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HOLOMORPHY ANGLES AND SECTIONAL CURVATURE IN HERMITIAN ELLIPTIC PLANES OVER FIELDS AND TENSOR PRODUCTS OF FIELDS

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Abstract. Holomorphy angles and sectional curvature in Hermitian elliptic planes over field C of complex numbers, skew field H of quaternions, and alternative skew field O of octonions are considered, and analogues of holomorphy angles in planes over tensor products of these fields and sectional curvature in these planes are found.

1. Planes over fields

The geometry of Hermitian elliptic planes over fields C, H and O is exposed in author's book [**1**, pp. 219–283 and 333–369]. These planes are symmetric Riemannian 4-space, 8-space, and 16-space, respectively. For each 2-direction in these spaces a holomorphy angle and the sectional curvature are determined. The holomorphy angle α for the 2-direction passing through point X and two orthogonal local unit vectors a and b is determined by formula

$$\textstyle\sum_i \bar{a}^i b^i = u \cos\alpha, \tag{1}$$

where $\sum_i \bar{a}^i b^i$ is Hermitian scalar product of vectors a and b and u is a unit element of the field for which $\bar{u} = -u$. The sectional curvature K of this 2-direction is equal to

$$K = (1 + 3\cos^2\alpha)/r^2, \tag{2}$$

where $1/r^2$ is the curvature of the plane.

A 2-direction is called holomorphic if $\alpha = 0$, and antiholomorphic if $\alpha = \pi/2$. Holomorphic 2-directions lie in straight lines of the planes, antiholomorphic 2-directions lie in normal real 2-chains of the planes, that is sets of points with real coordinates or figures obtained from these sets by motions in the planes. Straight

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lines in the planes of curvature $1/r^2$ are isometric to 2-spheres, 4-spheres, and 8-spheres of radius $r/2$, respectively, normal real 2-chains in the planes are isometric to real elliptic 2-planes, 4-spaces, and 8-spaces of curvature $1/r^2$, respectively.

Since sectional curvature of these planes in holomorphic 2-directions is equal to constant number $4/r^2$, these planes are called "planes of constant holomorphic sectional curvature".

Holomorphy angles were first met by E. Study in 1905 in his article [**2**], where he has defined complex Hermitian elliptic spaces. Study has found sectional curvature of 2-direction in this space in the form

$$K_0 = (1 + 3k^2)/4, \tag{3}$$

where $k \leqslant 1$ [**2**, p. 343], but he did not know the geometric meaning of k.

The tangent spaces to Hermitian elliptic planes over fields C, H, and O are Hermitian Euclidean planes over same fields exposed in book [**1**, pp. 168–204 and 339–340].

Holomorphy angles in complex Hermitian elliptic and Euclidean spaces were defined by many geometers under different names. K. Scharnhorst [**3**, pp. 97–99] mentioned that P. Shirokov called them "angles of inclination" (ugly naklona), S. Kobayashi and K. Nomizu called them "Kähler angles", G. Rizza called them "Characteristic deviation" (deviazione caratteristica), and "holomorphic deviation", Bang-Yen Chen called them "Wirtinger angles" and "slant angles".

Since Hermitian elliptic planes are symmetric Riemannian spaces, they are isometric to totally geodesic surfaces in groups of motions of these planes in their Killing–Cartan metric. These groups of motions are compact simple Lie groups of classes A_2, C_3, F_4, E_6, E_7, E_8, respectively. The Lie algebras $\mathbf{G}$ tangent to these groups are direct sums of Lie algebras $\mathbf{K}$ tangent to stabilizers of points of these planes and linear spaces $\mathbf{E}$ which can be regarded as tangent spaces to the planes. Vectors in spaces $\mathbf{E}$ can be regarded as local vectors in these planes.

The sectional curvature of 2-direction in a symmetric Riemannian space determined by two orthogonal unit local vectors a and b is equal to

$$K = \rho[[ab]a]b, \tag{4}$$

where $[ab]$ is the commutator of elements a and b of Lie algebra $\mathbf{G}$, and cd is the scalar product of vectors c and d in Euclidean space $\mathbf{E}$ [**1**, p. 246].

Since $\mathbf{K}$ is a Lie algebra, the commutator of two elements in this algebra is an element in this algebra, the commutator of two elements in subalgebra $\mathbf{K}$ and subspace $\mathbf{E}$ is an element in subspace $\mathbf{E}$, the commutator of two elements in subspace $\mathbf{E}$ is an element in subalgebra $\mathbf{K}$. This property can be written as inclusions

$$[\mathbf{KK}] \subset \mathbf{K}, \quad [\mathbf{KE}] \subset \mathbf{E}, \quad [\mathbf{EE}] \subset \mathbf{K}. \tag{5}$$

In the case of Hermitian elliptic planes over fields C, H, and O Lie algebra $\mathbf{G}$ consists of skew symmetric Hermitian (3×3)-matrices ($A_{ij} = -\bar{A}_{ji}$, $i, j = 0, 1, 2$) with zero trace ($A_{00}+A_{11}+A_{22} = 0$) and Lie algebra of the group of automorphisms of the group of motions in the plane. The last group is finite for the plane over C, is compact simple Lie group A_1 for the plane over H, and is compact simple Lie group G_2 for the plane over O.

Let us denote the (3×3)-matrix with 1 on the intersection of i-th line and j-th column and zeroes on all remaining entries by E_{ij}. Therefore $E_{ij}E_{kl}=\delta_{jk}E_{il}$.

Subspace $\mathbf{E}$ of the Lie algebra tangent to the group of motions in the plane can be reduced to the subspace containing matrices E_{0i} and E_{i0}, $i=1,2$. Matrices A and B in subspace $\mathbf{E}$ corresponding to local vectors a and b have the form

$$A=\textstyle\sum_i E_{0i}a^i-\sum_i E_{i0}\bar{a}^i,\quad B=\sum_j E_{0j}b^j-\sum_j E_{j0}\bar{b}^j,\quad i,j=1,2. \tag{6}$$

Since the commutator $[AB]$ of two matrices A and B is equal to difference $AB-BA$, the commutator of matrices (6) is equal to

$$[AB]=-\textstyle\sum_i E_{00}a^i\bar{b}^i-\sum_i\sum_j E_{ij}\bar{a}^ib^j+\sum_i E_{00}b^i\bar{a}^i+\sum_i\sum_j E_{ij}\bar{b}^ia^j. \tag{7}$$

and commutator $[[AB]A]$ has the form

$$\begin{aligned}&[[AB]A]=\\ &-\textstyle\sum_i\sum_j E_{0i}a^j\bar{b}^ja^i+\sum_i\sum_j E_{i0}\bar{a}^ib^j\bar{a}^j+\sum_i\sum_j E_{i0}b^j\bar{a}^ja^i-\sum_i\sum_j E_{i0}\bar{b}^ia^j\bar{a}^i\\ &-\textstyle\sum_i\sum_j E_{i0}\bar{a}^ia^j\bar{b}^j+\sum_i\sum_j E_{0i}a^j\bar{a}^jb^i+\sum_i\sum_j E_{i0}\bar{a}^ib^j\bar{a}^i-\sum_i\sum_j E_{i0}a^j\bar{b}^ja^i\end{aligned} \tag{8}$$

Formulas (7) and (8) show that matrix (7) corresponds to an element in Lie subalgebra $\mathbf{K}$ and matrix (8) corresponds to a vector in Euclidean subspace $\mathbf{E}$ according to inclusions (5).

Coordinates c^1 and c^2 of vector c corresponding to matrix (8) are coefficients at E_{01} and E_{02}, respectively, that is

$$c^i=-\textstyle\sum_j a^j\bar{b}^ja^i+\sum_j b^j\bar{a}^ja^i+\sum_j a^j\bar{a}^jb^i-\sum_j a^j\bar{b}^ja^i \tag{9}$$

Since the scalar product ab of vectors a and b in subspace $\mathbf{E}$ is equal to the real part of $\sum_i a^i\bar{b}^i$, the scalar product of vector c corresponding to matrix (8) by vector b in Euclidean subspace $\mathbf{E}$ is equal to the real part of scalar product of vectors c and b. This scalar product is equal to

$$\textstyle\sum_i c^i\bar{b}^i=-\sum_i\sum_j a^j\bar{b}^ja^i\bar{b}^i+\sum_i\sum_j b^j\bar{a}^ja^i\bar{b}^i+\sum_i\sum_j a^j\bar{a}^jb^i\bar{b}^i-\sum_i\sum_j a^j\bar{b}^ja^i\bar{b}^i \tag{10}$$

In the case of the plane over commutative field C this product can be rewritten in the form

$$\begin{aligned}\textstyle\sum_i c^i\bar{b}^i&=-\left(\textstyle\sum_i a^i\bar{b}^i\right)\left(\sum_j a^j\bar{b}^j\right)+\left(\sum_i\bar{b}^ia^i\right)\left(\sum_j\bar{a}^jb^j\right)\\ &\quad+\left(\textstyle\sum_i\bar{a}^ia^i\right)\left(\sum_j\bar{b}^jb^j\right)-\left(\sum_i\bar{a}^ib^i\right)\left(\sum_j\bar{a}^jb^j\right)\\ &=\cos^2\alpha+\cos^2\alpha+1+\cos^2\alpha=1+3\cos^2\alpha\end{aligned} \tag{11}$$

Since scalar product (11) is real, formula (4) shows that sectional curvature K of 2-direction in complex Hermitian elliptic plane is equal to

$$K=\rho(1+3\cos^2\alpha). \tag{12}$$

For finding constant ρ in the case of plane with curvature $1/r^2$, let us consider normal real 2-chains in this plane. Since these 2-chains are isometric to real elliptic plane with curvature $1/r^2$ and holomorphy angles of 2-directions in these 2-chains is equal to $\pi/2$, we obtain that $\rho=1/r^2$, that is sectional curvature (11) has the form (2).

In the case of the planes over skew fields H and O scalar product (10) can be reduced to form (11) by permutations of coordinates of vectors a and b, but permutation of two elements in fields H and O is equivalent to addition of commutator $[wz] = wz - zw$. Since real parts of these commutators are equal to zero, real part of scalar product (10) is equal to (11) and sectional curvatures of 2-directions in Hermitian elliptic planes over skew fields H and O, like this curvature in complex Hermitian elliptic plane, has the form (2).

In formula (4.36) in book [**1**, p. 246], which must coincide with formula (2), there is a misprint.

K. Scharnhorst [**3**, pp. 99–100] has mentioned that two planes in Euclidean 4-space representing two straight lines in complex Hermitian Euclidean plane isometric to this 4-space are isoclinic, that is straight lines at infinity of these planes are paratactic lines in elliptic 3-space containing points at infinity of the 4-space. Analogous property there is for 4-planes and 8-planes in Euclidean 8-space and 16-space representing two straight lines in quaternionic and octonionic Hermitian Euclidean planes isometric to these 8-space and 16-space which are isoclinic, that is 3-planes and 7-planes at infinity of these 4-spaces and 8-spaces are paratactic in elliptic 7-space and 15-space containing points at infinity of the 8-space and 16-space.

Analogously we can prove that two Euclidean 2-planes, 4-planes, and 8-planes tangent to two 2-spheres, 4-spheres, and 8-spheres representing two straight lines passing through point X in complex, quaternionic, and octonionic Hermitian elliptic planes, are isoclinic, that is intersections of these two 2-planes, 4-planes and 8-planes, respectively, with hyperplanes at infinity of Euclidean 4-space, 8-space, and 16-space tangent to corresponding Riemannian symmetric space at point X are two paratactic straight lines, 3-planes, and 7-planes in real elliptic 3-space, 7-space, and 15-space, respectively.

2. Planes over tensor products of fields

The first attempt to study holomorphy angles in Hermitian elliptic planes over tensor products of fields was undertaken in article [**4**] by author and R. P. Vyplavina.

The geometry of Hermitian elliptic planes over tensor products of fields C by C, C by H, and H by H is mentioned in book [**1**, p. 226]. The first of these planes admits interpretation as pair of complex Hermitian elliptic planes, its group of motions is isomorphic to direct product of two compact simple groups A_1. The second of these planes admits interpretation as manifold of straight lines in complex Hermitian elliptic 5-space, its group of motions is isomorphic to compact simple Lie group A_3. The third of these planes admits interpretation as manifold of 3-planes in real elliptic 11-space, its group of motions is isomorphic to compact simple group D_6.

The geometry of Hermitian elliptic planes over tensor products of fields C by O, H by O, and O by O is exposed in book [**1**, pp. 340–369]. The groups of motions of these planes are isomorphic to compact simple Lie groups E_6, E_7, and E_8, respectively.

The tangent spaces to Hermitian elliptic planes over tensor products of fields are Hermitian Euclidean planes over same tensor products.

The finding analogue of holomorphy angle and sectional curvature in these Hermitian planes is based on fact proved in book [**1**, pp. 237, 342, and 346] interpretation of straight lines in Hermitian elliptic planes over tensor product C by C, C by H, H by H, C by O, H by O and O by O as manifolds of straight lines in real elliptic 3-space, of straight lines in real elliptic 5-space, of 3-planes in real elliptic 7-space, of straight lines in real elliptic 9-space, of 3-planes in real elliptic 11-space, and of 7-planes in real elliptic 15-space, respectively. Another proof of this interpretation was given by E. Vinberg [**5**].

The groups of motions in Hermitian elliptic lines are isomorphic to groups of motions in corresponding real elliptic spaces, that is to compact semisimple Lie group D_2 and to compact simple Lie groups D_3, D_4, D_5, D_6, and D_8, respectively.

If a 2-direction in Hermitian plane over tensor product passes through point X, a and b are two orthogonal local vectors in this 2-direction issuing from point X, and through these vectors two straight lines XA and XB are drawn, intersecting polar straight line AB of point X at points A and B. These points A and B in planes over tensor products C, by C, C by H, and C by O are represented by two straight lines in real elliptic 3-space, 5-space, and 9-space, respectively. Points A and B in planes over tensor products H by H and H by O are represented by two 3-planes in real elliptic 7-space and 11-space, respectively. Points A and B in plane over tensor products O by O are represented by two 7-planes in real elliptic 15-space. In these cases the role of holomorphy angles is played by stationary distances of mentioned straight lines, 3-planes, and 7-planes divided by radius r of curvature of the Hermitian plane.

The number of these distances is equal to 2 for first three planes, to 4 for following two planes, and to 8 for last plane.

These distances can be calculated as follows. If coordinates of points A and B in Hermitian plane over tensor products of fields are a^i and b^i, the element α equal to divided by radius r of curvature of Hermitian plane distance between points A and B is determined by formula

$$\cos^2\alpha = \rho\left(\textstyle\sum_i \tilde{\bar{a}}^i b^i\right)\left(\textstyle\sum_j \tilde{\bar{b}}^j b^j\right)^{-1}\left(\textstyle\sum_k \tilde{\bar{b}}^k a^k\right)\left(\textstyle\sum_l \tilde{\bar{a}}^l a^l\right)^{-1}\rho^{-1}, \tag{13}$$

element ρ is such that right hand part of this equality has the form $a+iIb$, $a+iIb+jJc+kKd$ or $a+iIb+jJc+kKd+lLe+pPf+qQg+rRh$, where i,j,k,l,p,q,r are basis elements in first multiplier of the tensor product and I,J,K,L,P,Q,R are basis elements in second multiplier of this product. Expressions $a+iIb$, $a+iIb+jJc+kKd$ or $a+iIb+jJc+kKd+lLe+pPf+qQg+rRh$ can be regarded as asplit complex (double), quadruple, and eightfold numbers, respectively, that is elements of direct sums of 2, 4, and 8 fields of real numbers. These algebras are commutative and isomorphic to algebras of diagonal real (2×2)-matrices, (4×4)-matrices, and (8×8)-matrices, respectively. Expressions $\cos^2\alpha$ and α also are elements of these algebras, and stationary distances between straight lines, 3-planes, and 7-planes in real elliptic spaces are coordinates of elements $r\alpha$ in these algebras.

Lie algebras **G** tangent to groups of motions in in Hermitian elliptic planes over tensor products of fields are direct sums of Lie algebras **K** tangent to stabilizers of points in these planes and linear spaces **E** which can be regarded as tangent spaces to the planes. Vectors in spaces **E** can be regarded as local vectors in the planes.

Lie algebra **G** consists of skew symmetric Hermitian (3×3)-matrices ($A_{ij} = -\tilde{\bar{A}}_{ji}$, $i, j = 0, 1, 2$), where $\alpha \to \bar{\alpha}$ and $\alpha \to \tilde{\alpha}$ are transitions to conjugate elements in first and second multipliers of tensor product, with zero trace ($A_{00} + A_{11} + A_{22} = 0$) and Lie algebra of the group of automorphisms of the group of motions in the plane. The last group is finite for the plane over tensor product C by C, is compact simple Lie group A_1 for the plane over tensor product C by H, is compact simple Lie group G_2 for the plane over tensor product C by O, is direct product of two compact simple Lie groups A_1 for the plane over tensor product H by H, is direct product of compact simple Lie groups A_1 and G_2 for the plane over tensor product H by O, and is direct product of two compact simple Lie groups G_2 for the plane over tensor product O by O. Commutators of elements in algebras G were determined by E. Vinberg [**6**].

Subspace **E** of the Lie algebra tangent to the group of motions in the plane can be reduced to the subspace containing matrices E_{0i} and E_{i0}, $i = 1, 2$. Matrices A and B in subspace **E** corresponding to local vectors a and b have the form

$$A = \sum_i E_{0i} a^i - \sum_i E_{i0} \tilde{\bar{a}}^i, \quad \sum_j E_{0j} b^j - \sum_j E_{j0} \tilde{\bar{b}}^j, \quad i, j = 1, 2. \tag{14}$$

Since in Hermitian elliptic planes over tensor product of fields the metric of symmetric Riemannian spaces whose groups of motions are isomorphic to groups of motions in these Hermitian planes, the sectional curvature of 2-directions can be determined by two orthogonal unit local vectors a and b according to formula (4).

In Hermitian elliptic planes over tensor product of fields, unlike as in planes over fields, there are 2-directions with zero sectional curvature. These 2-directions are located in Cartan submanifolds of these planes, where matrices A and B corresponding to local vectors a and b commute one with another.

Let us find nonzero sectional curvature of 2-directions in planes over tensor products of fields located in normal complex, quaternionic, and octonionic 2-chains, that is sets of points with complex, quaternionic or octonionic coordinates or figures obtained from these sets by motions in the planes. Since these normal 2-chains are isometric to Hermitian elliptic planes over fields C, H, or O of the same curvature $1/r^2$ as considered elliptic Hermitian planes over tensor products of fields, sectional curvature of these 2-directions has the form (2).

Therefore sectional curvature of 2-directions in Hermitian elliptic planes over tensor product of fields C by C, C by H, and C by O is equal to

$$K = (2 + 3\cos^2 \alpha_0 + 3\cos^2 \alpha_1)/2r^2, \tag{15}$$

where α_0 and α_1 are divided by r stationary distances of two straight lines representing points A and B.

Sectional curvature of 2-directions in Hermitian elliptic planes over tensor products of fields H by H and H by O is equal to

$$K = (4 + 3\cos^2\alpha_0 + 3\cos^2\alpha_1 + 3\cos^2\alpha_2 + 3\cos^2\alpha_3)/4r^2, \tag{16}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are divided by r stationary distances of two 3-planes representing points A and B.

Sectional curvature of 2-directions in Hermitian elliptic planes over tensor product of fields O by O is equal to

$$\begin{aligned} K = (8 &+ 3\cos^2\alpha_0 + 3\cos^2\alpha_1 + 3\cos^2\alpha_2 + 3\cos^2\alpha_3 \\ &+ 3\cos^2\alpha_4 + 3\cos^2\alpha_5 + 3\cos^2\alpha_6 + 3\cos^2\alpha_7)/8r^2, \end{aligned} \tag{17}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ are divided by r stationary distances of two 7-planes representing points A and B.

Formulas (15), (16), and (17) can be written in unitary form

$$K = \textstyle\sum_i(\delta_i + 3\cos^2\alpha_i)\big/\big(\sum_i\delta_i\big)\, r^2, \tag{18}$$

where $\delta_i = 1$, $i = 0, 1$ for planes over tensor products C by C, C by H, and C by O, $i = 0, 1, 2, 3$ for planes over tensor products H by H and H by O, and $i = 0, 1, 2, 3, 4, 5, 6, 7$ for plane over tensor product O by O.

In the case of holomorphic 2-directions all angles α_i are zeros and formulas (15), (16), and (17) give $K = 4/r^2$.

In the case of antiholomorphic 2-directions all angles α_i are equal to 90° and formulas (15), (16), and (17) give $K = 1/r^2$.

In the case of 2-directions located in normal complex, quaternionic, and octonionic 2-chains all angles α_i are equal one to another that is straight lines, 3-planes and 7-planes in real elliptic spaces representing points A and B are paratactic, and if we denote this common value of α_i by α, formulas (15), (16), and (17) coincide with formula (2).

Numerator of right hand part of equality (18) is the trace of matrix $E + \cos^2 A$, where E and A are (2×2)-matrices, (4×4)-matrices, and (8×8)-matrices, respectively, E is unit matrix, A is diagonal matrix with eigenvalues α_i.

Formula (18) can be obtained also from formula (14). Analogously as formula (10) is obtained from formula (6), from formula (14) the formula differing from formula (10) by replacing coordinates $\bar{a}^i$ and $\bar{b}^i$ by coordinates $\tilde{\bar{a}}^i$ and $\tilde{\bar{b}}^i$ is obtained.

The matrix $E + \cos^2 A$ coincides with the matrix corresponding to the element $\rho\big(\sum_i c^i \tilde{\bar{b}}^i\big)\rho^{-1}$ of the subalgebra of the tensor product isomorphic to the direct sum of 2, 4 or 8 fields of real numbers.

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