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ON THE INDEX OF CACTUSES WITH n VERTICES

Bojana Borovićanin and Miroslav Petrović

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Abstract. Among all connected cactuses with n vertices we find a unique graph whose largest eigenvalue (index, for short) is maximal.

1. Introduction

We consider only simple graphs in this paper. Let G be a graph with n vertices, and let $A(G)$ be the $(0,1)$-adjacency matrix of G. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality we can write them in non-increasing order as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial of G is just $\det(\lambda I - A(G))$, and denoted by $P(G, \lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the index of G (or the spectral radius of G). If G is connected, then $A(G)$ is irreducible and it is well-known that $\lambda_1(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\lambda_1(G)$, by the Perron–Frobenius theory of non-negative matrices. We shall refer to such an eigenvector as the Perron vector of G.

The investigation of the index of graphs is an important topic in the theory of graph spectra. The reference [**9**] is an excellent survey which includes a large number of references on this topic.

Let $\mathcal{H}(n, n+t)$ be the set of all connected graphs with n vertices and $n+t$ edges ($t \geqslant -1$). The corresponding extremal index problems have been solved for certain values of t [**2, 3, 8, 13, 16, 17**].

The recent developments on this topic [**1, 4, 5, 6, 11, 14, 18**] also involve the problem concerning graphs with maximal or minimal index of a given class of graphs.

In this paper we study the index of cactuses with n vertices. We say that the graph G is a *cactus* if any two of its cycles have at most one common vertex (about

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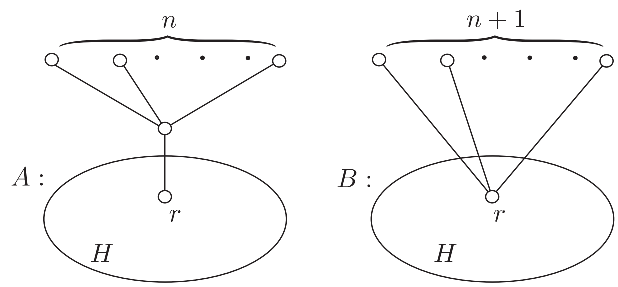

FIGURE 1

cactuses see e.g., [**15**]). If all cycles of the cactus G have exactly one common vertex we say that they form a *bundle*.

Denote by $C(n)$ the set of all connected cactuses with n vertices. In this paper we determine the graphs with the largest index in the class $C(n)$.

2. Preliminaries

Denote by C_n the cycle on n vertices. Let $G - x$ or $G - xy$ denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$.

For $v \in V(G)$, $d(v)$ denotes the degree of vertex v and $N(v)$ denotes the set of all neighbors of vertex $v \in G$. Also, by $d(v, w)$ we will denote the distance between vertices v and w in G.

In order to complete the proof of our main result we need the following lemmas.

LEMMA 2.1. [**18**] *Let G be a connected graph and let $\lambda_1(G)$ be the index of $A(G)$. Let u, v be two vertices of G and let $d(v)$ be the degree of the vertex v. Suppose that $v_1, v_2, \dots, v_s \in N(v) \setminus N(u)$ $(1 \leqslant s \leqslant d(v))$ and $x = (x_1, x_2, \dots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i $(1 \leqslant i \leqslant n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leqslant i \leqslant s)$. If $x_u \geqslant x_v$, then $\lambda_1(G) < \lambda_1(G^*)$.*

Lemma 1 was first given by Wu, Xiao and Hong and it is a stronger version of a similar lemma in [**16**].

LEMMA 2.2. [**16**] *Let H be any connected graph with at least two vertices. If A and B are the graphs as in Figure 1, then $P(A, \lambda) > P(B, \lambda)$ for $\lambda \geqslant \lambda_1(A)$. In particular, $\lambda_1(A) < \lambda_1(B)$.*

If H is a spanning subgraph of G we shall write $H \leqslant G$; in particular if it is a proper spanning subgraph, we then write $H < G$.

LEMMA 2.3. [**12**], [**10**, p. 50] *Let G be a connected graph. If H is connected and $H < G$, then $\lambda_1(H) < \lambda_1(G)$.*

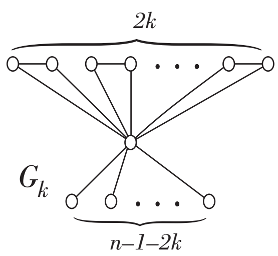

Figure 2.

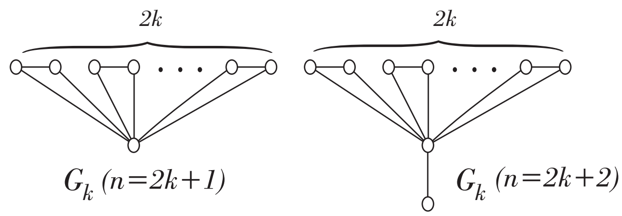

Figure 3.

3. Main result

Let G_k be the bundle with n vertices and k cycles of length 3 depicted in Figure 2. In particular, if $k = \lfloor \frac{n-1}{2} \rfloor$ we have a bundle with n vertices and k cycles of length 3 depicted in Figure 3.

Theorem 3.1. *Let G be a graph in $C(n)$. Then*

$$\lambda_1(G) \leqslant \lambda_1(G_k),$$

where G_k is the graph depicted in Figure 3 *($k = \lfloor \frac{n-1}{2} \rfloor$), and equality holds if and only if $G \cong G_k$.*

Proof. Choose $G \in C(n)$ such that the index of G is as large as possible. Denote the vertex set of G by $V(G) = \{v_1, v_2, \dots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \dots, x_n)$, where x_i corresponds to the vertex v_i $(1 \leqslant i \leqslant n)$.

We first prove that the graph G is a bundle. In order to do that we will prove the following two claims.

Claim 1. *Any two cycles of the graph G have one common vertex.*

Proof. Assume, on the contrary, that there are two disjoint cycles C_p and C_q. Then, there exists a path $v_1\, v_2 \cdots v_k$ joining the cycles C_p and C_q of length $k-1 \geqslant 1$, where the vertex v_1 belongs to the cycle C_p and the vertex v_k belongs to the cycle C_q. Note, any path joining the cycles C_p and C_q starts from v_1 and ends to v_k (in the opposite case G is not a cactus). Without loss of generality we may assume that $x_1 \geqslant x_k$. Denote by v_{k+1} and v_{k+2} neighbors of v_k which belong to C_q.

Let

$$G^* = G - \{v_k v_{k+1},\, v_k v_{k+2}\} + \{v_1 v_{k+1},\, v_1 v_{k+2}\}.$$

Then $G^* \in C(n)$ and by Lemma 1 we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. □

Hence, any two cycles have one common vertex.

Claim 2. *Any three cycles have exactly one common vertex.*

Proof. In the opposite case the graph G is not a cactus, because there exist cycles which have more than one common vertex. □

By Claims 1 and 2, all cycles of the graph G have exactly one common vertex, i.e., they form a bundle. Let us denote by v_1 the common vertex of all cycles in this bundle.

Secondly we prove that if G contains a tree T attached to a cycle at some vertex v (called the *root* of T) then T consists only of edges containing v. That is:

Claim 3. *Any tree T attached to a vertex v of one of the cycles in the graph G contains only vertices at distance one from its root v.*

Proof. In the opposite case, there exists a tree T (with root $v_i \in C_p$) and a vertex of T whose distance from v_i is greater than one. Let $v_j \in T$ be a vertex furthest from the root v_i. Then, $d(v_i, v_j) \geqslant 2$ and there exists a path $v_i \dots v_{j-2}v_{j-1}v_j$ joining v_i and v_j of length $\geqslant 2$. Now, if we take the vertex v_{j-2} as the root r of the graph A from Figure 1 and apply Lemma 2 we will get a graph $G^* \in C(n)$, such that $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. □

Hence, any tree T attached to a vertex of some cycle of G consists only of edges with exactly one common vertex.

We further prove the following claim.

Claim 4. *Any tree T of the graph G is attached to the common vertex v_1 of all cycles of the bundle.*

Proof. In the opposite case, there exists a tree T attached to a vertex v_i ($v_i \neq v_1$), and let $v_i \in C_p$. Let this tree T consist of vertices $y_1, y_2, \dots, y_k$ (at distance one from the root v_i) and $w_1, w_2, \dots, w_l \in N(v_1) \smallsetminus V(C_p)$. If $x_1 \geqslant x_i$, let

$$G^* = G - \{v_iy_1, v_iy_2, \dots, v_iy_k\} \ + \ \{v_1y_1, v_1y_2, \dots, v_1y_k\}.$$

If $x_1 < x_i$, let

$$G^* = G - \{v_1w_1, v_1w_2, \dots, v_1w_l\} \ + \ \{v_iw_1, v_iw_2, \dots, v_iw_l\}.$$

Then, in either case, $G^* \in C(n)$, and by Lemma 1, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. □

Hence G is a bundle with a unique tree attached to the common vertex of all cycles of G, and this tree contains only vertices at distance one from the root.

Finally, we prove:

Claim 5. *All cycles of G have length three.*

Proof. Suppose on the contrary that there exists a cycle C_p of length $p \geqslant 4$. Let $C_p = v_1v_2 \cdots v_pv_1$ and let $w_1, w_2, \dots, w_l \in N(v_1) \smallsetminus V(C_p)$. If $x_1 \geqslant x_2$, let

$$G^* = G - \{v_2v_3\} + \{v_1v_3\}.$$

If $x_1 < x_2$, let

$$G^* = G - \{v_1v_p, v_1w_1, \dots, v_1w_l\} + \{v_2v_p, v_2w_1, \dots, v_2w_l\}.$$

Then, in either case, $G^* \in C(n)$, and by Lemma 1, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. □

Combining arguments from Claims 1–5, we have that $G = G_k$, where G_k is a bundle with n vertices and k cycles of length 3 (Figure 2).

We notice that $G_0 = K_{1,n-1}$ and

$$G_0 < G_1 < \cdots < G_{\lfloor \frac{n-1}{2} \rfloor}.$$

By Lemma 3 we have

$$\lambda_1(G_0) < \lambda_1(G_1) < \cdots < \lambda_1(G_{\lfloor \frac{n-1}{2} \rfloor}).$$

So, we obtain that the graph G_k ($k = \lfloor \frac{n-1}{2} \rfloor$) depicted in Figure 3, is the graph with maximal index in the set $C(n)$ of all connected cactuses with n vertices. This completes the proof. □

Corollary 3.1. *The graph G_k depicted in Figure 2 is the graph with maximal index in the set of all connected cactuses with n vertices and k cycles* ($1 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$).

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Faculty of Science
University of Kragujevac
Kragujevac
Serbia
bojanab@kg.ac.yu

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Faculty of Science
University of Kragujevac
Kragujevac
Serbia
petrovic@kg.ac.yu