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SOME PROPERTIES OF VIBRATION PROPAGATION IN A VISCO–ELASTIC CONTINUUM DUE TO MOVING LOAD

Jiři Náprstek

1. Introduction

Let us consider the following problem. In a homogeneous isotropic continuum there is a void in the form of a circular cylinder. The material of the continuum, which is linear and visco–elastic, fills the whole infinite three dimensional space. Inthe afore mentioned void a certain load moves at a constant speed whose magnitude varies harmonically in time. Under these assumptions the motion of the continuum can be characterized by the equation – see (1), (5):

$$\mu^* g^{ks}\nabla_k \nabla_s u_j + (\lambda^* + \mu^*) g^{ks} \nabla_j \nabla_k u_s = \rho \frac{\partial^2 u_j}{\partial t^2} \qquad (1.1)$$

where

μ^*, λ^* – rheological operators corresponding with Lame constants in classical elasticity,

g^{ks} – components of the metric tensor.

In Eq.(1.1) the effect of the forces of gravity on the wave field has been neglected. Eq.(1.1) can be rewritten in the form of

$$\mu^* \Delta \vec{u} + (\lambda^* + \mu^*) \text{ grad div } \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \qquad (1.2)$$

The vector field u can be written, as it is known, in the form of the sum of a non–curl and non–potential fields, if the field u is sufficiently smooth and unambiguous and if for any $|r| \to \infty$ $|u| \to \vartheta$:

$$u = \text{grad } \Theta + \text{rot} \vec{A}; \text{div } \vec{A} = \vartheta \qquad (1.3)$$

If we take into account the form of the void, the vector potential can be vriten by means of two independent scalar functions [4], [6], so that we finally obtain for our particular case [6]

$$\vec{u} = \text{grad}\,\Theta + \text{grad}\,\Psi \times a_3 + h\,\text{grad}\,\frac{\partial \Omega}{\partial x^3} - h\rho\mu^{*-1} D^2 a_3 \Omega \tag{1.4}$$

where

h – dimensional equalization constant,
D – operator of the derivative according to time,
a_3 – unit vector in the void axis direction,
x^3 – contravariant coordinate in the void axis direction,
Θ, ψ, Ω – scalar potentials.

It can be shown that all three potentials Θ, ψ, Ω, by means of which the whole solution of the problem can be described, satisfy the equation [6]

$$L^* \Delta\varphi - \rho \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{1.5}$$

where L^* is the generally linear integro–differential operator according to time:

$$L^* \equiv \lambda^* + 2\mu^* \quad \text{for the potential } \Theta \text{ (a)}$$
$$L^* \equiv \mu^* \quad \text{for the potentials } \psi, \Omega \text{ (b)} \tag{1.6}$$

If we neglect the integral terms in the rheological relations, the operator (1.6) can be written in the form of

$$L^* \equiv \sum_{j=o}^{n} C_j \frac{\partial^j}{\partial t^j} \tag{1.7}$$

If the model of the inner mechanism consists of linear springs and pistons only, it holds that

$$n = 0 \quad \text{or} \quad n = 1 \tag{1.8}$$

2. General Solution of Equation for Potentials

If we substitute (1.7) into (1.5), we obtain

$$\sum_{j=0}^{n} C_j \frac{\partial^j}{\partial t^j} \Delta\varphi(r,\xi,z,t) - \rho \frac{\partial^2}{\partial t^2} \varphi(r,\xi,z,t) = 0 \tag{2.1}$$

(r, ξ, z being cylindrical coordinates according to Fig.1).

We shall introduce the transformation

$$\gamma = z - vt; \; t' = t \tag{2.2}$$

so that it holds

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \varphi}{\partial \gamma}; \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial t'} - v \frac{\partial \varphi}{\partial \gamma} \tag{2.3}$$

We shall substitute (2.3) into (2.1) and obtain

$$\sum_{j=o}^{n} C_j \left(\frac{\partial}{\partial t} - v \frac{\partial}{\partial \gamma}\right)^j \Delta \varphi (r,\xi,\gamma,t) - \rho \frac{\partial^2}{\partial t^2} \varphi (r,\xi,\gamma,t) = \theta \tag{2.4}$$

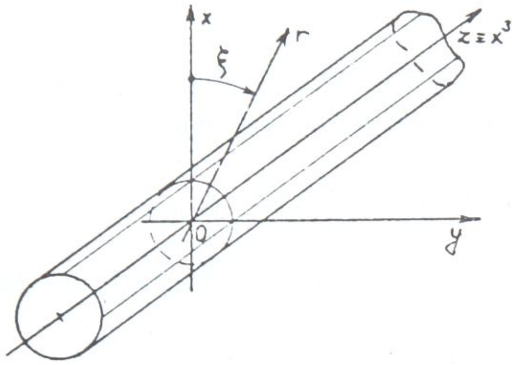

Fig. 1

Now we shall apply to (2.4) the Laplace transformation according to time whose lower limit will approach $-\infty$. Thus we shall indicate that we consider the action after an infinitely long time from its beginning, i.e. in the stationary state. After a number of considerations we shall arrive at equation

$$\sum_{j=o}^{n} C_j \left(i\omega - v \frac{\partial}{\partial \gamma}\right)^j \Delta \hat{\varphi} - \rho \left(i\omega - v \frac{\partial}{\partial \gamma}\right)^2 \hat{\varphi} = \theta \tag{2.5}$$

where it holds that

$$\varphi (r,\xi,\gamma,t) = \hat{\varphi} (r,\xi,\gamma)\, e^{i\omega t} \tag{2.6}$$

Let us presume that for γ approaching, in its absolute value, to infinity the boundary conditions become homogeneous. Practically it means that there is an interval $\gamma \in (\gamma_{min}, \gamma_{max})$ outside which no excitation occurs. Let us assume that the stimuli are described by the function integrable with the square in every moment within the mentioned interval. In such a case it is possible to introduce Fourier's transformation into (2.5) in the direction γ. After reorganization we obtain for the $\Phi(r, \zeta, p)$ image of the function $\varphi(r, \zeta, \gamma)$ equation

$$\Delta_r \Phi + \kappa^2 \Phi = \theta \quad \left(\Delta_r \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \xi^2}\right) \tag{2.7}$$

where κ^2 is a complex number having the following components:

$$R_e(\kappa^2) \doteq \frac{\rho(\omega - vp)^2 M - p^2(M^2 + N^2)}{M^2 + N^2}; \qquad I_m(\kappa^2) = \frac{\rho(\omega - vp)^2 N}{M^2 + N^2} \tag{2.8}$$

while

$$M = \sum_{j=o}^{n/2} C_{2j+1} (\omega - vp)^{2j+1} (-1)^j ; \quad N = \sum_{j=o}^{n/2} C_{2j} (\omega - vp)^{2j} (-1)^j \tag{2.9}$$

We shall write the unknown quantity $\Phi(r, \zeta, p)$ in form of

$$\Phi(r,\zeta,p) = \sum_{k=o}^{\infty} (F_k^c(\kappa_r) \cos k\zeta + F_k^s(\kappa_r) \sin k\xi) \tag{2.10}$$

where $F_k^c(\kappa_r)$, $F_k^s(\kappa_r)$ are unknown functions following out of the Bessel's equation of the same type; therefore, we cnan write summarily

$$F_k''(\kappa_r) + \frac{1}{r} F_k'(\kappa_r) + (\kappa^2 - \frac{k^2}{r^2}) F_k(\kappa_r) = 0 \tag{2.11}$$

We can write the common integral of Eq.(2.11) in the form of [2], [3]

$$F_k(\kappa_r) = c_k Z_k^{(1)}(\kappa_r) + d_k Z_k^{(2)}(\kappa_r) \tag{2.12}$$

where

$Z_k^{(1)}(\kappa_r)$ – cylindrical function of the first type

$Z_k^{(2)}(\kappa_r)$ – cylindrical function of the second type

c_k, d_k – integration constants

The parameter κ^2 is a complex number; the concrete form of Eq.(2.12) depends on the mutual relation of its real and imaginary parts. For this reason it is necessary to deal with the conditions under which the components of κ^2 will be positive, negative or zero.

3. Characteristics of Parameter κ^2

To be able to make a more detailed analysis of parameter κ^2 we shall limit the number of terms in L* according to the condition (1.8). Eq.(2.8) will acquire the form of

$$R_e(\kappa^2) = \frac{\rho(\omega - vp)^2 (C_o - \frac{p^2}{\rho} C_1^2) - p^2 C_o^2}{C_o^2 + C_1^2 (\omega - vp)^2} = A^2 \qquad \text{(a)}$$

$$I_m(\kappa^2) \doteq \frac{\rho(\omega - vp)^2 C_1}{C_o^2 + C_1^2 (\omega - vp)^2} = B^2 \qquad \text{(b)} \tag{3.1}$$

We shall make the analysis in the Cartesian space of (ω, p, A^2). From (2.1a) we obtain:

$$\omega = vp \pm \sqrt{\frac{C_o^2 (p^2 + A^2)}{\rho (C_o - \frac{C_1}{\rho}(p^2 + A^2))}} \tag{3.2}$$

The square root in (3.2) will be defined only, if

1. $-\sqrt{\frac{C_o \rho}{C_1^2} - A^2} < p < \sqrt{\frac{C_o \rho}{C_1^2} - A^2}$ (a)

2. $p^2 > - A^2$ (b) (3.3)

3. $A^2 < \frac{C_o \rho}{C_1^2}$ (c)

The condition (3.3a) limits the definition area p, the relation (3.3c) indicates that $Re(\kappa^2)$ is upper–bound by a positive value which it acquires at $|\omega| \to \infty$. The condition (3.3b) will find application only for negative A^2. It follows from (3.3a) that A^2 has an asymptotic parabolic cylindrical surface in the system(ω, p, A^2) whose surface lines are parallel with the axis ω and the directing parabola lies in the plane (p, A^2).

For various extents of damping the parabola has the same from and direction of the main axis. Only its origin for $C_1 \to 0$ shifts along the axis A^2 to $+\infty$.

In the Cartesian space (p, v, ω) (3.3a) defines a pair of asymptotic planes of the function $\omega(v, p)$ according to (3.2). For an undampened material these planes will separate in the direction p to $\pm \infty$.

Let us consider three cases of A^2, viz. $A^2 = 0$; $A^2 > 0$; $A^2 < 0$.

Ad 1:

This case is very important as the type of functions in (2.12) changes with the transition over the axis of $Re(\kappa^2) = 0$. For $v = 0$ (3.2) consists of two centrally symmetrical curves in the plane (p, ω) – see Fig. 2 – which can be inscribed into the areas limited by concurrent lines $\omega = \pm p\sqrt{C_o/\rho}$ and the condition (3.3a). The first addend describes the plane fotating in the positive direction with growing v. If v grows, the second addend in (3.2) remains constant.

The stationary points of the function $\omega_o(v,p)$ od not exist in real field, if

$$|v| < \sqrt{\frac{C_o}{\rho}} = v_s \tag{3.4.}$$

FUNCTION $\omega_0(v, p)$

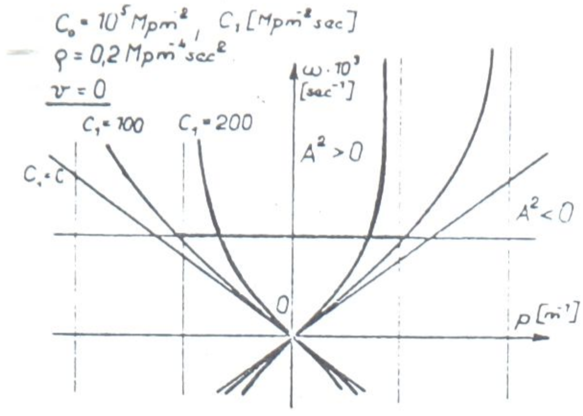

Fig. 2a

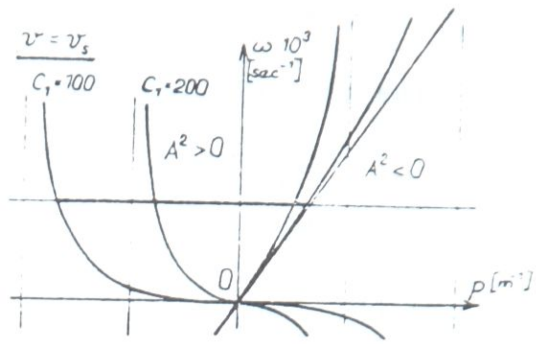

Fig. 2b

FUNCTION $\omega_{A^2 \neq 0}(v, p)$

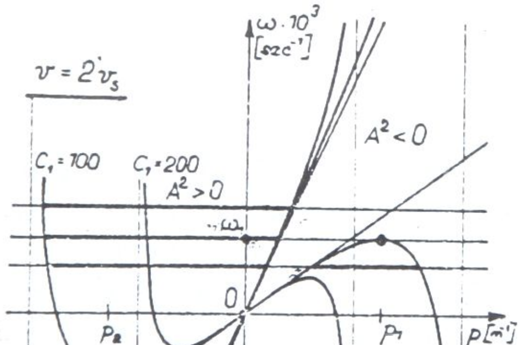

Fig. 2c

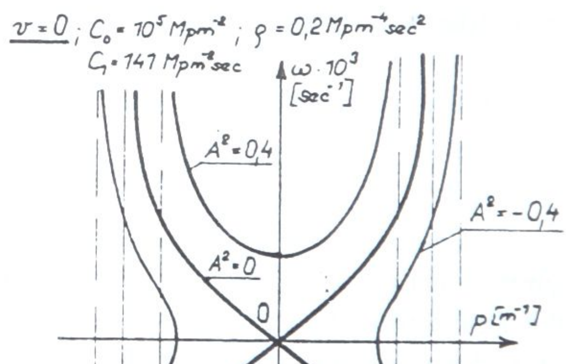

Fig. 3

i.e at the velocity of the load motion which is inferior to the velocity of urdulation propagation in an undampened material.

If equality applies to (3.4), an inflexion point of one branch of the function ω_o (v,p) /transsonic field/ will exist in point p = 0 (Fig. 2b). For

$$|v| \geqslant \sqrt{\frac{C_o}{\rho}} \tag{3.5}$$

(supersonic field (the function ω_o (v,p) has two stationary points

$$p_{1\,2} = \pm \sqrt{\rho \frac{C_o}{C_1^2} - \left(\frac{C_o \rho}{C_1^3 v}\right)^{\frac{2}{3}}} \tag{3.6}$$

which, too, will separate to $\pm\infty$ at $C_1 \to 0$. The respective values of the function ω_o(v,p) in $p_1{}'_2$ according to (3.6) in the branch with the extreme are

$$\omega_{1\,2} = \pm \frac{\left(C_o - v \sqrt[3]{\frac{C_o \rho}{v}}\right) \sqrt{\rho C_o - \left(\frac{C_o^2 \rho}{v}\right)^{\frac{2}{3}}}}{C_1 \sqrt[3]{\frac{C_o^2 \rho}{v}}} \tag{3.7}$$

It can be seen that (3.7) has a meaning only in the transsonic and the supersonic fields in Fig.20.

From these results it is possible to determine unambiguously for which ω, p A^2 will be positive, zero or negative. In Fig.2 the function ω_o (v,p) is shown for various C_1 and v. The values of $Re(\kappa^2)$ is positive in the fields comprizing the axis ω. Hence it follows that for subsonic and transsonic and velocities there exist for every $\omega \neq 0$ one field $p \in (p_1; p_2)$ in which $Re(\kappa^2) > 0$, while $p_1 < 0$; $p_2 > 0$.

For supersonic v several cases can occur in accordance with the magnitude of ω – see Fig. 2c. For $|\omega|$ in excess of (3.7) the situation is the same as in the subsonic case. For $\omega = \omega_1{}'_2$ according to (3.7) there comes yet another point in which $Re(\kappa^2) = 0$ and there originate three intervals p of negative $Re(\kappa^2)$. For ω $\omega \in (\omega_1 \omega_2)$ there are three intervals p in which $Re(\kappa^2) < 0$ and two in which $Re(\kappa^2) > 0$.

Ad 2: $A^2 > 0$

The contour lines of the function A^2 (v, ω, p) consist of two branches in the definition area p according to (3.3a). Its bounderies determine the vertical asymptotes of both branches of the function $\omega_{A^2>0}(c,p)$. For high $|\omega|$ the behavior of the function A^2 can be characterized by an asymptotic cylinder following out of (3.3a). The values of A^2 are upper–bound by the condition (3.3c)– graphically see Fig.3

$$p = \frac{C_o \rho}{C_1^2} - A^2 \qquad (3.8)$$

Ad 3: $A^2 < 0$

The conditions (3.3a), (3.3b) limit a pair of definition fields

$$-\sqrt{\frac{C_o \rho}{C_1^2} + |A^2|} < p \leqslant -\sqrt{|A^2|}; \quad \sqrt{|A^2|} \leqslant p < \sqrt{\frac{C_o \rho}{C_1^2} + |A^2|} \qquad (3.9)$$

For $A^2 \rightarrow = \infty$ the intervals (3.9) mutually separate and decrease the more speedily the greater inner damping. For very high p A^2 no longer depends on ω and for low velocities not even on v. For the calculation of A^2 it is possible to use (3.8). Graphically see Fig. 3.

Analogous considerations can be carried out also for $\mathrm{Im}(\kappa^2)$. Particular attention should be afforded to the parameter

$$\chi = \sqrt{A^2 + i\,B^2} = \alpha + i\beta \qquad (3.10)$$

In this raspect we refer to (6).

4. Types of Solution of Potential Equation

The type (2.12) of the solution of Eq.(2.11) depnds on the type of mutual relations of signs of $\mathrm{Re}(\kappa^2)$, $\mathrm{Im}(\kappa^2)$ and the magnitude of $|\kappa_r|$. If approximately $0.1 \leqslant |\kappa_r|| \leqslant 6$, it is possible to speak about a certain medium interval within which the solution of (2.12)is determined by cylindrical functions. If for $\mathrm{Re}(\kappa^2) > 0$ also $\mathrm{Im}(\kappa^2) < 0$, one integration constant is annuled, since the absolute value of one of the functions grows to infinity for $r \rightarrow \infty$, so that the boundary condition $\lim_{r \rightarrow \infty} |F_k(\kappa_r)| = 0$ could not be satisfied. If $\mathrm{Re}(\kappa^2) > 0$ and $\mathrm{Im}(\kappa^2) = 0$, the Sommerfeld condition applied to two dimensional space decides between both alternatives. The concrete form of hthe solution can be seen from the table in Fig.4. To make F_k continuous in the whole Gauss plane (κ), F_k must be multiplied with a certain constant O_k whose value can be determined from the same table. In the table H_k, H_k means Hankel's function of the first and the second types of the k–th order, and K_k the McDonald function.

For $|\kappa_r| > 6$ the cylindrical function begin to be numerically unstable. The solution is, therefore, written by means of asymptotic formulas of cylindrical functions in the form shown in the afore mentioned table (upper interval).

For $|\kappa_r| < 0.1$ the Bessel formula (2.11) nears Euler's equation and the cylindrical functions are numerically unstable, too. The solution is effecte by the smlal parameter method. Its results can be found in the same table (lower interval). The function $Q_k F_k(\kappa_r)$ is, consequently, an analytical function, whose main branch is holomorphous in the whole Gauss' plane (κ) with the exception of the positive demi–axis and origin where there is the bifurcation line.

TYPES OF THE SOLUTION – $F_k(\varkappa r)$

$\varkappa_k^2$	Q_k	$F_k(\varkappa_r)$; $0.1 < \lvert\varkappa_r\rvert < 6$	$F_k(\varkappa_r)$; $\lvert\varkappa_r\rvert \geq 6$
$A_k^2 > 0$ $B_k^2 = 0$	$\frac{\pi i e^{k\frac{\pi}{2}i}}{(-1)^{k+1}\cdot 2}$	$d_k Q_k H_k^{(2)}(\alpha r)$	$d_k Q_k \sqrt{\frac{2}{\pi\alpha r}}\, e^{-i(\alpha r - \frac{k\pi}{2} - \frac{\pi}{4})}$
$A_k^2 > 0$ $B_k^2 > 0$	$-\frac{\pi i e^{-k\frac{\pi}{2}i}}{(-1)^{k+2}\cdot 2}$	$d_k Q_k H_k^{(1)}[(\alpha + i\beta)r]$	$d_k Q_k \sqrt{\frac{2}{\pi\lvert\varkappa_r\rvert}} \cdot e^{i(\alpha r - \frac{k\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\operatorname{arctg}\frac{\beta}{\alpha})} \cdot e^{-\beta r}$
$A_k^2 > 0$ $B_k^2 < 0$	$\frac{\pi i e^{k\frac{\pi}{2}i}}{(-1)^{k+1}\cdot 2}$	$d_k Q_k H_k^{(2)}[(\alpha - i\beta)r]$	$d_k Q_k \sqrt{\frac{2}{\pi\lvert\varkappa_r\rvert}}\, e^{-i(\alpha r - \frac{k\pi}{2} - \frac{\pi}{4} + \frac{1}{2}\operatorname{arctg}\frac{\beta}{\alpha})} \cdot e^{-\beta r}$
$A_k^2 < 0$ $B_k^2 = 0$	1	$d_k Q_k K_k(\alpha r)$	$d_k Q_k \sqrt{\frac{\pi}{2\alpha r}} \cdot e^{-\alpha r}$
$A_k^2 < 0$ $B_k^2 > 0$	1	$d_k Q_k K_k[(\alpha - i\beta)r]$	$d_k Q_k \sqrt{\frac{\pi}{2\lvert\varkappa_r\rvert}}\, e^{i(\beta r + \frac{1}{2}\operatorname{arctg}\frac{\beta}{\alpha})} \cdot e^{-\alpha r}$
$A_k^2 < 0$ $B_k^2 < 0$	1	$d_k Q_k K_k[(\alpha + i\beta)r]$	$d_k Q_k \sqrt{\frac{\pi}{2\lvert\varkappa_r\rvert}}\, e^{-i(\beta r - \frac{1}{2}\operatorname{arctg}\frac{\beta}{\alpha})} \cdot e^{-\alpha r}$
	$k > 0$	$k > 2$	$k > 4$
$\lvert\varkappa_r\rvert < 0.1$	$d_k r^{-k}$	$d_k r^{-k}\left(1 + \varkappa^2 \frac{r^2 - R^2}{4(k-1)}\right)$	$d_k r^{-k}\left(1 + \varkappa^2 \frac{r^2 - R^2}{4(k-1)} + \varkappa^4 \frac{(r^2 - R^2)^2(k-1) + 2R^2(r^2 - R^2)}{32(k-1)^2(k-2)}\right)$

Fig. 4

FUNCTION $F_0(\varkappa_r)$

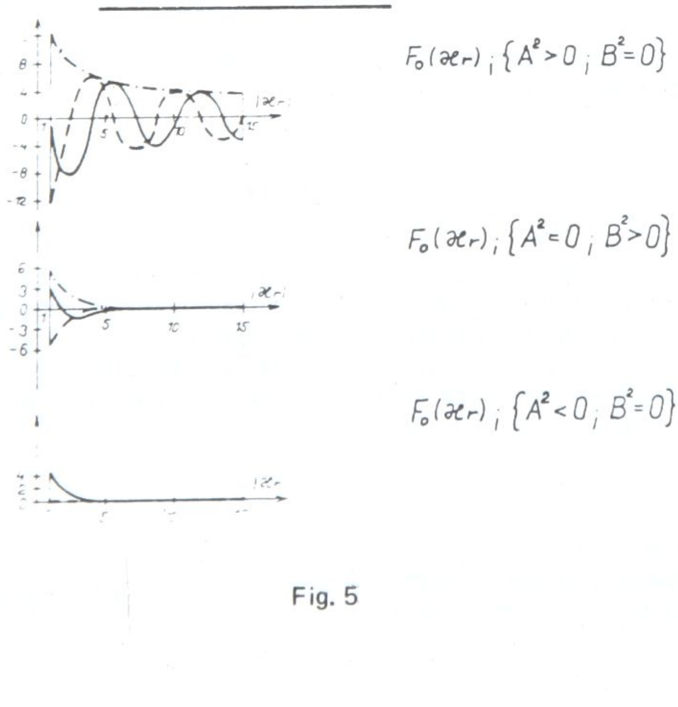

Fig. 5

From the solution in the upper interval the character of the decrement of F_k for growing r can be seen. The damping is clearly divided into two parts. The first part is due to the dispersal of energy in space, which is a relatively small decrement of the order of $r^{-1/2}$. A much stronger source of damping is that due to inner viscosity, that being of the order of $e^{-\beta r}$, if $Re(\kappa^2) \geqslant 0$. If $Re(\kappa^2) < 0$, then the overall damping is of the order of $r^{-1/2}\ e^{-\alpha r}$. In concrete conditions is $|\alpha| \gg |\beta|$, $\alpha > 0$ even for zero viscosity, when $\beta = 0$. Hence it follows that for $Re(\kappa^2) > 0$ F_k has a wave oharacter, the wave amplitdue decressing very slowly. Since the effect of undulation is determined chiefly by the character of F_k, it can be said that for $Re(\kappa^2) > 0$ undulation is dangerous. For $Re(\kappa^2) < 0$ the response is almost montonously and speedly sinking. It has a character of Raleygh's wave whose effect practically disappears at a very short distance from the emitter. – Fig. 5. For this reason it was necessary to deal in great detail with the properties of $Re(\kappa^2)$ in Chapter 3. If we imagine the load within the void written in the form of Fourier's integral in the direction of the void axis, the solution need not be effected in the whole interval $p\epsilon(-\infty, +\infty)$, but only in the fields where $Re(\kappa^2) \geqslant 0$. This measure considerably reduces the requirements of computer time, a fact ascertained during the application to the calculation of vibrations due to the operation of Prague Underground [5].

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SUMMARY

The author deals with the characteristics of the propagation of the waves of mechanical stresses in a linear visco–elastic three–dimensional continuum. In this continuum there is a void in the form of a circkular cylinder. The wave motion is due to a load moving in the said void, its magnitude being variable in time. A number of new effects is shown originating as a result of the viscous character of the medium for various combinations of material constants, load frequency and the velocity of its motion.

Author's adress
Ing Jiří Náprstek CSc.,
Institute of Theoretical and Applied Mechanics,
Praha, 2, Vyšehradska 49, Czechoslovakia