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KNESER'S THEOREM FOR WEAK SOLUTIONS OF AN INTEGRAL EQUATION WITH WEAKLY SINGULAR KERNEL

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ABSTRACT. We prove that the set of all weak solutions of the Volterra integral equation (1) is nonempty, compact and connected.

Assume that $D = [0, a]$ is a compact interval in $\mathbb{R}$, E is a sequentially weakly complete Banach space, $B = \{x \in E : \|x\| \leqslant b\}$. We prove the existence of a weak solution of the integral equation

$$x(t) = \int_0^t K(t,s) f(s, x(s))\, ds, \tag{1}$$

where

1° $f : D \times B \mapsto E$ is a weakly-weakly continuous function such that $\|f(t,x)\| \leqslant M$ for $(t,x) \in D \times B$;

2° $K(t,s) = \dfrac{H(t,s)}{(t-s)^r}$, $0 < r < 1$, where H is a real continuous function.

Moreover, we study the topological structure of the set of all weak solutions of (1). In what follows we shall need the following result of W. Mydlarczyk given in [6].

THEOREM 1. *Let $\alpha > 0$ and let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a nondecreasing function such that $g(0) = 0$, $g(t) > 0$ for $t > 0$. Then the equation*

$$u(t) = \int_0^t (t-s)^{\alpha-1} g(u(s))\, ds \quad (t \geqslant 0)$$

has a nontrivial continuous solution if and only if

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{1/\alpha} ds < \infty \quad (\delta > 0).$$

Let $c = \max_{t,s\in D} |H(t,s)|$. Choose a positive number d such that $d \leqslant a$ and $M \cdot c \cdot \dfrac{d^{1-r}}{1-r} < b$. Denote by $L = M \cdot c \cdot \dfrac{d^{1-r}}{1-r}$. Hence $L < b$.

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Let $J=[0,d]$. Denote by $C_w(J,E)$ the space of weakly continuous functions $J\mapsto E$ endowed with the topology of weak uniform convergence. Moreover, denote by β the measure of weak noncompactness introduced by De Blasi [2].

Let us recall that for any nonvoid, bounded subset A of a Banach space E, $\beta(A)=\inf\{\varepsilon>0:$ there exists a weakly compact set K such that $A\subset K+\varepsilon B\}$, where B is the norm unit ball. Recall that β has the following properties:

1° $A\subset B \;\Rightarrow\; \beta(A)\leqslant\beta(B)$;
2° $\beta(\bar{A}^w)=\beta(A)$, where $\bar{A}^w$ denotes the weak closure of A;
3° $\beta(A)=0 \;\Leftrightarrow\; \bar{A}^w$ is weakly compact;
4° $\beta(A\cup B)=\max(\beta(A),\beta(B))$;
5° $\beta(\operatorname{conv}A)=\beta(A)$;
6° $\beta(A+B)\leqslant\beta(A)+\beta(B)$;
7° $\beta(\lambda A)=|\lambda|\beta(A)$, $(\lambda\in\mathbb{R})$;
8° $\beta\big(\bigcup_{|\lambda|\leqslant h}\lambda A\big)=h\beta(A)$.

Let V be a subset of $C_w(J,E)$. Put $V(t)=\{u(t):u\in V\}$ and $V(T)=\{u(t): u\in V,\ t\in T\}$. Let us recall the well known Ambrosetti type

Lemma 1. *If the set V is strongly equicontinuous and uniformly bounded, then*
(a) *the function $t\mapsto\beta(V(t))$ is continuous on J;*
(b) *for each compact subset T of J one has $\beta(V(T))=\sup\{\beta(V(t)):t\in T\}$.*

Let $\tilde{B}$ denote the set of all weakly continuous functions $J\mapsto B$. We shall consider $\tilde{B}$ as a topological subspace of $C_w(J,E)$. Put

$$F(x)(t)=\int_0^t K(t,s)f(s,x(s))\,ds,\quad (x\in\tilde{B},\ t\in J).$$

Arguing similarly as in [4, p. 132–133] we can prove that the set $F(\tilde{B})$ is strongly equicontinuous. On the other hand, from the following Krasnoselskiĭ type

Lemma 2. *For any $\varphi\in E^*$, $\varepsilon>0$ and $z\in\tilde{B}$ there exists a weak neighbourhood U of 0 in E such that $|\,\varphi(f(t,z(t))-f(t,w(t)))\,|\leqslant\varepsilon$ for $t\in J$ and $w\in\tilde{B}$ such that $w(s)-z(s)\in U$ for all $s\in J$.* [**8**]

It follows that F is a continuous mapping from $\tilde{B}$ into $C_w(J,E)$.

For given $\varepsilon>0$ denote by S_ε the set of all $z\in\tilde{B}$ such that $\|z(t)-F(z)(t)\|<\varepsilon$ for all $t\in J$.

Lemma 3. *For each ε, $0<\varepsilon<b-L$, the set S_ε is nonempty and connected in $C_w(J,E)$.*

Proof. For any positive integer n we define $F_n(x)(t)=F(x)(r_n(t))$ $(x\in\tilde{B}$, $t\in J)$, where

$$r_n(t)=\begin{cases}0, & \text{if } 0\leqslant t\leqslant d/n\\ t-d/n, & \text{if } d/n\leqslant t\leqslant d.\end{cases}$$

Put

$$\sup_{t\in J,\ x\in\tilde{B}}\|F(x)(t)-F(x)(r_n(t))\|=w_n. \tag{2}$$

Because the set $F(\tilde{B})$ is equicontinuous, we have $w_n \mapsto 0$ as $n \mapsto \infty$. Moreover, there exists a unique $z_n \in \tilde{B}$ such that $z_n = F_n(z_n)$. It is clear from (2) that $z_n \in S_\varepsilon$ for sufficiently large n. Fix $u_0, u_1 \in S_\varepsilon$. Put

$$\eta = \max\Big(\sup_{t\in J}\|u_0(t) - F(u_0)(t)\|,\ \sup_{t\in J}\|u_1(t) - F(u_1)(t)\|\Big)$$

and $\delta = \varepsilon - \eta$. Fix a positive integer n such that $2w_n < \delta$. Let

$$a_\lambda = \lambda(u_1 - F_n(u_1)) + (1-\lambda)(u_0 - F_n(u_0)) \quad \text{for } 0 \leqslant \lambda \leqslant 1.$$

It follows from (2) that

$$\|u_i(t) - F_n(u_i)(t)\| \leqslant \|u_i(t) - F(u_i)(t)\| + \|F(u_i)(t) - F_n(u_i)(t)\| \leqslant \eta + w_n \qquad (i = 0, 1).$$

Hence

$$\|a_\lambda(t)\| \leqslant \eta + w_n \quad \text{for} \quad t \in J \quad \text{and} \quad 0 \leqslant \lambda \leqslant 1. \tag{3}$$

Arguing similarly as in [8, p. 122] we can prove that for each $\lambda \in [0,1]$ there exists a unique u_λ such that $u_\lambda = a_\lambda + F_n(u_\lambda)$ and u_λ depends continuosly on λ. Since

$$\int_0^t \frac{ds}{(t-s)^r} = \frac{t^{1-r}}{1-r},$$

we have

$$\|F(x)(t)\| = \left\|\int_0^t K(t,s) f(s, x(s))\, ds\right\| \leqslant c \cdot M \cdot \frac{d^{1-r}}{1-r} = L \quad \text{for} \quad x \in \tilde{B},\ t \in J.$$

From this and inequalities (2)–(3) we obtain

$$\|u_\lambda(t)\| \leqslant \|a_\lambda(t)\| + \|F(u_\lambda)(r_n(t))\| \leqslant \eta + w_n + L < \eta + \delta + L = \varepsilon + L < b$$

and

$$\begin{aligned} \|u_\lambda(t) - F(u_\lambda)(t)\| &= \|a_\lambda(t) + F_n(u_\lambda)(t) - F(u_\lambda)(t)\| \\ &\leqslant \|a_\lambda(t)\| + \|F_n(u_\lambda)(t) - F(u_\lambda)(t)\| \\ &\leqslant \eta + 2w_n < \eta + \delta = \varepsilon \quad (t \in J,\ 0 \leqslant \lambda \leqslant 1), \end{aligned}$$

so that $u_\lambda \in S_\varepsilon$. From this we conclude that for any $u_0, u_1 \in S_\varepsilon$ there exists a continuous curve in S_ε connecting u_0 and u_1, which proves that S_ε is arcwise connected. □

The main result of the paper is the following

THEOREM 2. *Let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $g(0) = 0$, $g(t) > 0$ for $t > 0$ and*

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{1/(1-r)} ds = \infty \quad (\delta > 0). \tag{4}$$

If 1° and 2° hold and

$$\beta(f(J \times X)) \leqslant g(\beta(X)) \quad \textit{for} \quad X \subset B, \tag{5}$$

then the set S of all weak solutions of (1) *defined on J is nonempty, compact and connected in $C_w(J,E)$.*

PROOF. 1. First we shall show that the set S is nonempty. By Lemma 3 there exists a sequence (u_n) such that $u_n \in \tilde{B}$ and

$$\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - F(u_n)(t)\| = 0. \tag{6}$$

Let $V = \{u_n : n \in N\}$. Since

$$\begin{gathered} V \subset \{u_n - F(u_n) : n \in N\} + F(V) \\ V(t) \subset \{u_n(t) - F(u_n)(t) : n \in N\} + F(V)(t), \\ F(V)(t) \subset V(t) - \{u_n(t) - F(u_n)(t) : n \in N\}, \end{gathered}$$

it follows from (6) that the set V is strongly equicontinuous and

$$\beta(V(t)) = \beta(F(V)(t)) \quad \text{for} \quad t \in J. \tag{7}$$

Hence, by Lemma 1, the function $t \mapsto v(t) = \beta(V(t))$ is continuous on J.

Fix $t \in J$ and $\varepsilon > 0$, and choose $\eta > 0$ such that

$$\left\| \int_{t-\eta}^{t} \frac{H(t,s)}{(t-s)^r} f(s,x(s))\, ds \right\| \leqslant \int_{t-\eta}^{t} \frac{|H(t,s)|}{(t-s)^r} M\, ds < \varepsilon \quad \text{for all } x \in \tilde{B}.$$

From the continuity of the function $\dfrac{H(t,s)}{(t-s)^r}\, g(v(s))$ on $[0, t-\eta]$ it follows that there exists $\delta > 0$ such that

$$\left| \frac{H(t,\tau)}{(t-\tau)^r}\, g(v(q)) - \frac{H(t,s)}{(t-s)^r}\, g(v(s)) \right| < \varepsilon \tag{8}$$

if $|\tau - s| < \delta$, $|q - s| < \delta$, $q, s, \tau \in [0, t-\eta]$. Divide the interval $[0, t-\eta]$ into n parts $0 = t_0 < t_1 < \ldots < t_n = t - \eta$ so that $\Delta t_i = t_i - t_{i-1} < \delta$ $(i = 1, \ldots, n)$. Put $T_i = [t_{i-1}, t_i]$. By Lemma 1 for each i there exists $s_i \in T_i$ such that $\beta(V(T_i)) = v(s_i)$ $(i = 1, \ldots, n)$. Put

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$$\int_T \frac{H(t,s)}{(t-s)^r} f(s, V(s))\, ds = \left\{ \int_T \frac{H(t,s)}{(t-s)^r} f(s,x(s))\, ds : x \in V \right\}.$$

Because

$$\begin{aligned} \int_0^{t-\eta} \frac{H(t,s)}{(t-s)^r} f(s,V(s))\, ds &\subset \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{H(t,s)}{(t-s)^r} f(s,V(s))\, ds \\ &\subset \sum_{i=1}^{n} \Delta t_i \overline{\mathrm{conv}} \left\{ \frac{H(t,s)}{(t-s)^r} f(s,x(s)) : s \in T_i,\ x \in V \right\}, \end{aligned}$$

we get

$$\begin{aligned}
&\beta\left(\int_0^{t-\eta} \frac{H(t,s)}{(t-s)^r} f(s,V(s))\,ds\right)\\
&\quad\leqslant \sum_{i=1}^{n} \Delta t_i \beta\left(\overline{\mathrm{conv}}\left\{\frac{H(t,s)}{(t-s)^r} f(s,x(s)) : s\in T_i, x\in V\right\}\right)\\
&\quad= \sum_{i=1}^{n} \Delta t_i \beta\left(\left\{\frac{H(t,s)}{(t-s)^r} f(s,x(s)) : s\in T_i, x\in V\right\}\right)\\
&\quad= \sum_{i=1}^{n} \Delta t_i \max_{s\in T_i} \frac{|H(t,s)|}{(t-s)^r}\, \beta(f(J\times V(T_i))\\
&\quad\leqslant \sum_{i=1}^{n} \Delta t_i \frac{|H(t,\tau_i)|}{(t-\tau_i)^r}\, g(\beta(V(T_i))) \leqslant \sum_{i=1}^{n} \Delta t_i \frac{|H(t,\tau_i)|}{(t-\tau_i)^r}\, g(v(s_i)).
\end{aligned}$$

Here $\tau_i \in T_i$ is a number such that $\dfrac{|H(t,\tau_i)|}{(t-\tau_i)^r} = \max\limits_{s\in T_i} \dfrac{|H(t,s)|}{(t-s)^r}$. Furthermore, from inequality (8) we infer that

$$\sum_{i=1}^{n} \Delta t_i \frac{|H(t,\tau_i)|}{(t-\tau_i)^r}\, g(v(s_i)) \leqslant \int_0^{t-\eta} \frac{|H(t,s)|}{(t-s)^r}\, g(v(s))\,ds + \varepsilon(t-\eta). \tag{9}$$

Since

$$F(V)(t) \subset \int_0^{t-\eta} \frac{H(t,s)}{(t-s)^r} f(s,V(s))\,ds + \int_{t-\eta}^{t} \frac{H(t,s)}{(t-s)^r} f(s,V(s))\,ds$$

and

$$\beta\left(\left\{\int_{t-\eta}^{t} \frac{H(t,s)}{(t-s)^r} f(s,x(s))\,ds : x\in V\right\}\right) \leqslant 2\varepsilon,$$

from inequalities (8) and (9) it follows that

$$\begin{aligned}
\beta(F(V)(t)) &\leqslant \int_0^{t-\eta} \frac{|H(t,s)|}{(t-s)^r}\, g(v(s))\,ds + \varepsilon(t-\eta) + 2\varepsilon\\
&\leqslant \int_0^{t} \frac{|H(t,s)|}{(t-s)^r}\, g(v(s))ds + \varepsilon t + 2\varepsilon.
\end{aligned}$$

As the last inequality is satisfied for every $\varepsilon > 0$, we get

$$\beta(F(V)(t)) \leqslant \int_0^{t} \frac{|H(t,s)|}{(t-s)^r}\, g(v(s))\,ds.$$

Therefore, by (7),

$$\beta(V(t)) \leqslant \int_0^{t} \frac{|H(t,s)|}{(t-s)^r}\, g(v(s))\,ds$$

i.e.,

$$v(t) \leqslant \int_0^{t} \frac{|H(t,s)|}{(t-s)^r} g(v(s))\,ds \quad \text{for} \quad t\in J.$$

Applying Theorem 1 with $\alpha = 1 - r$ and theorem on integral inequalities [3, Lemma 1] from this we deduce that $v(t) = 0$ for $t \in J$. Therefore, by Lemma 1

$$\beta(V(J)) = \sup\{\beta(V(t)) : t \in J\} = 0$$

i.e., V is relatively compact in $C_w(J,E)$. Hence we can find a subsequence (u_{n_k}) of (u_n) which converges in $C_w(J,E)$ to a limit u. As F is continuous, from this and (6) we conclude that $u = F(u)$. This proves that the set S is nonempty.

2. Further, since F is continuous, S is closed in $C_w(J,E)$. As $S = F(S)$, we have $\beta(S(t)) = \beta(F(S)(t))$ for $t \in J$. Arguing similarly as in 1, we can show that S is compact in $C_w(J,E)$.

Now we shall prove that S is connected. Suppose that S is not connected in $C_w(J,E)$. As S is compact, there are nonempty compact sets S_1, S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, and there are two disjoint open sets U_1, U_2 such that $S_1 \subset U_1$, $S_2 \subset U_2$. Let $U = U_1 \cup U_2$. We choose n_0 such that $1/n_0 < b - L$. Suppose that for each $n \geqslant n_0$ there exists $u_n \in S_{1/n} \setminus U$. Put $V = \{u_n : n \in N\}$. Because $\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - F(u_n)(t)\| = 0$, by repeating the argument from 1 we can prove that there exists $u_0 \in \overline{V}$ such that $u_0 = F(u_0)$, i.e., $u_0 \in S$. Furthermore, $\overline{V} \subset C_w(J,E) \setminus U$, as U is open, so that $u_0 \in S \setminus U$, a contradiction. Therefore there exists $k \in N$ such that $S_{1/k} \subset U$. Since $U_1 \cap S_{1/k} \neq \emptyset \neq U_2 \cap S_{1/k}$, this shows that $S_{1/k}$ is not connected, which contradicts Lemma 3. Hence S is connected. $\square$

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