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ON MINIMAL ORDERED STRUCTURES

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Abstract. We partially describe minimal, first-order structures which have a strong form of the strict order property.

An infinite first-order structure is *minimal* if its each definable (possibly with parameters) subset is either finite or co-finite. It is *strongly minimal* if the minimality is preserved in elementarily equivalent structures. While strongly minimal structures were investigated more closely in a number of papers beginning with [**4**] and [**1**], there are a very few results on minimal but not strongly minimal structures. For some examples see [**2**] and [**3**].

In this paper we shall consider minimal, ordered structures. A first-order structure $\mathbb{M}_0 = (M_0, \dots)$ is *ordered* if there is a binary relation $<$ on M_0, which is definable possibly with parameters from M_0, irreflexive, antisymmetric, transitive and has arbitrarily large finite chains. We usually distinguish (one) such relation by absorbing the involved parameters into the language and assuming that $<$ is an interpretation of a relation symbol from the language, in which case we write $\mathbb{M}_0 = (M_0, <, \dots)$.

Two basic examples of minimal, ordered structures are $(\omega, <)$ and $(\omega + \omega^*, <)$ (where ω^* is reversely ordered ω and $+$ denotes the (ordinal) sum of partial orders). We can modify a basic example by replacing the original order by a new one, so that the structure remains minimal, ordered. For example, we can change $<$ by taking a finite set of elements from the domain and rearranging them arbitrarily, leaving the order of the other elements unchanged; e.g. we can take $0 \in \omega$ and make it bigger than all other elements, or incompatible to the others. Also, we can simply reverse the original order. Note that the 'new' order obtained in either way is inter-definable with the original one, so that the structure remains unchanged. Further, we can enlarge basic structure as follows: starting with $(\omega, <)$ we can replace each $n \in \omega$ by some (large enough) finite set $L(n)$ and define for $a \in L(n)$ and $b \in L(m)$: $a < b$ iff $n < m$. For example (details are left to the reader):

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EXAMPLE 0.1. Let $L = \{(i,n) \in \omega \times \omega \mid i \leq n\}$, and $(i,n) < (j,m)$ iff $n < m$. Then $\mathbb{L} = (L,<)$ is a minimal ordered structure.

EXAMPLE 0.2. Let $U = \{(i,n^*) \in \omega \times \omega^* \mid i \leq n\}$, let L be as in the previous example, and define: $(i,n) < (j,m^*)$ for all i,j,n,m; $(i,n) < (j,m)$ iff $n < m$ and $(i,n^*) < (j,m^*)$ iff $m < n$. Then $(L \cup U, <)$ is a minimal, ordered structure.

Note that in both examples $(L,<)$ is directed upwards and has no ascending chains of order type $\omega + 1$, whilst $(U,<)$ is directed downwards and has no descending chains of order type $(\omega+1)^*$.

Our main result is a characterization of $(M_0,<)$ in a minimal, ordered structure $\mathbb{M}_0 = (M_0,<,\dots)$. It is done in Section 1 below where we we show that, after possibly changing $<$ on a finite subset of M_0 (as described above), or reversing the original order, we can partition M_0 into two pieces $L(M_0)$ and $U(M_0)$ such that $L(M_0)$ is upwards directed and unbounded, and has no ascending chains of order type $\omega + 1$ and either:

$U(M_0) = \emptyset$ (i.e., $(M_0,<)$ is similar to Example 0.1 above); or

$U(M_0)$ is downwards directed and unbounded, and has no descending chains of order type $(\omega+1)^*$ (i.e., $(M_0,<)$ is similar to Example 0.2).

In Section 2 we give a model-theoretic characterization of minimal, ordered structures $\mathbb{M}_0 = (M_0,<,\dots)$ which are similar to Example 0.1. in the above sense.

1. Properties of order

Throughout this section we fix a minimal ordered structure $\mathbb{M}_0 = (M_0,<,\dots)$.

Let $p(x)$ be the set of all formulas in a free variable x (possibly with parameters from M_0), defining a co-finite subset of M_0. By minimality, $p(x)$ is a complete 1-type with parameters from M_0; moreover, it is the only type in $S_1(M_0)$ which is not already realized in $\mathbb{M}_0$ by an element of M_0. We write simply p instead of $p(x)$.

If $\mathbb{M}_0 \prec \mathbb{M} = (M,<,\dots)$, then by $p(M)$ we denote the set of realizations of p in M.

LEMMA 1.1. (a) $\{x \in M_0 \mid x$ *has immediate successor and predecessor*$\}$ *is co-finite.*

(b) *Every realization of p in an elementary extension of $\mathbb{M}_0$ has immediate successors and predecessors which are necessarily realizations of p.*

PROOF. Suppose that $m \in M_0$ does not have immediate successors. Then we have (exactly) one of the following two cases:

(I) m is a maximal element; (II) $\mathbb{M}_0 \models (\forall x > m)(\exists y)(m < y < x)$.

Firstly, we show that there are finitely many possibilities for m in case (I). Note that the set of maximal elements is a definable antichain; the existence of arbitrary large finite chains in M_0 implies that the antichain can not be co-finite. Therefore, by minimality, it must be finite.

Suppose our m is in case (II) above. Choose an infinite descending chain above m, of order type ω^*. Suppose that an element from the chain falls into case (II), and, then choose a descending chain above it of order type ω^*. We now have a chain of order type $\omega^* + \omega^*$ (above m), contradicting the minimality assumption.

We conclude that infinitely many elements of the (first) chain fall into neither of cases (I) or (II), so they have immediate successors.

Thus the set of elements of M_0 having immediate successors (which is definable) is infinite; by minimality it has to be co-finite. Dually, the set of elements of M_0 having immediate predecessors is co-finite, finishing the proof of (a).

To prove (b) note that it easily follows from (a) that any realization of p has immediate successors, so it remains to show that among them there are no elements of M_0. For, let $m \in M_0$. Note that the set of all immediate predecessors of m is a definable antichain which is, by minimality, finite. Hence, the 'formula' $'m$ is an immediate successor of x' does not belong to p, completing the proof of the Lemma. $\square$

LEMMA 1.2. *Suppose that* $\mathbb{M}_0 \prec \mathbb{M} = (M, <, \dots)$. *Then at least one of the following conditions holds:*

(MIN) *Every definable (with parameters) subset of* M *has a minimal element;*
(MAX) *Every definable (with parameters) subset of* M *has a maximal element.*

PROOF. Suppose that neither (MAX) nor (MIN) hold in $\mathbb{M}$. Let $\phi(x, \bar{m})$ define a subset of M without minimal element, and let $\psi(x, \bar{n})$ define a subset of M without maximal element, where $\bar{m}$ and $\bar{n}$ are tuples of elements of M. Note that $\phi(x, \bar{y})$ (considered as a formula in variable x, with $\bar{y}$ fixed) defines a subset without minimal elements' is expressible by a first-order formula in variables $\bar{y}$, call it $\phi_1(\bar{y})$ and similarly define $\psi_1(\bar{z})$. Thus

$$\mathbb{M} \models (\exists \bar{y})(\exists \bar{z})(\phi_1(\bar{y}) \wedge \psi_1(\bar{z})),$$

and hence:

$$\mathbb{M}_0 \models (\exists \bar{y})(\exists \bar{z})(\phi_1(\bar{y}) \wedge \psi_1(\bar{z})).$$

We conclude that there are $\bar{m}_0$ and $\bar{n}_0$ in M_0 such that $\phi(x, \bar{m}_0)$ defines a subset of M_0 without a minimal element, and $\psi(x, \bar{n}_0)$ defines a subset of M_0 without a maximal element. Clearly both sets are infinite so, by minimality of $\mathbb{M}_0$, both of them are co-finite and so is their intersection, call it D. D is definable, infinite and has neither minimal nor maximal elements. It follows that for any $d \in D$ both sets $\{x \mid x < d\}$ and $\{x \mid x > d\}$ are infinite, contradicting minimality of $\mathbb{M}_0$. $\square$

REMARK 1.1. Satisfaction of (MIN) (or (MAX)) in a structure can be expressed by a set of first-order sentences; so if it is satisfied in some structure, it must be satisfied in all structures elementary equivalent to it.

REMARK 1.2. The set of minimal (maximal) elements of a definable subset of M_0 is finite: clearly it is a definable antichain, and if it were infinite it would have to be co-finite, contradicting the existence of arbitrary large finite chains in M_0.

PROPOSITION 1.1. M_0 *is countable.*

PROOF. Let $\mathrm{Lev}(0)$ be the set of minimal and $\mathrm{Lev}^*(0)$ the set of maximal elements of M_0. Inductively define $\mathrm{Lev}(n+1)$ as the set of minimal and $\mathrm{Lev}^*(n+1)$ as the set of maximal elements of $M_0 \setminus \bigcup_{i \leq n}(\mathrm{Lev}^*(i) \cup \mathrm{Lev}(i))$.

Note that if $m \in M_0 \setminus \bigcup_{i\in\omega}(\mathrm{Lev}^*(i) \cup \mathrm{Lev}(i))$, then both $\{x \in M_0 \mid x < m\}$ and $\{x \in M_0 \mid m < x\}$ are infinite, contradicting the minimality assumption. Therefore $M_0 = \bigcup_{i\leq n}(\mathrm{Lev}^*(i) \cup \mathrm{Lev}(i))$. But by the previous remark each $\mathrm{Lev}(i)$ (and also $\mathrm{Lev}^*(i)$) is finite so that the union is countable. □

Definition 1.1. We define

$$\begin{aligned} L_<(M_0) &= \{m \in M_0 \mid (m < x) \in p\} \\ U_<(M_0) &= \{m \in M_0 \mid (x < m) \in p\} \\ I_<(M_0) &= \{m \in M_0 \mid (x \perp m) \in p\}. \end{aligned}$$

Usually, we operate with a single ordering relation $<$ inside a structure, in which case the subscript '$<$' is omitted, i.e., we write simply $L(M_0)$, $U(M_0)$ and $I(M_0)$.

Note that any element of M_0 must belong to either $L(M_0)$ or $U(M_0)$ or $I(M_0)$, i.e.:

$$M_0 = L(M_0) \cup U(M_0) \cup I(M_0) \text{ and } M = p(M) \cup L(M_0) \cup U(M_0) \cup I(M_0).$$

Also we have $L(M_0) < p(M) < U(M_0)$ and $I(M_0) \perp p(M)$, in which sense we consider the set of realizations of p in $\mathbb{M} = (M, <, \dots) \succ \mathbb{M}_0$ as the 'middle' part of M; then 'L' in $L(M_0) = \{x \in M \mid x < p(M)\}$ indicates that the 'lower' part of M_0 (actually of $\mathbb{M}$) is in question. Similarly, U indicates the 'upper' and I the 'incompatible' part of $\mathbb{M}$.

Note that $m \in L(M_0)$ iff $\{x \in M_0 \mid m < x\}$ is co-finite or, equivalently, infinite. Similarly for $U(M_0)$ and $I(M_0)$. This fact shall be used often in what follows.

Lemma 1.3. (a) *Every nonempty subset of $U(M_0)$ has a maximal element.*

(b) *If $U(M_0)$ is infinite, then $\mathbb{M}_0$ satisfies* (MAX).

Dually:

(a′) *Every nonempty subset of $L(M_0)$ has a minimal element.*

(b′) *If $L(M_0)$ is infinite, then $\mathbb{M}_0$ satisfies* (MIN).

Proof. (a) Suppose, on the contrary, that a nonempty subset of $U(M_0)$ does not have a maximal element. Then there is $m \in U(p)$ such that $\{x \in M_0 \mid m < x\}$ is infinite and hence (by minimality) co-finite. Then $\{x \in M_0 \mid x < m\}$ is finite, contradicting $(x < m) \in p$.

(b) Suppose $U(M_0)$ is infinite and we prove that (MAX) holds. Let $\phi(x)$ define a nonempty subset D of M_0. If D is finite it clearly has a maximal element. Assume D is infinite. By minimality D must be co-finite and thus $E = D \cap U(M_0) \neq \emptyset$. By (a) E has a maximal element, say d, which is also maximal in D (if $d < m$ and $m \in D$, then, since $(x < d)$ is in p, $(x < m)$ must be in p, too, i.e., $m \in U(M_0)$ and $m \in E$). □

We shall describe $L(M_0)$, $U(M_0)$ and $I(M_0)$ in more detail.

Lemma 1.4. *If $\mathbb{M}_0$ satisfies* (MIN), *then $L(M_0)$ is infinite, directed upwards (i.e., every finite subset of $L(M_0)$ has a strict upper bound in $L(M_0)$) and has no ascending chains of order type $\omega + 1$.*

Dually, if $\mathbb{M}_0$ *satisfies* (MAX)*, then* $U(M_0)$ *is infinite, directed upwards and has no descending chains of order type* $(\omega+1)^*$.

PROOF. Suppose that $\mathbb{M}_0 \prec \mathbb{M} = (M, <, \dots)$ and $a \in M$ is a realization of p.

(1) Firstly, we prove that if $m_1, m_2, \dots, m_k \in L(M_0)$, then there is $n \in L(M_0)$ such that $m_1, m_2, \dots, m_k < n$.

Note that $m_1, m_2, ..., m_k < a$ and let $D(a) = \{x \in M \mid m_1, m_2, ..., m_k < x < a\}$. By Lemma 1.1 a has a predecessor b realizing p. Then $m_1, m_2, \dots, m_k < b$, whence $b \in D(a)$ and $D(a) \neq \emptyset$ (showing that for any realization c of p $D(c) \neq \emptyset$). By (MIN) $D(a)$ has a minimal element n. But $D(n) = \emptyset$ so n is not a realization of p and from $n < a$ we infer $n \in L(M_0)$

(2) Now we show that $L(M_0) \neq \emptyset$, which combined with (1) implies that there is an infinite chain of order type ω in $L(M_0)$; in particular, $L(M_0)$ is infinite.

By (MIN) (in $\mathbb{M}$) the set $\{x \in M \mid x < a\}$ has a minimal element m. By Lemma 1.1 m can't realize p, so $m \in M_0$, i.e., $L(M_0) \neq \emptyset$.

To complete the proof it remains to note that any chain of order-type $\omega + 1$ in $L(M_0)$ would, by (1), produce a chain of order-type $\omega + \omega$ contradicting the minimality. □

LEMMA 1.5. *If one of* $U(M_0)$ *and* $L(M_0)$ *is finite, then* $I(M_0)$ *is finite, too.*

PROOF. Without loss of generality, suppose $U(M_0)$ is finite. Then $M_0 \setminus U(M_0)$ is definable and let A be the set of its maximal elements. A is finite by Remark 1.2. By Lemma 1.4 no element of $L(M_0)$ is maximal, so $A \subseteq I(M_0)$. Thus, for $m \in A$ we have $(x < m) \notin p$, and $\{x \in M_0 \mid x \leq m\}$ is finite. Therefore $B = \bigcup_{m \in A} \{x \in M_0 \mid x \leq m\}$ is finite.

We claim that $I(M_0) \subseteq B$. Otherwise, there would exist $n \in I(M_0)$ with no maximal elements above it. But the last implies that $\{x \in M_0 \mid n < x\}$ is infinite, i.e., $(n < x) \in p$ and $n \in L(M_0)$, contradicting $n \in I(M_0)$. □

PROPOSITION 1.2. $I(M_0)$ *is finite.*

PROOF. If either of $U(M_0)$ or $L(M_0)$ is finite we are done by previous lemma, so suppose that both of them are infinite. By Lemma 1.3 both (MIN) and (MAX) hold in M_0.

The proof goes as follows: assuming that $I(M_0)$ is infinite we shall find another definable ordering relation $\lhd$ on M_0, such that: $(M_0, \lhd, \dots)$ is also minimal ordered, $I_\lhd(M_0)$ is also infinite, but $U_\lhd(M_0) = \emptyset$, contradicting the previous lemma.

Suppose that $I(M_0)$ is infinite. Let $\mathbb{M}_0 \prec \mathbb{M} = (M, <, \dots)$ and let $a \in p(M)$. For $x \in M$ define $\mathrm{Succ}(x)$ to be the set of all immediate successors of x and:

$$D(x) = \{t \in M \mid x < t \text{ and } (\exists y \in \mathrm{Succ}(x))(y \perp t)\}.$$

For $x, y \in M$ with $D(x), D(y) \neq \emptyset$ define: $x \lhd y$ iff $D(y) \subset D(x)$.

We continue with a sequence of claims:

(1) $\emptyset \neq D(a) \subset p(M)$. In particular $\{x \mid D(x) \neq \emptyset\}$ is co-finite.

We leave to the reader to verify that $\mathrm{Succ}(a)$ has at least two elements in which case $\mathrm{Succ}(a) \subseteq D(a)$ and $D(a) \neq \emptyset$. To prove the inclusion suppose $b \in D(a)$; i.e., there is $c \in \mathrm{Succ}(a)$ such that $a < b$ and $b \perp c$. By Lemma 1.1 $c \models p$. Now, $c < U(M_0)$ and $b \perp c$ imply $b \notin U(M_0)$. Also $a < b$ implies $b \notin L(M_0) \cup I(M_0)$ and altogether $b \models p$, completing the proof of (1).

(2) $a \lhd b$ implies $b \in p(M)$, i.e., $U_\lhd(M_0) = \emptyset$.

$a \lhd b$ means $\emptyset \neq D(b) \subset D(a) \subset p(M)$ which, by (1), implies $D(b) \cap p(M) \neq \emptyset$. If $b \in M_0$ were true, then $D(b)$, being definable in $\mathbb{M}_0$, would have to contain the whole $p(M)$, contradicting $D(b) \subset D(a) \subset p(M)$.

(3) If $m \in M_0$ and $m \lhd a$, then $m \in L(M_0)$. Thus $L_\lhd(M_0) \subseteq L(M_0)$.

$m \lhd a$ implies $D(m) \supset D(a)$, so by (1) $D(m) \cap p(M) \neq \emptyset$. But $D(m)$ is definable in $\mathbb{M}_0$ so $D(m) \supseteq p(M)$ implying $m < p(M)$, i.e., $m \in L(M_0)$.

(4) If $m \in M_0$ and $m \lhd a$, then $\mathrm{Succ}(m) \cap I(M_0) \neq \emptyset$.

By (3) $m \in L(M_0)$ so $\mathrm{Succ}(m) \subseteq L(M_0) \cup I(M_0)$. If $\mathrm{Succ}(m) \subset L(M_0)$, then, since $L(M_0)$ is directed, we have $\mathrm{Succ}(m) < p(M)$ and $D(m) \cap p(M) = \emptyset$ which contradicts $m \lhd a$.

(5) Conversely, if $m \in L(M_0)$ and $\mathrm{Succ}(m) \cap I(M_0) \neq \emptyset$, then $m \lhd a$.

If $i \in I(M_0) \cap \mathrm{Succ}(m)$, then $m < p(M)$ and $i \perp p(M)$ imply $p(M) \subseteq D(m)$ so $m \lhd a$ by (1).

(6) If $m \in L(M_0)$, then there exists $n \in L(M_0)$ such that $m \leq n$ and $n \lhd a$. Thus $\{x \in L(M_0) \mid x \lhd a\}$ is infinite

$m \in L(M_0)$ implies that $\{x \mid m < x\}$ is co-finite, so it must contain some $i \in I(M_0)$. Going upwards along a chain connecting m and i we can find along the way some $n \in L(M_0)$ having immediate successor in $I(M_0)$ which, by (5), implies $n < a$.

(7) If $m \in L(M_0)$ and $m \lhd a$, then there is $n \in L(M_0)$ such that $m \lhd n \lhd a$.

By (3), $D(m) \supseteq p(M)$ is infinite, hence is co-finite by minimality. Let $D(m) = M \smallsetminus \{m_1, m_2, \dots, m_k\}$. Then $m_i \notin D(m) \supset D(a)$ implies $m_i \notin D(a) \neq \emptyset$ so, by minimality, the set $\{x \mid m_i \notin D(x) \neq \emptyset\}$ must be co-finite. Therefore, the set $E = \bigcap_{i=1}^k \{x \mid m_i \notin D(x) \neq \emptyset\}$ is co-finite, too. Now, if $x \in E$, then $m_i \notin D(x)$ so $M \smallsetminus \{m_1, m_2, \dots, m_k\} = D(m) \supseteq D(x) \neq \emptyset$, which is $m \unlhd x$. Further, from (6) $F = \{x \in L(M_0) \mid x \lhd a\}$ is infinite so $E \cap F$ is infinite, too. Thus for all $x \in E \cap F$ we have $m \unlhd x \lhd a$ and the conclusion follows.

To finish the proof of the proposition note that by (7) we have that $(M_0, \lhd, \dots)$ is a minimal ordered structure. By (3) any element from $U(M_0) \cup I(M_0)$ is not in $L_\lhd(M_0)$, and since by (2) $U_\lhd(M_0) = \emptyset$ we conclude that $I_\lhd(M_0) \supseteq U(M_0) \cup I(M_0)$. In particular $I_\lhd(M_0)$ must be infinite, completing the proof of the proposition. $\square$

Summing altogether results of this section we come to a closer description of $(M_0, <)$ in a minimal, ordered structure $\mathbb{M}_0 = (M_0, <, \dots)$. First of all, by Proposition 1.2, $I(M_0)$ is finite, so after rearranging its elements and possibly replacing '$<$', as described in the introduction, we may assume $I(M_0) = \emptyset$. Now, we have two cases depending on whether one of $L(M_0)$ and $U(M_0)$ is finite or not. In the first case suppose that $U(M_0)$ is finite (if $L(M_0)$ is finite, reverse the order). Then after rearranging its elements, we may assume $U(M_0) = \emptyset$. Thus we have:

Type(ω) $(M_0, <)$ has no maximal elements, it is directed upwards and has no increasing chains of order type $\omega + 1$.

In the other case both $L(M_0)$ and $U(M_0)$ are infinite, so by Lemma 1.4 we have:

Type ($\omega + \omega^*$) M_0 can be partitioned into $L(M_0)$ and $U(M_0)$, such that: $(L(M_0), <)$ has no maximal elements, is directed upwards and has no increasing chains of order type $\omega + 1$ and $(U(M_0), <)$ has no minimal elements, it is directed downwards and has no decreasing chains of order type $\omega^* + 1$.

2. Model-theoretic properties

We recall Pillay's notion of semi-isolation. Let $\mathbb{N} = (N, \dots)$ be a first-order structure, $q \in S_1(\emptyset)$ and $a, b \in q(N)$. Then b is *semi-isolated* over a (or a semi-isolates b) if there is a formula $\phi(x, y)$ (without parameters) such that $\mathbb{N} \models \phi(a, b)$ and whenever $\mathbb{N} \models \phi(a, c)$, then $c \in q(N)$. Semi-isolation is reflexive and transitive. For transitivity, if a semi-isolates b is witnessed by $\phi(x, y)$, and b semi-isolates c is witnessed by $\psi(y, z)$, it is straightforward to check that $(\exists y)(\phi(x, y) \wedge \psi(y, z))$ witnesses that a semi-isolates c.

The following theorem is inspired by [**5**], see also Proposition 2.1 in [**6**].

THEOREM 2.1. *Let $\mathbb{M}_0 = (M_0, \dots)$ be a minimal structure. Then the following two conditions are equivalent:*

(A) $\mathbb{M}_0$ *is ordered of Type* (ω).

(B) *There exists* $\mathbb{M} \succ \mathbb{M}_0$ *such that semi-isolation is not symmetric on* $p(M)$.

PROOF. (A) $\Rightarrow$ (B). Suppose $\mathbb{M}_0 = (M_0, <, \dots)$ is ordered by $<$ so that $M_0 = L(M_0)$. Let $\mathbb{M} = (M, <, \dots) \succ \mathbb{M}_0$ be $\aleph_1$-saturated and let $a \in p(M)$. Consider the set $C = \{\mathrm{tp}(m/M_0 \cup \{a\}) \mid m \in M_0\} \subset S_1(M_0 \cup \{a\})$. C is infinite so, by compactness, it has an accumulation point $q \in S_1(M_0 \cup \{a\})$. Let $b \in M$ be a realization of q. We shall show that a is semi-isolated over b, and that b is not semi-isolated over a.

Note that $(x < a) \in \mathrm{tp}(m/M_0 \cup \{a\})$ for all $m \in M_0$, so $(x < a) \in q$. Thus $\mathbb{M} \models b < a$ and b semi-isolates a via $b < y$. Further, note that any formula $\phi(x, a)$ (with parameters from $M_0 \cup \{a\}$) which is satisfied by b must belong to some $\mathrm{tp}(m/M_0 \cup \{a\})$. Therefore, $\phi(x, a)$ can not witness that b is semi-isolated over a, and b is not semi-isolated over a. Semi-isolation is not symmetric on $p(M)$.

(B) $\Rightarrow$ (A). Suppose $\mathbb{M} = (M, \dots) \succ \mathbb{M}_0 = (M_0, \dots)$ and $a, b \in p(M)$ are such that a semi-isolates b (witnessed by $\phi(x, y)$) but b does not semi-isolate a. Also, by replacing $\phi(x, y)$ by $\phi(x, y) \wedge \neg\phi(y, x)$ if necessary, we may assume that $\phi(x, y)$ is asymmetric, i.e., $\mathbb{M} \models \phi(x, y) \Rightarrow \neg\phi(y, x)$. For $x, y \in M$ define:

$$x < y \text{ iff } \mathbb{M} \models \phi(x, y) \wedge (\forall t)(\phi(y, t) \rightarrow \phi(x, t)).$$

We will prove a sequence of claims.

(1) $<$ is irreflexive and transitive.

Asymmetry of $\phi(x, y)$ implies that $<$ is irreflexive; checking transitivity is straightforward and is left to the reader.

(2) $a < c$ implies $c \models p$, i.e., $U(M_0) = \emptyset$.

If $a < c$, then $\mathbb{M} \models \phi(a, c)$ and, since $\phi(x, y)$ witnesses semi-isolation, we have $c \models p$.

(3) $\mathrm{tp}(a/M_0 \cup \{b\})$ is in accumulation point of $\{\mathrm{tp}(m/M_0 \cup \{b\}) \mid m \in M_0\}$.

Otherwise, there is a formula $\psi(x, b) \in \mathrm{tp}(a/M_0 \cup \{b\})$ (with possible parameters from M_0 not displayed) which is not satisfied by any $m \in M_0$. Thus $\psi(x, b) \vdash p(x)$ and a is semi-isolated over b; a contradiction.

(4) $a < b$.

Choose, by (3), a sequence $\{m_k \mid k \in \omega\}$ of distinct elements of M_0, so that $\{\mathrm{tp}(m_k/M_0 \cup \{b\}) \mid k \in \omega\}$ converges to $\mathrm{tp}(a/M_0 \cup \{b\})$ (in $S_1(M_0 \cup \{b\})$). From $\phi(x, b) \in \mathrm{tp}(a/M_0 \cup \{b\})$ we derive that for all but finitely many $k \in \omega$ $\phi(x, b) \in \mathrm{tp}(m_k/M_0 \cup \{b\})$, i.e., $\mathbb{M} \models \phi(m_k, b)$ and $\phi(m_k, x) \in p$. Further, if $\mathbb{M} \models \phi(b, c)$, then $c \models p$, so $\mathbb{M} \models \phi(m_k, c)$ and:

$$\mathbb{M} \models \phi(m_k, b) \wedge (\forall t)(\phi(b, t) \rightarrow \phi(m_k, t)).$$

From the convergence of $\mathrm{tp}(m_k/M_0 \cup \{b\})$ to $\mathrm{tp}(a/M_0 \cup \{b\})$ we get:

$$\mathbb{M} \models \phi(a, b) \wedge (\forall t)(\phi(b, t) \rightarrow \phi(a, t)).$$

We have just showed that $(M_0, <)$ is ordered and that $U(M_0) = \emptyset$, completing the proof of the theorem. $\square$

Condition (B) in the theorem does not mention any specific order which is definable in the structure. Therefore if it is satisfied then a minimal, ordered structure can not be of Type (ω) with respect to one ordering, and of Type $(\omega+\omega^*)$ with respect to some other. We get:

Corollary 2.1. *If $\mathbb{M}_0$ is a minimal ordered structure, then Type of ordering of $(M_0, <)$ does not depend on the particular choice of $<$.*

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