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STATIC STABILITY OF A VISCOELASTIC CURVED BEAM

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Linearized theory is applied to the problem of the static stability of viscoelastic solid curved beam. By means of the discussion of the eigenvalues of the integral equation of the problem, the critical parameters of static stability are determined as the functions of the time.

We shall consider the class of viscoelastic materials for which an instantaneous modulus of elasticity E° can be defined and whose rate of deformation under constant load tends to zero as $t \to \infty$. Solids of this type will be referred to as V material.

The stress–strain relations for a V material, given in the integral form, are [3]

$$\sigma(t) = E^\circ\epsilon(t) - \dot{\Psi}(t) * [E^\circ\epsilon(t)], \qquad E^\circ\epsilon(t) = \sigma(t) + \dot{\Phi}(t) * \sigma(t)^{1)} \tag{1}$$

for any $t > 0$.

The relaxation and creep functions of a V material are

$$\Psi(t) = \sum_{j=1}^{n} \psi_j \left[1 - e^{-\frac{t}{\theta_j}}\right], \qquad \Phi(t) = \sum_{j=1}^{n} \varphi_j \left[1 - e^{-\frac{t}{\vartheta_j}}\right], \tag{2}$$

and are determined by 2n independent parameters ψ_j, θ_j or φ_j, ϑ_j, $(j = 1,2,\ldots,n)$.

The relaxation coefficient is given by

$$\psi = \lim_{t \to \infty} \Psi(t), \qquad 0 < \psi < 1. \tag{3}$$

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1) $f_1(t) * f_2(t) = \int_{\tau 0}^{t} f_1(t-\tau) f_2(\tau)\, d\tau; \quad \dot{f}_1(t) = \frac{\partial}{\partial(t-\tau)} f_1(t-\tau)$

The creep coeficient is given by

$$\varphi = \lim_{t \to \infty} \Phi(t), \qquad 0 < \varphi < M, \tag{4}$$

where M is a finite number.

The relaxation factor is

$$\psi' = 1 - \psi, \qquad 0 < \psi' < 1, \qquad \text{and} \qquad \varphi = \frac{\psi}{\psi'} . \tag{5}$$

Equation (1) remains valid as $t \to \infty$, i.e. when the creep process of the V material is finished,

$$\sigma^{\infty} = \psi' E^{o} \epsilon^{\infty} . \tag{6}$$

Referring to the stress–strain relation at the instant when the load starts to act ($t = 0^{+}$)

$$\sigma^{o} = E^{o} \epsilon^{o} \tag{7}$$

which describes the ,,elastic state'' of the V material at $t = 0^{+}$ it may be said that eq. (6) describes another ,,elastic state'' of the material with ,,modulus of elasticity''

$$E^{\infty} = \psi' E^{o}, \tag{8}$$

which is approached as $t \to \infty$.

Two limiting cases are:

$\psi' = 1$, the case of an ideal elastic material obeying Hooke's law;

$\psi' = 0$, the case of the class of viscoelastic materials for which the rate of deformation does not decrease with time, and which includes Maxwell's material.

The stability of a V material curved beam, lying in a plane, of variable cross–sectional characteristics, arbitrarily supported, will be examined applying linearized theory. The usual assumptions of the Theory of Structures appropriate for plane arcs are taken into account, including Bernoulli's hypothesis. The external load, lying in the plane of the beam axis, consists of:

distributed forces q_r in the radial and q_φ in the tangential direction, whose signs are positive towards the center of curvature and in the direction of increasing arc lenght coordinate s along the beam axis, respectively;
a distributed moment q_m, closkwise positive.

The external load has the form

$$q_\lambda (s,t) = \alpha p_\lambda(s)\, H(t)^{1)}, \qquad (\lambda = r, \varphi, m), \tag{9}$$

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1) H(t) is the Heaviside step function

where α is a parameter of the external load, and the $p_\lambda(s)$, $(\lambda = r, \varphi, m)$ are arbitrary functions of s.

It is assumed that the resultant of external forces acting on any given element of the beam do not change in magnitude with deformation of the beam. It is also assumed that the external load does not change its direction during the deformation. The influence of the deformation on the equilibrium conditions of the beam is not neglected (i.e. the system is considered as „flexible'') but using assumption that the displacements are small in relation to the lenght of the axis and radius of curvature of the beam, the equations representing the equilibrium conditions may be linearized. Then the problem of bending of the beam becomes linear in the unknown functions, but not in the load [5] [3]. Hence in the „flexible system'' any influence $Z_w(s,t)$ can be determined from the expression

$$Z_{wq}(s,t) = Z_{owq}(s,t) + \sum_{\mu} \int_{S} Z^{o}_{w\mu}(s,u)\, q_{d\mu}(u,t)\, du + \sum_{\mu} \int_{S} \dot{Z}_{w\mu}(s,u,t)\, q_{d\mu}(u,t)\, du,^{1)} \qquad (\mu = r,\varphi,m), \tag{10}$$

where

$Z_{owq}(s,t)$ is the corresponding influence for a curved beam of a V material determined from the equilibrium of the undeformed element (i.e. in the „rigid system''), indicated by the suffix o; $q_{d}\mu(s,t)$, $(\mu = r, \varphi, m)$ is the extra loading component which accounts for the influence of deformation on the equilibrium conditions;

$Z_w\mu(s,u,t)$ is the influece function expressing the influence of the corresponding generalized force $p_\mu(s,t) = \delta(s-u)H(t)$[2] $(\mu = r, \varphi, m)$ on the beam. If this influence refers to a generalized displacement then

$$Z_{w\mu}(s,u,t) = [1 + \Phi(t)]\, Z^{o}_{w\mu}(s,u), \qquad (\mu = r,\varphi,m). \tag{11}$$

$Z^{o}_{w\mu}(s,u)$ is the corresponding influence function for the equivalent elastic beam (E^{o}). This relation results from the Boltzmann's principle.

The extra loading component involves the unknown functions of the problem. When the axial dilatation of the beam is neglected, the extra loading is given by the distributed moments

$$q_{d\mu}(s,t) = \alpha N_o(s)\, \beta_q(s,t), \tag{12}$$

1) The index w designates the type of influence: cross–sectional force, or any generalized displacement; the index q denotes that the influence is due to the entire external load (9).
2) $\delta(s-u)$ is the Dirac delta function.

where

$N_o(s)$ is the axial force due to the given external loading (9) in the beam as a „rigid system''; it is positive if it exerts pressure on the element of the beam;

$\beta_q(s,t)$ is the unknown function of the rotation of the cross sectional plane due to the external load (9). It is positive clockwise.

From eq. (10) – (12), the integral equation of the problem can be written in terms of the unknown fuction $\beta_q(s,t)$

$$\beta_q\ (s,t) - \alpha \int_S Z^o_{\beta m}\ (s,u)\ N_o\ (u)\ \beta_q\ (u,t)\ du -$$

$$- \alpha\ \dot{\Phi}(t) * \int_S Z^o_{\beta m}\ (s,u)\ N_o\ (u)\ \beta_q\ (u,t)\ du = 0.^{1)} \tag{13}$$

From Maxwell's reciprocal theorem it follows that the kernel $Z^o{}_{\beta m}(s,u)$ is symmetrical:

$$Z^o_{\beta m}\ (s,u) \equiv Z^o_{\beta m}\ (u,s). \tag{14}$$

Since the function $\beta_q(s,t)$ satisfies the conditions of the Hilbert –Schmidt's theorem [7], this function can be expanded into the sum of its Fourier series with respect to the orthonormal system $B_k(s)$, of eigenfunctions of the symmetric kernal $Z^o{}_{\beta m}(s,u)$, which is absolutely and uniformly convergent

$$\beta_q\ (s,t) = \sum_{k=1}^{\infty} f_k(t)\ B_K(s), \tag{15}$$

where f_k are unknown functions.

When the series (15) is set in the equation (13) the result is

$$B_k(s) - \alpha \frac{1}{f_k\ (t)}\ [\delta\ (t) + \dot{\Phi}(t)] * f_k\ (t) \int_S Z^o_{\beta m}\ (s,u)\ N_o(u)\ B_k\ (u)\ du = 0,$$

$(k = 1,2, \ldots)$.

These equations can be reduced to pure Fredholm equations. The determination of the critical parameters of stability, consists of the consideration of the eigenvalues of fredholm equation: These eigenvalues are assumed to exist:

The equation of static stability problem of the equivalent elastic bema (E^o), which is also Fredholm equation, is obtained as a special case of (13), if it is put that $\Phi(t) \equiv 0$

1) Since the stability problem is considered, we put $\beta_{oq}(s,t) \equiv 0$.

$$\beta_q (s) - \alpha \int_S Z^o_{\beta m} (s,u)\ N_o(u)\ \beta_q\ (u)\ du = 0, \tag{17}$$

If the eigenvalues of this equation exist, they are $\alpha^*_k o$, and can be determined by using the known methods [1].

Since the kernel $Z^o_{\beta m}(s,u)$ of eq. (16) and (17) is the same, it follows:

a) eigenvalues of eq. (16) exist under the same conditions as in the case of eq. (17);

b) if exist, the both set of eigenvalues coincide, then

$$\alpha^*_k (t)\ [\delta\ (t) + \dot{\Phi}\ (t)] * f_k\ (t) = \alpha^{*o}_k\ \delta\ (t) * f_k\ (t), \tag{18}$$

(k = 1,2, . . .), i.e.

$$\alpha^*_k (t) = \frac{\alpha^{*o}_k}{1 + \Phi(t)}, \qquad (k = 1,2, ...). \tag{19}$$

From the consequence a) it is seen that the loss of stability, for a beam of V material, is possible, if it is possible for the equivalent elastic beam, under the same external loading . It is known, that the loss of stability for the elastic beam occurs as a consequence of the property of the kernel, when its eigenvalues are real and positive [5].

The kernel $Z^o_{\beta m}(s,u)$ is symmetrical so that its eigenvalues are always real. If the axial force $N_o(s)$ is positive, (pressure) for any s along the axis of the beam, the Kernel $Z^o_{\beta m}(s,u)N_o(s)$ is positive definite, so that its eigenvalues are real and positive. In this case, the loss of static stability for a beam of V material is possible.

If the axial force $N_o(s)$ is negative (tension) throughout the beam, the eigenvalues are real and negative. So the loss of stability for a beam of V material is not possible.

From the consequence b) the relation between the critical parameters of stability for a beam of V material α^*_k, and the corresponding critical parameters for the equivalent elastic beam α^{*o}_k, under the same external loading is obtained.

In the interval $0 \leqslant t < \infty$, the critical parameters are the functions of the time, because as a result of the external loading in the course of time the creep process takes place.

From eq. (19) the bounds of the critical parameters of stability in the interval $0 \leqslant t < \infty$ can be determined

$$\alpha^{*o}_k \geqslant \alpha^*_k \geqslant \psi' \alpha^{*o}_k, \qquad (k = 1,2, ...). \tag{20}$$

At $t = 0^+$ the critical parameters are the same as for the equivalent elastic beam of modulus of elasticity E^o, and as $t \to \infty$ become the same as those of the equivalent elastic beam with modulus of elasticity $E^{\infty} = \psi' E^o$.

This conclusion becomes evident in the light of the „elastic states" of the V material as $t = 0^+$ and $t \to \infty$, with „moduli of elasticity" E^o and E^{∞} respectively.

If α is within the bounds (20) for k=1, the time when the curved beam of V material will loss stability can be determined from (19) for k=1 and (2). For the case of V material represented by Zener model the creep function is

$$\Phi(t) = \frac{\psi}{\psi'}\left[1 - e^{-\psi'\frac{t}{\theta}}\right], \tag{21}$$

and static instability takes place in at time

$$t^* = \frac{\theta}{\psi'} \ln \frac{\psi\alpha}{\alpha - \psi'\alpha_1^{*o}}, \qquad \alpha_1^{*o} \geqslant \alpha \geqslant \psi'\alpha_1^{*o}. \tag{22}$$

In the special case of an elastic material (E^o), $\psi' = 1$, so that the upper and lower bounds of the critical parameters coincide. The solution derived here can also be applied to the class of viscoelastic materials whose deformation rate does not decrease with time, $\psi' = 0$. Hence the problem is also solved for the special case of a Maxwell's material, whose creep function is

$$\Phi(t) = \frac{t}{\theta}\ . \tag{23}$$

The critical parameters of stability at arbitrary time t, and the bounds within which they lie for $0 \leqslant t < \infty$ are then

$$\alpha_k^*(t) = \frac{\alpha_k^{*o}}{1 + \frac{t}{\theta}}, \qquad \alpha_k^{*o} \geqslant \alpha_k^* \geqslant 0, \qquad (k = 1,2, \ldots). \tag{24}$$

In this case, the stability problem is reduced to the determination of the critical time

$$t^* = \theta\left[\frac{\alpha_1^{*o}}{\alpha} - 1\right], \qquad \alpha_1^{*o} \geqslant \alpha \geqslant 0, \tag{25}$$

when static stability is lost.

All the conclusions are valid also for a straight arbutrarily supported beam, of variable cross–section.

In Euler's four cases the critical load at arbitrary time t and the bounds within which they lie for $0 \leqslant t < \infty$, for the beam of the V material, are

$$P_K^*(t) = \frac{P^o_{KE}}{1 + \Phi(t)}, \qquad P^o_{KE} \geqslant P_K^* \geqslant \psi' P^o_{KE}, \qquad (k = 1,2, \ldots), \tag{26}$$

where P^o_{kE} is the k^{th} Euler's critical load for the equivalent elastic bema (E^o).

For the straight beam, made of Maxwell's material, the critical load and its bounds are

$$P_K^*(t) = \frac{P^o_{KE}}{1 + \frac{t}{\theta}}, \qquad P^o_{KE} \geqslant P_k^* \geqslant 0, \qquad (k = 1,2, \ldots), \tag{27}$$

The critical time is

$$t^* = \theta \left[\frac{P^o_{1E}}{P} - 1 \right], \quad P^o_{1E} > P > 0. \tag{28}$$

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Содержание

В границах линейной теории, исследовается статическая устойчивость криволинейных стержней, произвольно опертых, с произвольно изменяющимися характеристиками поперечного сечения, от вязкоупругого твердого материяла.

Исследованием собственых значений интегрального уравнения проблема статический устойчивости, определены крытические параметри нагрузки, которые являются функциями от времени. Найдены и границы крытических параметров в $t = 0^+$ и $t \to \infty$.

Для нагрузки параметар которой находится в этих границах, возможно оптеделить момент времени, когда наступает потеря статической устойчивости явзкоупругого стержня.