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SOME RESULTS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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ABSTRACT. We investigate the generalization of the starlikeness of complex order and the generalization of convexity of complex order for the analytic functions in the unit disc $D = \{z : |z| < 1\}$.

1. Introduction

Let Ω be the family of functions $\omega(z)$ regular in the unit disc D and satisfying the condition $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$. For arbitrary fixed numbers A, B, $-1 \leqslant B < A \leqslant 1$, denote by $P(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in D, such that $p(z) \in P(A, B)$ if and only if $p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}$ for some functions $\omega(z) \in \Omega$ and for every $z \in D$. This class was introduced by Janowski [6].

Further let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ be analytic functions in the unit disc D. Then we say that the function $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, such that $f(z) = g(\omega(z))$, $\omega(z) \in \Omega$, for all $z \in D$. In particular, if $g(z)$ is univalent in D, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$.

Next we consider the following class of functions defined in D. Let $CS^*(A, B, b, q)$ denote the family of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ regular in D, such that $f(z) \in CS^*(A, B, b, q)$ if and only if

$$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where $b \neq 0$, b is a complex number, $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q \in \{0, 1\}$ with $f^{(0)}(z) = f(z)$ and $\omega(z) \in \Omega$. The definition

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of the class $CS^*(A,B,b,q)$ is equivalent to $f(z) \in CS^*(A,B,b,q)$ if and only if

$$(1.1)\qquad \begin{aligned} &1+\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}+q-1\right) \prec \frac{1+Az}{1+Bz} \quad \text{for all } z\in D,\ \ B\neq 0\\ &1+\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}+q-1\right) \prec 1+Az, \quad \text{for all } z\in D,\ \ B=0 \end{aligned}$$

The geometric meaning of (1.1) is that the image of D by

$$1+\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}+q-1\right)$$

is inside the open disc centered on the real axis with diameter end points

$$\begin{array}{llll} \dfrac{1-A}{1-B} & \text{and} & \dfrac{1+A}{1+B}, & B\neq 0\\ 1-A & \text{and} & 1+A, & B=0 \end{array}$$

Some examples of functions in the classes $CS^*(A,B,b,0)$, $CS^*(A,B,b,1)$, $CS^*(1,-1,b,0)$, $CS^*(1,-1,b,1)$ respectively, are the following

$$\text{for } q=0,\ f(z)=\begin{cases} z(1+Bz)^{b(A-B)/B} & B\neq 0\\ ze^{Abz} & B=0\end{cases},$$

$$\text{for } q=1,\ f(z)=\begin{cases} \int_0^z (1+B\zeta)^{b(A-B)/B}d\zeta & B\neq 0\\ \int_0^z e^{bA\zeta}d\zeta & B=0\end{cases},$$

$$\text{for } A=1, B=-1, q=0,\ f(z)=\frac{z}{(1-z)^{2b}},$$

$$\text{for } A=1, B=-1, q=1,\ f(z)=\int_0^z (1-\zeta)^{-2b}d\zeta,$$

Clearly we have the following classes:

(i) For $q=0$, $A=1$, $B=-1$, $CS^*(1,-1,b,0)$ is the class of starlike functions of complex order. This class was introduced by Aouf [1].

(ii) For $q=1$, $A=1$, $B=-1$, $CS^*(1,-1,b,1)$ is the class of convex functions of complex order. This class was introduced by Nasr and Aouf [2].

(iii) For $q=0$, $A=1$, $B=-1$, $b=1$, $CS^*(0,1,-1,1)=S^*$ is the class starlike functions. This class is well known [3], [4].

(iv) For $q=1$, $A=1$, $B=-1$, $b=1$, $CS^*(1,-1,1,1)=C$ is the class convex function. This class is well known [3], [4].

We note that by giving special values to b (which are $b=1-\alpha$, $0\leqslant\alpha<1$; $b=1-(1-\alpha)(\cos\lambda)e^{-i\alpha}$, $0\leqslant\alpha<1$, $|\lambda|<\pi/2$; $b=(1-(\cos\lambda)e^{-i\lambda})$ we obtain very important subclasses of starlike functions and convex functions [3], [4].

2. Some results for the class $CS^*(A,B,b,q)$

We need the following lemmas.

LEMMA 2.1. [**5**] *Let $\omega(z)$ be a non-constant and analytic function in the unit disc D with $\omega(0)=0$. If $|\omega(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_1, then $z_1\omega'(z_1)=k\omega(z_1)$ and $k\geqslant 1$.*

LEMMA 2.2. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an analytic functions in the unit disc D. If $f(z)$ satisfies*

$$\begin{cases} \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec \frac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0 \\ \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec Az = F_2(z), & B = 0 \end{cases} \tag{2.1}$$

then $f(z) \in CS^*(A, B, b, q)$ and the result is sharp as the function

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ z^{1-q}e^{Abz}, & B = 0. \end{cases}$$

PROOF. Let $B \neq 0$. We define a function $\omega(z)$ by

$$\frac{f^{(q)}(z)}{z^{1-q}} = (1 + B\omega(z))^{\frac{b(A-B)}{B}}, \tag{2.2}$$

where $(1 + B\omega(z))^{\frac{b(A-B)}{B}}$ has the value 1 at the origin. Then $\omega(z)$ is analytic in D, $\omega(0) = 0$ and

$$\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{(A-B)z\omega'(z)}{1 + B\omega(z)}. \tag{2.3}$$

Now it is easy to realize that the subordination (2.1) is equivalent to $|\omega(z)| < 1$, for all $z \in D$. Indeed assume the contrary: There exist $z_1 \in D$ such that $|\omega(z_1)| = 1$. Then by I. S. Jack's lemma $z_1\omega'(z_1) = k\omega(z_1)$, $k \geqslant 1$ and for such z_1 we have

$$\frac{1}{b}\left(z_1\frac{f^{(q+1)}(z_1)}{f^{(q)}(z_1)} + q - 1\right) = k\frac{(A-B)\omega(z_1)}{1 + B\omega(z_1)} \notin F_1(D)$$

because $|\omega(z_1)| = 1$ and $k \geqslant 1$. But this is a contradiction to the condition (2.1) of this lemma and so the assumption is wrong i.e., $|\omega(z)| < 1$ for all $z \in D$.

On the other hand we have

$$\begin{aligned} \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec \frac{(A-B)z}{1+Bz} &\Leftrightarrow \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{(A-B)\omega(z)}{1 + B\omega(z)} \\ &\Leftrightarrow 1 + \frac{1}{b}\left(\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \end{aligned} \tag{2.4}$$

The equivalencies (2.4) show that $f(z) \in CS^*(A, B, b, q)$.

Let B =0. Define a function by $\dfrac{f^{(q)}(z)}{z^{1-q}} = e^{Ab\omega(z)}$. Then $\omega(z)$ is analytic in D and $\omega(0) = 0$ and

$$\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = Az\omega'(z). \tag{2.5}$$

Similarly by using I. S. Jack's lemma we obtain

$$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = 1 + A\omega(z). \tag{2.6}$$

The equality (2.6) shows that $f(z) \in CS^*(A, B, b, q)$.

The sharpness of the result follows from the fact that for

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ z^{1-q}e^{Abz}, & B = 0 \end{cases}$$

we receive

$$\left(z\frac{f_*^{(q+1)}(z)}{f_*^{(q)}(z)} + q - 1 \right) = \begin{cases} \frac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0 \\ Az = F_2(z), & B = 0 \end{cases} \qquad \square$$

Lemma 2.3. *If $f(z) \in CS^*(A,B,b,q)$, then the set of the values of $\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\right)$ is the disc with the centre $C(r)$ and the radius $\rho(r)$, where*

$$C(r) = \frac{(1-q) + \left[(q-1)B^2 - b(AB-B^2)\right] r^2}{1-B^2r^2}, \quad \rho(r) = \frac{|b|(A-B)}{1-B^2r^2}, \quad B \neq 0$$

$$C(r) = 1, \qquad \rho(r) = |Ab|r, \qquad B = 0$$

Proof. If $p(z) \in P(A,B)$, then

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \leqslant \frac{(A-B)r}{1-Br^2}. \tag{2.7}$$

The inequality (2.7) was proved by Janowski [6].

By using the definition of the class $CS^*(A,B,b,q)$ and the inequality (2.7) we get

$$\left| 1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1 \right) - \frac{1-ABr^2}{1-B^2r^2} \right| \leqslant \frac{(A-B)r}{1-B^2r^2}. \tag{2.8}$$

After a brief calculations from (2.8) we obtain

$$\left| z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - \frac{(1-q) + \left[(q-1)B^2 - b(AB-B^2)\right] r^2}{1-B^2r^2} \right| \leqslant \frac{|b|(A-B)r}{1-B^2r^2}, \quad B \neq 0$$

$$\left| z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1 \right| \leqslant |Ab|r, \qquad B = 0. \quad \square$$

Theorem 2.1. *If $f(z) \in CS^*(A,B,b,q)$, then*

$$\begin{aligned} M_1(A,B,r) \leqslant |f^{(q)}(z)| \leqslant M_2(A,B,r), \quad & B \neq 0 \\ N_1(A,r) \leqslant |f^{(q)}(z)| \leqslant N_2(A,r) < \quad & B = 0 \end{aligned} \tag{2.9}$$

where

$$M_1(A,B,r) = r^{1-q}(1-Br)^{\frac{(A-B)(|b|+\operatorname{Re} b)}{2B}}(1+Br)^{\frac{(A-B)(\operatorname{Re} b-|b|)}{2B}},$$

$$M_2(A,B,r) = r^{1-q}(1-Br)^{\frac{(A-B)(|b|-\operatorname{Re} b)}{2B}}(1+Br)^{\frac{(A-B)(|b|+\operatorname{Re} b)}{2B}},$$

$$N_1(A,r) = r^{1-q}e^{-|Ab|r}, \quad N_2(A,r) = r^{1-q}e^{|Ab|r}$$

These bounds are sharp because the extremal function is

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ z^{1-q}e^{Abz}, & B = 0 \end{cases}$$

PROOF. By using Lemma 2.3 and after a brief calculations we get

$$\frac{(1-q)-|b|(A-B)r+\left[(q-1)B^2-\operatorname{Re}b(AB-B^2)\right]r^2}{1-B^2r^2}\leqslant \operatorname{Re}z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}$$
$$\leqslant\frac{(1-q)+|b|(A-B)r+\left[(q-1)B^2-\operatorname{Re}b(AB-B^2)\right]r^2}{1-B^2r^2},\quad B\neq 0$$
$$(1-q)-|Ab|r\leqslant \operatorname{Re}z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\leqslant(1-q)+|Ab|r,\quad B=0$$

Since

$$\operatorname{Re}z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}=\frac{\partial}{\partial r}\log\left|f^{(q)}(re^{i\theta})\right|,\quad |z|=r$$

and using preceding inequalities we obtain

$$\frac{(1-q)-|b|(A-B)r+\left[(q-1)B^2-\operatorname{Re}b(AB-B^2)\right]r^2}{r(1-B^2r^2)}\leqslant\frac{\partial}{\partial r}\log\left|f^{(q)}(re^{i\theta})\right|$$
$$\leqslant\frac{(1-q)+|b|(A-B)r+\left[(q-1)B^2-\operatorname{Re}b(AB-B^2)\right]r^2}{r(1-B^2r^2)},\quad B\neq 0$$
$$\frac{(1-q)}{r}-|Ab|\leqslant\frac{\partial}{\partial r}\log\left|f^{(q)}(re^{i\theta})\right|\leqslant\frac{(1-q)}{r}+|Ab|,\quad B=0$$

Integrating both sides of these inequalities from 0 to r we obtain (2.9). □

COROLLARY 2.1. *For $q=0$, $A=1$, $B=-1$, $b=1$ we obtain*

$$\frac{r}{(1+r)^2}\leqslant|f(z)|\leqslant\frac{r}{(1-r)^2}.$$

This is the distortion theorem of starlike functions. The result is well known [3], [4].

COROLLARY 2.2. *For $q=1$, $A=1$, $B=-1$, $b=1$ we get*

$$\frac{1}{(1+r)^2}\leqslant|f'(z)|\leqslant\frac{1}{(1-r)^2}.$$

This is the distortion theorem for the derivative of convex function. This result is well known [3], [4].

COROLLARY 2.3. *For $q=0$, $A=1$, $B=-1$ the following result is obtained*

$$\frac{r}{(1+r)^{(\operatorname{Re}b+|b|)}(1-r)^{(\operatorname{Re}b-|b|)}}\leqslant|f(z)|\leqslant\frac{r}{(1-r)^{(\operatorname{Re}b+|b|)}(1+r)^{(|b|-\operatorname{Re}b)}}.$$

This is the distortion theorem for the starlike functions of complex order.

COROLLARY 2.4. *For $q=1$, $A=1$, $B=-1$ the following result is obtained*

$$\frac{1}{(1-r)^{(\operatorname{Re}b-|b|)}(1+r)^{(\operatorname{Re}b+|b|)}}\leqslant|f'(z)|\leqslant\frac{1}{(1-r)^{(|b|-\operatorname{Re}b)}(1+r)^{(|b|+\operatorname{Re}b)}}.$$

This is the distortion theorem for the derivative of convex functions of complex order.

Corollary 2.5. (Generalized radius problem) *The radius of starlikeness for the class $CS^*(A,B,b,q)$ is given by*

$$R_{cs} = \frac{2}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4\left[(1-q)B^2 - \operatorname{Re} b(AB-B^2)\right]}}.$$

This is a generalization of the radius of starlikeness for the class $CS^*(1,-1,b,0)$, and a generalization of the radius of convexity for the class $CS^*(1,-1,b,1)$.

Proof. By using Lemma 2.3 and after simple calculations we get

$$\operatorname{Re}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\right) \geqslant \frac{(1-q) - |b|(A-B)r + \left[(q-1)B^2 - \operatorname{Re} b(AB-B^2)\right] r^2}{1-B^2r^2}. \tag{2.10}$$

Hence for $R < R_{cs}$ the left-hand side of the preceding inequality is positive, which implies that

$$R_{cs} = \frac{2}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4\left[(1-q)B^2 \operatorname{Re} b(AB-B^2)\right]}}.$$

Also note that the inequality (2.10) becomes an equality for the function

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ z^{1-q}e^{Abz}, & B = 0. \end{cases}$$

It follows that

$$R_{cs} = \frac{2}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4\left[(1-q)B^2 - \operatorname{Re} b(AB-B^2)\right]}},$$

and the proof is complete. □

We note that, by giving special values to A, B, b, q, we obtain the radius of starlikeness and the radius of convexity for the important subclasses of univalent functions. For example:

For $A = 1$, $B = -1$, $q = 0$ and $b = 1$ we obtain $R_{cs} = 1$. This means that the radius of starlikeness for the class of starlike functions is 1.

For $A = 1$ $B = -1$, $q = 0$ the following radius is obtained

$$R_{cs} = \frac{1}{|b| + \sqrt{|b|^2 - 2\operatorname{Re} b + 1}}.$$

This is the radius of starlikeness for the class of starlike functions of complex order. This radius was obtained by Aouf [1].

Similarly by using Lemma 2.3 and after a simple calculations we get

$$\operatorname{Re}\left(1 + z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\right) \geqslant \frac{(2-q) - |b|(A-B)r - \left[(2-q)B + (A-B)\operatorname{Re} b\right] r^2}{1-B^2r^2}.$$

Therefore, a generalization of the radius of convexity is

$$R_{CC} = \frac{2}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 + 4B\left[B + (A-B)\operatorname{Re} b\right]}}.$$

Similarly, if we take $A = 1$ and $B = -1$, then we obtain

$$R_{CC} = \frac{1}{|b| + \sqrt{|b|^2 - 2\operatorname{Re} b + 1}},$$

This is the radius of convexity for the class of convex functions of complex order that was obtained by Nasr and Aouf [2].

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