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ON BORN'S RELATIVISTIC RIGIDITY AND SOME PROPERTIES OF MHD STEADY FLOWS

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Abstract

We consider, in this paper, a vector field in Space–time V_4, which is steady in the relativistic sense, i.e. which does not depend on a timelike parameter. This means that there exists, in every point of V_4, a frame such that corresponding vector of the field considered is steady with respect to its proper time. We choose this frame among those which see considered vector in its „original" length, i.e. which perform motions locally orthogonal to it. So we obtain a property of that kind of frames, to have either Born's rigidity, in the direction parallel to the vector considered, or to move geodesically.

In the second part of the paper we consider the steadiness of some physical variables with respect to their proper time, in a MHD continuum. We draw the conslusion, for an incompressible fluid in a magnetic field with steady covariant components, that the deformation tensor is two–dimensional and lies in the plane locally orthogonal to the four velocity and the magnetic field. We obtain also necessary and sufficient conditions for the steadiness of the magnetic field in a compressible fluid.

We show, in the third part, for an energetically undetermined MHD continuum, in stationary Space–time (four velocity is a Killing vector) with a steady magnetic field, how is varying the intensity of the vorticity along magnetic field lines, and draw a consequence in the case of a perfect fluid.

1. We consider Space–time V_4 with a metric given by the condition that the quadratic form

$$\phi \equiv g_{\alpha\beta}\, \lambda^{\alpha} \lambda^{\beta}$$

is positive for spacelike vectors λ^a negative and null for, respectively timelike and light vectors. In this part of the paper we do not restrict ourselves to consider only Special or General Relativity. In the latter case, metric tensor is supposed to be continuous, and to have continuous first derivatives. All the other physical quantities are supposed to be derivable until second order, no matter continuously or not.

The steadiness of a variable (its independece of a timelike parameter, given by a tangent unit vector ξ^a) is expressed by the nullity of its Lie derivative in the direction ξ^a. We shall consider as completely steady a specelike vector field h^a, given in a domain of V_4, if Lie derivatives of both its covariant and contravariant components are equal to zero:

$$\mathcal{L}_\xi (h_\beta) \equiv \xi^\alpha \nabla_\alpha h_\beta + h_\alpha \nabla_\beta \xi^\alpha = 0 \qquad (1.1)$$

$$\mathcal{L}_\xi (h^\beta) \equiv \xi^\alpha \nabla_\alpha h^\beta - h^\alpha \nabla_\alpha \xi^\beta = 0$$

wherefrom

$$(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)\, h^\beta = 0 \qquad (1.2)$$

Δ is the symbol of covariant derivation and $\mathcal{L}_\xi$ the symbol of Lie derivative in the direction given by ξ.

The expression inside the brackets in (1.2) being the Lie derivative of the gravitational potential $g_{\alpha\beta}$, that system means that the components of $g_{\alpha\beta}$ parallel to h^a do not depend on the proper time of a local observer's frame with velocity ξ^a. That condition can be formulated as:

$$\mathcal{L}_\xi (g_{\alpha\beta}) = \mathcal{L}_\xi (g_{\gamma\delta})\, S^\gamma_\alpha S^\delta_\beta \qquad (1.3)$$

where S^γ_α is the projection tensor orthogonal to h^a $(S^\gamma_\alpha h_\gamma = 0)$:

$$S^\gamma_\alpha \equiv \frac{1}{h^2} (h_\alpha h^\gamma - h^2 \delta^\gamma_\alpha)$$

Explicitely (1.3) can be written as follows:

$$\tau_{\alpha\beta} \equiv h_\alpha h_\beta h^\gamma h^\delta (\nabla_\gamma \xi_\delta + \nabla_\delta \xi_\gamma) - h^2 [h_\alpha h^\gamma (\nabla_\beta \xi_\gamma + \nabla_\gamma \xi_\beta) + \qquad (1.3')$$

$$+ h_\beta h^\gamma (\nabla_\gamma \xi_\alpha + \nabla_\alpha \xi_\gamma)] = 0 \tag{1.3'}$$

which can be easily verified. If we multiply (1.3') by h^a we obtain (1.2); conversely, it follows from (1.2) that the left hand side in (1.3') must vanish. So these relations are equivalent.

Timelike unit vector ξ^a is the four velocity of a rigid motion (in the sense of Born) if it satisfies next relations:

$$\sigma_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + u_\alpha \tilde{w}_\beta - u_\beta \tilde{w}_\alpha = 0 \tag{1.4}$$

(see [1], [3], [5]) where

$$\tilde{w}^\alpha \equiv \xi^\beta \nabla_\beta \xi^\alpha \tag{1.5}$$

is the four acceleration of an observer's frame. If considered vector h^a is steady with respect to a frame with velocity ξ^a, chosen in such a way that h_a has the same spatial length as in the comoving frame (cf [3]), four velocity ξ^a will be orthogonal to four vector h_a. We shall now restrict ourselves to such local frames. In space three velocity corresponding to ξ^a can be expressed by means of two components, one of them being equal to the spatial velocity of the comoving frame (because h^a can be a vector generated in a material medium, and attached to its motion), the other orthogonal to it, with arbitrary intensity and direction. We have so:

$$h^\alpha \xi_\alpha = 0 \tag{1.6}$$

We must point out that property (1.6) has to be understood in the sense that that vector field h^a is, in every point of the domain considered, orthogonal to a field ξ_a, which represents four velocities of hypothetic local observers. So the derivatives of (1.6) must also be null.

If we multiply by h^a the left hand side of (1.4), without the assumption that Born's rigidity (1.4) holds, we obtain from (1.2) and (1.6):

$$\sigma_{\alpha\beta} h^\alpha = h^\alpha \tilde{w}_\alpha \xi_\beta$$

From (1.6) and the second relation (1.1) we have

$$h^\alpha \tilde{w}_\alpha = 0$$

Hence

$$\sigma_{\alpha\beta} h^\alpha = 0 \tag{1.7}$$

This means that we have Born's rigidity of the observer's frame in the direction parallel to the vector h^a. Suppose, conversely, that a local frame has Born's rigidity in a direction parallel with a vector which is steady with respect to it. We can easily conclude that this frame must either perform a geodesic motion (ω_a=0) or have four velocity orthogonal to considered four vector. Let us remark that h^a is not steady in general with respect to its comoving frame.

It is simple to verify that if only contravariant coordinates of $\vec{h}$ are steady, together with the condition (1.6) of „orthogonal" motion and Born's rigidity of the frame in the direction parallel to it (1.7), we obtain full steadiness of $\vec{h}$ as a consequence. But this is not the case when we associate the steadiness of *covariant* coordinates only to the other two conditions mentioned above.

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2. We shall identify the vector h^a, considered in the preceding section, with the magnetic field vector of a MHD continuum, with a constant magnetic permeability.

Magnetic field vector must satisfy Maxwell's equations for such a medium. We shall take into consideration only the group of these equations which does not involve electric current. They read (cf [8], [9]

$$\nabla_\alpha (h^\alpha u^\beta - h^\beta u^\alpha) = 0 \tag{2.1}$$

Where u^a is the four velocity of this continuum; it is a unit vector orthogonal to h_a. Of all the other equations describing our fluid we shall take in consideration only the equation of conservation of the mass, which is

$$\nabla_\alpha (ru^\alpha) = 0 \tag{2.2}$$

where r is the proper density of matter. The equations of motion and thermodynamic conditions will not be considered.

We shall formulate now the steadiness of covariant coordinates h_a only, with respect to the *proper* time given by u_a:

$$\mathcal{L}_u h_\beta \equiv u^\alpha \nabla_\alpha h_\beta + h_\alpha \nabla_\beta u^\alpha = 0 \tag{2.3}$$

The deformation tensor $\sigma_{a\beta}$, given by the left hand side of (1.4), when formulated with respect to u_a gives:

$$\sigma_{\alpha\beta}\, h^\alpha = h^\alpha \nabla_\alpha u_\beta + h^\alpha \nabla_\beta u_\alpha + h^\alpha w_\alpha u_\beta \tag{2.4}$$

After multiplying Maxwell's equations (2.1) by u^β and obtaining therefrom the scalar product $h^a w_\alpha$ ($w_\alpha \equiv u^\gamma \nabla_\gamma u_a$), we shall obtain, after substitution of that expression in (2.3) and (2.4):

$$\sigma_{\alpha\beta}\, h^\alpha = \nabla_\alpha u^\alpha \cdot h_\beta \tag{2.5}$$

We have the conclusion that, in the case of steadiness of the covariant magnetic field vector in MHD, this vector is an eigenvector of the deformation tensor $\sigma_{a\beta}$, the corresponding eigenvalue $\nabla_a u^a$ representing the specific rate of change of the proper volume with respect to proper time, as it is shown in Synge's monograph(cf [3]). Magnetic field vector gives then a characteristic direction of the deformation.

If multiplying now Maxwell's equations by h^β, and taking in account right hand sides of (2.4) and (2.5), we shall have

$$u^\alpha \partial_\alpha (h^2) + \nabla_\alpha u^\alpha \cdot h^2 = 0$$

After substituting of $\nabla_a u^a$ from the equation of continuity (2.2), we obtain

$$u^\alpha \partial_\alpha (r h^{-2}) = 0 \tag{2.6}$$

This means that the quantity $r\, h^{-2}$ remains constant along world lines. This result is analoguous to that given by Carstoiu in his paper [7] for classical MHD. Let us remark that if we add incompressibility ($\nabla_a u^a = 0$) to previous conditions, we shall obtain as a consequence:

$$\sigma_{\alpha\beta}\, h^\alpha = 0$$

or the deformation is always locally orthogonal to h^a.

We shall examine now if conditions (2.5) and (2.6) are sufficient for the steadiness of h_a. Let us write:

$$u^\alpha \partial_\alpha (r^{-1}\; h^2) = 0 \tag{2.7}$$

$$\sigma_{\alpha\beta}\, h^\beta = \varphi h_\alpha \tag{2.8}$$

The first of the above equations gives, because of $r \neq 0$:

$$r u^\alpha \partial_\alpha (h)^2 - h^2\; u^\alpha \partial_\alpha r = 0$$

which, in virtue of (2.2), becomes:

$$u^\alpha \partial_\alpha (h)^2 + h^2\; \nabla_\alpha u^\alpha = 0$$

On the other hand, we have from relations (2.8):

$$2h^{\alpha} h^{\beta} \nabla_{\alpha} u_{\beta} = \varphi h^2$$

and from Maxwell's equations:

$$h^{\alpha} h^{\beta} \nabla_{\alpha} u_{\beta} - \nabla_{\alpha} u^{\alpha} (h)^2 - \frac{1}{2} u^{\alpha} \partial_{\alpha} (h)^2 = 0$$

Wherefrom we conclude that

$$h^{\alpha} h^{\beta} \nabla_{\alpha} u_{\beta} = \frac{1}{2} h^2 \nabla_{\alpha} u^{\alpha}$$

Hence

$$\varphi = \nabla_{\alpha} u^{\alpha}$$

After substituting this value, obtained for φ from (2.9), and using again Maxwell's equations, we shall have:

$$\mathcal{L}_u h_{\alpha} + h^{\beta} w_{\beta} \cdot u_{\alpha} - \Delta_{\beta} h^{\beta} \cdot u_{\alpha} = 0$$

But from these equations we have

$$\nabla_{\beta} h^{\beta} = w_{\beta} h^{\beta}$$

So finally

$$\mathcal{L}_u h_{\alpha} = 0$$

If the quantity rh^{-2} is constant along world lines of a MHD continuum, h^{a} being an eigenvector of the deformation tensor, then corresponding eigenvalue is $\nabla_{\mu} u^{a}$, and covariant coordinates h_a remain steady along these world lines. So relations (2.7) and (2.8) are necessary and sufficient.

We can remark only that all obtained conclusions can be applied to the magnetic induction vector $b_a = \mu h_a$ instead of h_a, for a variable magnetic permeability μ.

We recall that, if adding the conditions of steadiness of the contravariant coordinates h^{a} with respect to proper time to Maxwell's equations (2.1), we should obtain at once the incompressibility and the local orthogonality of the magnetic field and the acceleration vectors:

$$\nabla_{\alpha} u^{\alpha} = 0$$

$$\nabla_{\alpha} h^{\alpha} = 0 \quad \Leftrightarrow \quad h^{\alpha} w_{\alpha} = 0 \qquad (2.9)$$

3. We shall now restrict ourselves to the case of a stationary Space–time, this condition being given by:

$$\mathcal{L}_u (g_{\alpha\beta}) \equiv \Delta_\alpha u_\beta + \nabla_\beta u_\alpha = 0 \qquad (3.1)$$

The gravitational potential $g_{\alpha\beta}$ is so subjected to stationarity with respect to the proper time, determined by u_α, of the rest frame corresponding to the magnetic field. Adding the condition that one type of coordinates of the magnetic field, say covariant ones h_α, are also subjected to the condition of stationarity (or *steadiness* as we used to write in this paper) which is given by (2.3), the other type of coordinates will be automatically steady, as a consequence of (3.1). Under conditions (2.3) and (3.1) we shall examine the properties of the vorticity tensor, which is given (cf Ehlers [4]) by:

$$w_{\alpha\beta} \equiv \nabla_\beta u_\alpha - \nabla_\alpha u_\beta + u_\beta w_\alpha - u_\alpha w_\beta \qquad (3.2)$$

The consequence of (3.1) being incompressibility ($\nabla_a u^a = 0$) and geodesic motion ($w^\alpha = 0$) which is obvious, the scalar ω^2 will reduce to:

$$\omega^2 \equiv \omega_{\alpha\beta}\, \omega^{\alpha\beta} = 4 g^{\alpha\beta} (\nabla_\alpha u_\gamma \cdot \nabla_\beta u^\gamma)$$

The variation of that scalar along a magnetic field line will be locally:

$$h^\epsilon \partial_\epsilon (\omega^2) = 8 g^{\alpha\beta} h^\epsilon \Delta_\alpha u_\gamma \cdot \nabla_\epsilon \nabla_\beta u^\gamma \qquad (3.3)$$

Because of the Ricci identity

$$\nabla_\epsilon \nabla_\beta u_\gamma = \nabla_\beta \nabla_\epsilon u_\gamma + R_{\delta\gamma\epsilon\beta} u^\delta$$

where $R_{\delta\gamma\epsilon\beta}$ is the Riemann–Christoffel curvature tensor of Space–time, we can write (3.3) as

$$h^\epsilon \partial_\epsilon (\omega^2) = 8 g^{\alpha\beta} [\nabla_\beta (h^\epsilon \nabla_\epsilon u^\gamma) - \nabla_\beta h^\epsilon \cdot \nabla_\epsilon u^\gamma] \cdot \nabla_\alpha u_\gamma + 4 \mathbf{R}_{\delta\gamma\epsilon\beta} u^\delta \omega^{\gamma\beta} h^\epsilon \qquad (3.3')$$

Maxwell's equations (2.1) take the form

$$h^\alpha \nabla_\alpha u_\beta - u^\alpha \nabla_\alpha h_\beta = 0$$

as a consequence of steadiness. The first term at the right hand side of (3.3') can be written, when using the above relation, as:

$$g^{\alpha\beta} [\nabla_\beta (h^\epsilon \nabla_\epsilon u^\gamma) - \nabla_\beta h^\epsilon \cdot \nabla_\epsilon u^\gamma] = g^{\alpha\beta} u^\epsilon \nabla_\alpha u_\gamma \cdot \nabla_\beta \nabla_\epsilon h^\gamma \qquad (3.4)$$

Where use was made of (3.1) also. Using the fact that covariant components of h_a must be also steady, we have after transformations

$$g^{\alpha\beta} \nabla_\alpha u_\gamma \theta^\epsilon \cdot u^\epsilon \nabla_\beta \nabla_\epsilon h^\gamma = - g^{\alpha\beta} \nabla_\alpha u^\gamma \cdot h^\epsilon \nabla_\beta \nabla_\gamma u_\epsilon \qquad (3.4')$$

and

$$g^{\alpha\beta} \nabla_\alpha u^\gamma \cdot h^\epsilon \nabla_\beta \nabla_\gamma u_\epsilon = g^{\alpha\beta} \nabla_\alpha u^\gamma \cdot h^\epsilon \nabla_\gamma \nabla_\beta u_\epsilon +$$

$$+ R_{\delta\epsilon\beta\gamma} u^\delta h^\epsilon g^{\alpha\beta} \nabla_\alpha u^\gamma \qquad (3.5)$$

We obtain again

$$g^{\alpha\beta} \nabla_\alpha u^\gamma \cdot h^\epsilon \nabla_\gamma \nabla_\beta u_\epsilon = g^{\alpha\beta} \nabla_\alpha u^\gamma \cdot u^\delta \nabla_\beta \nabla_\delta h_\gamma \qquad (3.6)$$

The right hand side term in (3.6) is identical with the one in (3.4). After substituting (3.4'). and (3.6) in (3.5) we have:

$$g^{\alpha} \nabla_\alpha u^\gamma \cdot u^\delta \nabla_\beta \nabla_\delta h_\gamma = \frac{1}{4} R_{\delta\epsilon\beta\gamma} u^\delta h^\epsilon \omega^{\beta\gamma}$$

And finally, from that relation, (3.4) and (3.3'):

$$h^\epsilon \partial_\epsilon (\omega^2) = 2u^\delta h^\epsilon \omega^{\beta\gamma} (R_{\delta\epsilon\beta\gamma} + 2R_{\delta\beta\epsilon\gamma}) \qquad (3.7)$$

This represents an expression which gives a possibility for the generalisation of the Ferraro theorem of isorotation(cf [6]).Let us take in account the relation given by Ehlers (cf [4]) and used in [11]:

$$R_{\alpha\beta} u^\alpha u^\beta = u^\alpha \partial_\alpha \theta + \frac{1}{3}\theta^2 - \nabla_\alpha w^\alpha + \frac{1}{4}(\sigma^2 - \omega^2)$$

where $\theta \equiv \Delta_\alpha u^\alpha$, and $\sigma^2 \equiv \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ is a scalar corresponding to the deformation tensor $\sigma_{a\beta}$ given by (1.4) (the factor 1/4 depends on the definition of σ,γ). The above formula reduces in our case to:

$$R_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{4}\omega^2 \qquad (3.8)$$

$R_{a\beta}$ being Ricci's curvature tensor. Its relation with the energy tensor $T_{a\beta}$ of any continuum is given by the Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = - \mathcal{H} T_{\alpha\beta}$$

In the case of a perfect magnetohydrodynamic fluid the left hand side of (3.8) reduces to a scalar function of the density, pressure and the intensity of the magnetic field (cf [11]). If we take into account the incompressibility of our fluid, presssure must be the function of density only. We have, under the condition that magnetic field and pressure have constant intensities along magnetic field lines, varying possibly from one line to another,

$$h^{\alpha} \partial_{\alpha} (\omega^2) = 0 \tag{3.9}$$

as a consequence of (3.8). This represents the „isorotation" along magnetic field lines. This gives, in virtue of (3.7)

$$u^{\delta} h^{\epsilon} \omega^{\beta\gamma} (R_{\delta\epsilon\beta\gamma} + 2R_{\delta\beta\gamma\epsilon}) = 0 \tag{3.10}$$

as a special consequence.

So we have obtained relation (3.9) without any assumption on the steadiness of the vorticity tensor itself, and with a more accurate definition of steady—state than it was the case in [10] and [11].

Let us remark that steadiness was analyzed for general charged fluids with given energy tensors by Pham Mau Quan (cf [9] for neutral perfect fluid), with many mathematical developments. Our aim was only to obtain several, special consequences given above.

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О РЕЛАТИВИСТИЧЕСКОЙ ЛЕСНОСТИ БОРНА И НЕКОТОРЫХ ОСОБЕННОСТЯХ M×2 УСТАНОВИЩИХСЯ ТЕЧЕНИИ.

Содержание

Рассматриваются поля стационарных (установившихся) векторов по отношению к одному временно-подобному параметру, который подбирается

определенным образом. Во второй части, в случае одной МГД сплошной среды, получаются некоторые следствия отношения деформации и магнитного поля, если использовать только уравнение непрерывности. Наконец рассматривается стационарное пространство-время с установившимся магнитным полем и получается одна локальная формуляция теоремы Ферраро (магнитогидродинамической изоротации).