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ON SOME RESULTS FOR λ-SPIRALLIKE AND λ-ROBERTSON FUNCTIONS OF COMPLEX ORDER

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Abstract. We give some results of various kinds concerning λ-spirallike functions of complex order and λ-Robertson functions of complex order in the unit disc $U = \{z : |z| < 1\}$. They represent extensions and generalizations of many previous results. We mainly used the subordination method.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

For a function $f(z)$ belonging to the class A we say that $f(z)$ is λ-spirallike of complex order in U if and only if

$$\operatorname{Re}\left\{\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda\right]\right\} > 0, \tag{1.2}$$

for some real λ, $|\lambda| < \pi/2$, $b \neq 0$, complex. We denote this class by $S^\lambda(b)$. It was introduced and studied by Al-Oboudi and Haidan [1].

Also for a function $f(z)$ belonging to the class A we say that $f(z)$ is λ-Robertson function of complex order in U if and only if

$$\operatorname{Re}\left\{\frac{1}{b\cos\lambda}\left[e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right) - (1-b)\cos\lambda - i\sin\lambda\right]\right\} > 0, \tag{1.3}$$

for some real λ, $|\lambda| < \pi/2$, $b \neq 0$, complex. We denote this class by $C^\lambda(b)$.

If follows from (1.2) and (1.3) that

$$f(z) \in C^\lambda(b) \text{ if and only if } zf'(z) \in S^\lambda(b). \tag{1.4}$$

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We note that:

(1) $S^\lambda(1) = S^\lambda$, is the class of λ-spirallike univalent functions defined by Spacek [16], $S^0(b) = S(b)$, is the class of starlike functions of complex order studied by Nasr and Aouf [7], $S^\lambda(1-\alpha) = S^\lambda(\alpha)$, $0 \leqslant \alpha < 1$, is the class of λ-spirallike functions of order α studied by Libera [4] and $S^0(1-\alpha) = S^*(\alpha)$, $0 \leqslant \alpha < 1$, is the class of starlike functions of order α, studied by Robertson [12].

(2) $C^\lambda(1) = C^\lambda$, is the class of λ-Robertson functions studied by Robertson [13], $C^\lambda(1-\alpha) = C^\lambda(\alpha)$, $0 \leqslant \alpha < 1$, is the class of λ-Robertson functions of order α studied by Chichra [3] and $C^0(b) = C(b)$, is the class of convex functions of complex order studied by Waitrowski [18], Nasr and Aouf [8] and Aouf [2] and $C^0(1-\alpha) = C(\alpha)$, $0 \leqslant \alpha < 1$, is the class of convex functions of order α studied by Robertson [12].

The object of this paper is to obtain some results for the classes $S^\lambda(b)$ and $C^\lambda(b)$ using mainly the method of subordination. In that sense, we give some definitions, notations and lemmas we need in the next part.

Let f and F be analytic in the unit disc U. The function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

The general theory of differential subordinations was introduced by Miller and Mocanu [5]. Some classes of the first-order differential subordinations were considered by the same authors in [6]. Namely let $\psi : C^2 \to C$ be analytic in a domain D, let h be univalent in U, and let $p(z)$ be analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then $p(z)$ is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z). \tag{1.5}$$

The univalent function q is said to be a dominant of the differential subordination (1.5) if $p \prec q$ for all p satisfying (1.5). If $\tilde{q}$ is a dominant of (1.5) and $\tilde{q} \prec q$ for all dominants q of (1.5), then $\tilde{q}$ is said to be the best dominant of (1.5).

First we cite the following lemma on differential subordinations due to Miller and Mocanu [6].

Lemma 1. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \qquad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

(i) *Q is starlike (univalent) in U with $Q(0) = 0$ and $Q'(0) \neq 0$,*

(ii) $\operatorname{Re}\left\{z\dfrac{h'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\dfrac{\theta'(q(z))}{\phi(q(z))} + z\dfrac{Q'(z)}{Q(z)}\right\} > 0$, $z \in U$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.6}$$

then $p \prec q$, and q is the best dominant of (1.6).

For the proof of Theorem 2, we need the following result of Robertson [14].

LEMMA 2. *Let $f(z) \in A$ be univalent in U. For $0 \leqslant t \leqslant 1$, let $F(z,t)$ be analytic in U, let $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let r be positive real number for which*

$$F(z) = \lim_{t \to 0^+} \frac{F(z,t) - F(z,0)}{zt^r}$$

exists. Let $F(z,t)$ be subordinate to $f(z)$ in U for $0 \leqslant t \leqslant 1$. Then $\operatorname{Re}\left\{\dfrac{F(z)}{f'(z)}\right\} \leqslant 0$, *$z \in U$. If, in addition, $F(z)$ is also analytic in U and $\{F(0)\} \neq 0$, then*

$$\operatorname{Re}\left\{\frac{f'(z)}{F(z)}\right\} < 0, \quad z \in U. \tag{1.7}$$

2. Results and consequences

First we use the differential subordinations to obtain:

THEOREM 1. *Let $f \in S^\lambda(b)$ ($|\lambda| < \pi/2$, $b \neq 0$, complex), then*

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}}, \tag{2.1}$$

where $a \neq 0$ is complex and either $|2abe^{-i\lambda}\cos\lambda + 1| \leqslant 1$ or $|2abe^{-i\lambda}\cos\lambda - 1| \leqslant 1$, and this is the best dominant.

PROOF. If we put $q(z) = (1-z)^{-2abe^{-i\lambda}\cos\lambda}$, $\phi(w) = (abe^{-i\lambda}\cos\lambda)^{-1}w^{-1}$ and $\theta(w) = 1$ in Lemma 1, then it is easy to check that the conditions (i) and (ii) in that lemma are satisfied. Namely, $q(z)$ is univalent in U [15], while

$$h(z) = \theta(q(z)) + zq'(z)\phi(q(z)) = \frac{1+z}{1-z}\cdot$$

Consequently, for $p(z) = 1 + p_1 z + \cdots$ analytic in U with $p(z) \neq 0$ for $0 < |z| < 1$, from (1.6) we get

$$1 + \frac{e^{i\lambda}}{ab\cos\lambda}\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} \Rightarrow p(z) \prec q(z)\cdot \tag{2.2}$$

Now, if in (2.2) we choose $p(z) = \left(\dfrac{f(z)}{z}\right)^a$, then we have

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}},$$

which evidently completes the proof of Theorem 1. □

REMARK 1. (1) Putting (i) $\lambda = 0$, (ii) $b = 1$, and (iii) $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, in Theorem 1, we get the results obtained by Obradović, Aouf and Owa [10] for the classes $S(b)$, S^λ and $S^\lambda(\alpha)$, respectively.

(2) Putting (i) $\lambda = 0$ and $b = 1$, (ii) $\lambda = 0$ and $b = 1-\alpha$, $0 \leqslant \alpha < 1$, in Theorem 1, we get the corresponding results for the classes S^* and $S^*(\alpha)$, especially, the well-known results for the classes S^* and $S^*(\alpha)$ when $a = 1$.

From Theorem 1 and using (1.4), we directly get:

COROLLARY 1. *Let $f(z) \in C^{\lambda}(b)$, ($|\lambda| < \pi/2$, $b \neq 0$, complex), and let $a \neq 0$ be a complex number and either $|2abe^{-i\lambda}\cos\lambda + 1| \leqslant 1$ or $|2abe^{-i\lambda}\cos\lambda - 1| \leqslant 1$. Then*

$$(f'(z))^a \prec (1-z)^{-2abe^{-i\lambda}\cos\lambda}$$

and this is the best dominant.

Putting $b = 1-\alpha$, $0 \leqslant \alpha < 1$, in Corollary 1, we get the following result for the class $C^{\lambda}(\alpha)$:

COROLLARY 2. *Let $f(z) \in C^{\lambda}(\alpha)$ ($|\lambda| < \pi/2$, $0 \leqslant \alpha < 1$), and let a be a complex number such that*

$$|2a(1-\alpha)\cos\lambda e^{-i\lambda} - 1| \leqslant 1 \ \text{or}\ |2a(1-\alpha)\cos\lambda e^{-i\lambda} + 1| \leqslant 1.$$

Then

$$(f'(z))^a \prec (1-z)^{-2a(1-\alpha)e^{-i\lambda}\cos\lambda} \tag{2.3}$$

and this is the best dominant.

Putting $\lambda = 0$ in Corollary 2 we get the result obtained by Obradović, Aouf and Owa [10].

If we put $a = -\dfrac{e^{i\lambda}}{2b\cos\lambda}$ in Theorem 1, we get:

COROLLARY 3. *Let $f(z) \in S^{\lambda}(b)$ ($|\lambda| < \pi/2$, $b \neq 0$), then*

$$\left(\frac{z}{f(z)}\right)^{\frac{e^{i\lambda}}{2b\cos\lambda}} \prec (1-z), \tag{2.4}$$

and this is the best dominant.

From (2.4), we have the following inequality for $f(z) \in S^{\lambda}(b)$

$$\left|\left(\frac{z}{f(z)}\right)^{\frac{e^{i\lambda}}{2b\cos\lambda}} - 1\right| \leqslant |z|, \quad z \in U. \tag{2.5}$$

REMARK 2. (i) Putting $\lambda = 0$ in (2.5), we get the result obtained by Obradović, Aouf and Owa [10], (ii) Putting $b = 1-\alpha$, $0 \leqslant \alpha < 1$ and $\lambda = 0$ in (2.5), we get the result obtained by Obradović, Aouf and Owa [10] and Todorov [17].

By using Lemma 2 we give a criterion for a function $f(z) \in A$ to be in the class $S^{\lambda}(b)$.

THEOREM 2. *Let $f(z) \in A$ with $f(z)/z \neq 0$ in U, and let the function*

$$g(z) = \frac{e^{i\lambda}}{b\cos\lambda}\left[f(z) - (1 - be^{-i\lambda}\cos\lambda)\int_0^z \frac{f(s)}{s}ds\right] = z + \cdots, \tag{2.6}$$

be univalent in U, where $|\lambda| < \pi/2$ and $b \neq 0$, is a complex number. If the function

$$G(z,t) = \frac{e^{i\lambda}}{b\cos\lambda}\left[(1 - tbe^{-i\lambda}\cos\lambda)f(z) - (1 - be^{-i\lambda}\cos\lambda)(1-t^2)\int_0^z \frac{f(s)}{s}ds\right] \tag{2.7}$$

is subordinate to $g(z)$ for a fixed $b, |\lambda| < \pi/2$, and for each $0 \leqslant t \leqslant 1$, then $f(z) \in S^{\lambda}(b)$.

Proof. It is evident that $G(z,0) \equiv g(z)$ and $G(0,t) \equiv 0$. In Lemma 2, we choose $r = 1$ and $F(z,t)$ to be the function $G(z,t)$ defined by (2.7). Then we have

$$G(z) = \lim_{t \to 0^+} \frac{G(z,t) - G(z,0)}{zt} = \frac{1}{z} \lim_{t \to 0} \frac{\partial G(z,t)}{\partial t} = -\frac{f(z)}{z}$$

and $G(z)$ is analytic in U with $\operatorname{Re}\{G(0)\} = -1 \neq 0$. Since from (2.6)

$$g'(z) = \frac{e^{i\lambda}}{b\cos\lambda}\Big[f'(z) - (1 - be^{-i\lambda}\cos\lambda)\frac{f(z)}{z}\Big],$$

then from (1.7) we have $\operatorname{Re}\left\{\dfrac{g'(z)}{G(z)}\right\} < 0$, $z \in U$, which is equivalent to

$$\operatorname{Re}\Big\{1 + \frac{e^{i\lambda}}{b\cos\lambda}\Big(\frac{zf'(z)}{f(z)} - 1\Big)\Big\} > 0 \quad z \in U,$$

i.e., $f(z) \in S^{\lambda}(b)$. □

Remark 3. (1) Putting (i) $\lambda = 0$ (ii) $\lambda = 0$ and $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, in Theorem 2, we get the results for the classes $S(b)$ and $S^*(\alpha)$ obtained by Obradović, Aouf and Owa [10] and Obradović [9], respectively.

(2) Putting $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, Theorem 2 we get the following result for the class $S^{\lambda}(\alpha)$ ($|\lambda| < \pi/2$, $0 \leqslant \alpha < 1$).

Corollary 4. *Let $f(z) \in A$, and let the function $g(z)$ defined by*

$$g(z) = \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\Big[f(z) - \big(1 - (1-\alpha)e^{-i\lambda}\cos\lambda\big)\int_0^z \frac{f(s)}{s}ds\Big] = z + \cdots$$

be univalent in U, where $|\lambda| < \pi/2$ and $0 \leqslant \alpha < 1$. If the function

$$G(z,t) = \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\Big[\big(1 - t(1-\alpha)e^{-i\lambda}\cos\lambda\big)f(z) \\ - \big(1 - (1-\alpha)e^{-i\lambda}\cos\lambda\big)(1-t^2)\int_0^z \frac{f(s)}{s}ds\Big],$$

is subordinate to $g(z)$ in the unit disc U for fixed λ ($|\lambda| < \pi/2$) and α ($0 \leqslant \alpha < 1$), and for each t ($0 \leqslant t \leqslant 1$), then $f(z)$ is in the class $S^{\lambda}(\alpha)$.

Remark 4. The result obtain in Corollary 4 corrected the result obtained by Obradović and Owa [11, Theorem 3] for the class S^{λ}.

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