Rotating Nonhomogeneous Fluids

It would be useful to consider briefly at the end the problem of oscillations of a disk in a nonhomogeneous fluid rotating round the axis z with the constant angular velocity ω. A possibility would be offered with that to investigate the mutual influence of the rotation, that is of the Coriolis force appearing on this occasion, and the stratification of the fluid. It is known [10] that in general case the equilibrium of rotating nonhomogeneous fluids is not possible. The equilibrium is possible only in case when the centrifugal force can be neglected regarding to the gravitational force, what is meaningful for relatively small angular velocities near the axis of rotation.

The Euler equation contains in that case en additional term – Coriolis force and it is:

$$\rho \frac{D\vec{v}}{Dt} + 2\rho [\vec{\omega}, \vec{v}] = - g\rho \vec{k} - \nabla p, \vec{\omega} = \omega \vec{k}$$

while the other governing equations remain unchanged. In case of small perturbations this equation becomes:

$$\tilde{\rho}\,\frac{\partial \vec{v}'}{\partial t} + 2\omega\,\tilde{\rho}\,[\vec{k}, \vec{v}'] = -\,g\rho'\,\vec{k} - \nabla p'.$$

It is obtained now in the same way as earlier instead of the equation (1):

$$(1 - \frac{4\omega^2}{\beta_1^2})(\frac{\partial^2 \tilde{\psi}_1}{\partial Z^2} + \frac{Q_o'}{Q_o}\frac{\partial \tilde{\psi}_1}{\partial Z}) + (1 + \frac{1}{\beta^2}\frac{Q_o'}{Q_o})(\frac{\partial^2 \tilde{\psi}_1}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{\psi}_1}{\partial R} - \frac{\tilde{\psi}_1}{R^2}) = 0,$$

or, after introducing of the coordinate ζ:

$$(1 - \frac{4\omega^2}{\beta_1^2})\,\frac{\partial^2 \tilde{\psi}_1}{\partial \zeta^2} + (Q_o^2 + \frac{Q_o Q_o'}{\beta^2})(\frac{\partial^2 \tilde{\psi}_1}{\partial R^2} + \frac{1}{R}\,\frac{\partial \tilde{\psi}_1}{\partial R} - \frac{\tilde{\psi}_1}{R^2}) = 0.$$

It is seen that this equation becomes an equation with constant coefficients at the same specific law of stratification (4), as the equation (2). However, its type does not depend now only on the choice of the sign in the term given by (3), but also on the ratio of the frequency of forced oscillations β_1 and the double angular velocity of the fluid. For example, in case of the upper sign the equation will be of elliptic type if $\beta_1 > 2\,\omega$, and of hyperbolic type if $\beta_1 < 2\omega$. At any rate, it can be reduced by means of the transform:

$$\zeta_1 = \zeta / \sqrt{\left|1 - \frac{4\omega^2}{\beta_1^2}\right|}$$

to the equation (2) and be solved in the same way. Quite analogous solutions are obtained, so that it will not be discussed particularly.

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SCHWINGUNGEN EINER SCHEIBE IN GESCHICHTETER FLUSSIGKEIT BEI EINEM SPEZIFISCHEN STRATIFIKATIONSGESETZ

Vladan D. Djordjević

Zusammenfassung

In der Arbeit wurde gezeigt, wie die Grundleichungen von kleinen erzwungenen Schwingungen von Körpern in geschichteter Flüssigkeit bei einem spezifischen Stratifikationsgesetz (verschieden von exponential!) zu den Gleichungen mit konstanten Koeffizienten zurückgeführt werden können. Die Lösungen von elliptischen und hyperbolischen Schwingungen einer Scheibe wurden durch die Methode der Integraltransformationen erhalten. Es wurde des Bestehen eines Resonanzefektes in einer räumlich begrenzten geschichteten Flüssigkeit gezeigt. Es wurde auch der komplexe Fall von Schwingungen in rotierender geschichteter Flüssigkeit analysiert und auf einige interessante Folgerungen, die aus der Zusammenwirkung der Rotation und der Stratifikation der Flüssigkeit folgen, hingewiesen.