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CLASSIFYING SPACES OF MONOIDS – APPLICATIONS IN GEOMETRIC COMBINATORICS

Maja D. Rabrenović

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Abstract. We give two examples of application of the theory of simplicial sets in geometric combinatorics. The emphasis is on monoids and their classifying spaces.

1. Introduction

The theory of simplicial sets (spaces) has been developed in algebraic topology and, today, is a standard tool in this field. Surprisingly enough, some aspects of this theory have recently found applications in geometric combinatorics [W-Z-Ž], [Z-Ž]. It is believed that this is just the beginning and that more and more applications will follow.

We give two examples of how this theory can be applied in problems of relevance for combinatorics. In Proposition 9 we present a new proof of contractibility of Higher-dimensional Dunce Hats D_{2n} [A-M-S], and in Proposition 13 we determine the classifying space $B(K \cap \mathbb{Z}^n)$, where $K \subset \mathbb{R}^n$ is a polyhedral, convex cone. The emphasis is on monoids and their classifying spaces. The exposition is directed towards nonspecialists. Our hope is to demonstrate that, besides order complexes of posets and classifying spaces of groups, classifying spaces of some other small categories are as useful and applicable.

2. Fundamental concepts, constructions and results

Simplicial sets, geometric realization. A simplicial set X is a contravariant functor from the category Δ of all positive finite ordinals and monotone (non-decreasing) maps to Set. The objects of the category Δ are ordered sets of the form $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$. The set $X([n])$ will be denoted by X_n. Its elements are called n-dimensional simplices (or, just, n-simplices).

The geometric realization of a simplicial set X is the topological space $|X|$ with the underlaying set $\coprod_{n=0}^{\infty}(\Delta_n \times X_n)/R$, where Δ_n is the n-dimensional geometric

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simplex, R is the weakest equivalence relation identifying points $(s,x) \in \Delta_n \times X_n$ and $(t,y) \in \Delta_m \times X_m$ such that $y = X(f)x$, $s = \Delta_f(t)$ for some monotone map $f : [m] \to [n]$ and induced lineal map $\Delta_f : \Delta_m \to \Delta_n$ which takes values $\Delta_f(e_i) = e_{f(i)}$ on vertices of Δ_m.

The geometric realization $|\cdot|$ is a functor from the category Δ°Set of simplicial sets and simplicial maps to the category Top of topological spaces. Note that the values of this functor are CW complexes. As a consequence, if the simplicial sets X and Y are sufficiently regular (e.g., if one of them is locally finite or both are at most countable), then the geometric realization passes through direct product, i.e., $|X \times Y| = |X| \times |Y|$.

Also, if $sk_n X$ is the n-th skeleton of X, that is the simplicial subset of X generated by its all nondegenerate simplices of dimension less or equal n, then $|X| = |\varinjlim sk_n X| = \varinjlim |sk_n X| = \varinjlim sk_n |X|$. Here, on the right side is the n-th skeleton of the CW complex $|X|$.

Even these two properties indicate (correctly) that in the category Δ°Set one can develop a homotopy theory similar to the one on CW complexes possessing all the usual properties.

Nerve of a Category. For each small category C there exists a special simplicial set called the *nerve* of C. This is a construction of fundamental importance in algebraic topology, e.g., it is essential for the Quillen's approach to algebraic K-theory [Q].

Definition 1. The *nerve* of a small category C is the simplicial set $N(C)$, where n-simplices are the composable strings $a_0 \xrightarrow{\varphi_0} a_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} a_n =: (a_i, \varphi_i)$ of arrows in C.

The map from $N(C)_n$ to $N(C)_m$, corresponding to the monotone map, $f : [m] \to [n]$ transforms the string (a_i, φ_i) into the string $b_0 \xrightarrow{\psi_0} b_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{m-1}} b_m$ such that $b_i = a_{f(i)}$, and $\psi_i = 1$ if $f(i) = f(i+1)$, while $\psi_i = \varphi_{f(i+1)-1} \circ \cdots \circ \varphi_{f(i)}$ if $f(i) < f(i+1)$.

Definition 2. Let $F : C \to D$ be a functor of small categories C and D. The *nerve* of F, $N(F)$, is the simplicial map (i.e., the morphism of simplicial sets) which maps a simplex (a_i, φ_i) into the simplex $(F(a_i), F(\varphi_i))$.

Properties of the nerve. It is easy to see that N is a functor from the category SCat of small categories to the category Δ°Set. Here are some of its properties:

a) A category C and its nerve are related as follows: If $X = N(C)$, then $X_0 = \operatorname{Ob} C$ and $X_1 = \operatorname{Mor} C$ (up to an isomorphism).

More precisely, if $\partial_n^i : [n-1] \to [n]$ is the only strictly increasing mapping whose image does not contain $i \in [n]$, a 1-simplex $x \in X_1$ represents a morphism from $X(\partial_1^1)(x)$ to $X(\partial_1^0)(x)$. Also, if x_1 and x_2 are two 1-simplices in X with common vertex $X(\partial_1^0)(x_1) = X(\partial_1^1)(x_2)$, then, in X, there exists a unique 2-simplex $x_2 \circ x_1$ such that $X(\partial_2^2)(x_2 \circ x_1) = x_1$ and $X(\partial_2^0)(x_2 \circ x_1) = x_2$ (This is a specific feature of the nerve of a category; not every simplicial set need to have this property.). The

composition of morphisms represented by x_1 and x_2 is represented by the simplex $X(\partial_2^1)(x_2 \circ x_1)$. The identity arrow $1_x : x \to x$ for $x \in X_0$ is represented by the element $X(s_0^0)(x)$, where s_0^0 is the only mapping from [1] to [0] in Δ.
b) $N(C \times C') = N(C) \times N(C')$ [G-M, II.4.21].
c) The functor N is fully faithful: the mapping $F \to N(F)$ induces a bijection $\mathrm{Hom}_{SCat}(C, C') \xrightarrow{\sim} \mathrm{Hom}_{\Delta^\circ \mathrm{Set}}(N(C), N(C'))$.

Monoid as category. If C is a monoid, then C is also a category with one single object, say $*$, the morphisms being the elements of C, where the unit 1 of C is identity on $*$, and the composition is the multiplication in C. Let $N(C)$ be the nerve of the category C, and $B(C) = |N(C)|$ the associated classifying space, i.e., its geometric realization.

EXAMPLE 3. It is well known that, for a group G, $B(G) \simeq K(G, 1)$ [H, §2.3b]. Here $K(G, 1)$ stands for any CW complex X satisfying $\pi_1(X) = G$ and $\pi_k(X) = 0$ for $k \neq 1$, i.e., for so called an Eilenberg–MacLane complex of type $(G, 1)$ [H, §4.1d, §1.4b] also [Sw, §6.43] So, specially, $B(\mathbb{Z}) \simeq K(\mathbb{Z}, 1) \simeq S^1$ [H, §1.4b].

We say that C is *contractible*, (*connected*,..., etc), if the associated classifying space $B(C)$ is contractible, (connected, ..., etc). The same definitions are used for any small category C, e.g. if $C = C_P$ is the small category of a poset P. Then the classifying space $B(C_P)$ is the order complex $\Delta(P)$ of P.

Nerve of a category and homotopy. [G-M, II.4, Exercise 1]
a) Let C and C' be two categories, $F_0, F_1 : C \to C'$ two functors, and $\tau : F_0 \to F_1$ a functor-morphism (a natural transformation) between them. Then $NF_0, NF_1 : NC \to NC'$ are homotopic as mappings of simplicial sets (τ is used to construct a homotopy explicitly).
b) Adjoint functors $F : C \to C'$, $G : C' \to C$ lead to homotopically inverse mappings $NF : NC \to NC'$, $NG : NC' \to NC$.
c) If C has either an initial or a final object, then NC is contractible (i.e., the identity mapping is homotopic to a point).

A Comma Category. A prominent place in category theory and algebraic topology is reserved for the comma categories. In full generality they were introduced in the sixties by Lawvere. Here we shall need only a special case.

DEFINITION 4. Let $F : C \to D$ be a functor and b an object in D. The category $(b \downarrow F)$ of *objects F-under b* is the category whose objects all pairs (f, c), where c is an object of C and $f : b \to F(c)$ is an arrow in D. Its arrows $h : (f, c) \to (f', c')$ are commutative triangles defined by arrows $h : c \to c'$ in C for which $f' = F(h) \circ f$. The identities and the composition is given by the identities and the composition of arrows h in C.

Finally, here is the famous Quillen's Theorem A which provides sufficient conditions for a functor to be a homotopy equivalence.

Its consequence is even more famous Theorem B, which has been consistently used by Quillen in his approach to K-theory.

THEOREM 5. [Q] *Let $F : C \to D$ be a functor. If the comma category $(b \downarrow F)$ is contractible for each object b of D, then the functor F is a homotopy equivalence.*

General references for the theory of simplicial sets are [G-M], [M], [G-Z] and [Q]. References [W-Z-Ž] and [Z-Ž] are expositions of related theory of diagrams of spaces with the emphasis on the technique which have proven to be useful in geometric combinatorics.

3. Applications

3.1. Higher-dimensional Dunce Hats. We start with the definition of a *black hole* in a monoid C.

DEFINITION 6. Let C be a monoid, i.e., a semigroup with the unit element. We say that an element $a \in C$ is a *black hole* in C, or a bh in C, if

$$az = a \quad \text{for each } z \text{ in } C. \tag{1}$$

EXAMPLE 7. Let $C_k = \{1, x, \dots, x^k\}$ be the monoid generated by a single element x subject to the relation $x^{k+1} = x^k$, or $C_k = \langle x \mid x^{k+1} = x^k \rangle$. Then $a := x^k$ is a bh in this monoid.

PROPOSITION 8. *A monoid C with a bh is contractible.*

PROOF. If C is a monoid with a bh a, then

$$az = a = 1a \quad \text{for all } z \in C. \tag{2}$$

Thus, the bh a defines a natural transformation between the identity functor $I_C : C \to C$ on C and the constant (in $*$) functor on C. Therefore, C is contractible. $\square$

Schori introduced in [S] a sequence D_n of interesting configuration spaces. It was proved in [A-M-S], that D_2 is the well-known Dunce Hat, and that each D_{2n}, for $n \geqslant 1$, is a contractible, not collapsible polyhedron. Consequently, D_{2n}, for $n \geqslant 2$, was refereed to as a Higher-dimensional Dunce Hat.

Our basic observation, based on the original definition of D_n given in [A-M-S] ($D_n = \mathbb{I}_0^1(n+2) = \{A \in \mathbb{I}(n+2) \mid 0 \in A \text{ and } 1 \in A\}$, where $\mathbb{I}(n+2)$ is the $(n+2)$-fold symmetric product of the closed real unit interval $\mathbb{I} = [0,1]$.), is that the space D_n is homeomorphic to the geometric realization of the n-skeleton of the nerve of a small category C which is a monoid generated by a single non-unit, idempotent element. Using this, we present here an alternative proof of the contractibility of D_{2n}.

PROPOSITION 9. *Let $D_n := |sk_n(N(C))|$ for $n = 0, 1, \dots,$ where C is the monoid $C = \langle x \mid x^2 = x \rangle$. Then D_{2n} is conctractible for each n.*

PROOF. For short, put $X = N(C)$ and X^n is its n-skeleton $sk_n(X)$. Since x is a bh in C, $|X|$ is contractible. Therefore, all $\pi_n(|X|)$ and $\widetilde{H}_n(|X|)$ are trivial. From the definition of X we see that, for each n, X_n has only one nondegenerate simplex x_n, described as the string of n arrows $* \xrightarrow{x} * \xrightarrow{x} * \cdots * \xrightarrow{x} *$. In other words, $x_n = (x, x, \dots, x)$ (n-factors). By definition, for the "i-th face – the strictly increasing function $\partial_n^i : [n-1] \to [n]$ not taking the value $i \in [n]$ we have:

$$X(\partial_n^i)(x_n) = \begin{cases} (\hat{x}, x, \dots, x), & \text{for } i = 1 \\ (x, \dots, x, \hat{x}), & \text{for } i = n \\ (x, \dots, xx, \dots, x), & \text{for } i = 2, \dots, n-1 \end{cases} = x_{n-1}.$$

Therefore, for explicit calculation of homology groups of X we will use the chain complex (C_n, d_n) such that each C_n is free Abelian group generated by $X_{(n)} = \{x_n\}$ [G-M,I.7, Exercise 1.g].

The boundary operator $d_n : C_n \to C_{n-1}$, evaluated on the generator yields

$$d_n(x_n) = \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x_n),$$

so that

$$\mathrm{Im}(d_n) = \begin{cases} 0, & \text{for } n \text{ odd} \\ C_{n-1}, & \text{for } n \text{ even} \end{cases} \quad \text{and} \quad \mathrm{Ker}(d_n) = \begin{cases} C_n, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

Thus, for n even, $\widetilde{H}_k(X^n) = 0$ and so, also, $\widetilde{H}_k(|X^n|) = 0$ for each k. As the n-th homotopy group of a CW complex depends only on its $n+1$ skeleton [Sw,6.11, a consequence of the Simplicial Approximation Lemma 6.8], $\pi_k(|X^n|) = 0$ for each $n \geqslant 2$ and $k < n$, i.e., each $|X^n|$, for $n \geqslant 2$, is a 1-connected CW complex.

So, for $n \geqslant 1$, the inclusion $i_{2n} : X^{2n} \to X$ of simplicial sets induces the map $|i_{2n}| : |X^{2n}| \to |X|$ of 1-connected CW complexes – their geometric realizations, such that $|i_{2n}|_{*_k} : \widetilde{H}_k(|X^{2n}|) \to \widetilde{H}_k(|X|)$ is an isomorphism for all $k \geqslant 0$ (as all homology groups are 0). Therefore, using the Hurewicz Isomorphism Theorem [Sw, 10.25 or the Whitehead's Theorem 10.28], $|i_{2n}| : |X^{2n}| \to |X|$ is a homotopy equivalence, so $|X^{2n}|$ is contractible for each n. □

3.2. Homotopy type of conical submonoids of $\mathbb{Z}^n$. In the following proposition we show that the additive monoids $\mathbb{N}$ and $\mathbb{Z}$ of natural numbers and integers, viewed as small categories, have the same homotopy types. Much more general results of this nature, the so called "group completion theorems" are known in algebraic topology.

PROPOSITION 10. *Let $i : \mathbb{N} \hookrightarrow \mathbb{Z}$ be the inclusion of the additive monoids of natural numbers and integers. Then i is a homotopy equivalence* (*in the sense that $B(i) : B(\mathbb{N}) \hookrightarrow B(\mathbb{Z})$ is a homotopy equivalence*).

PROOF. In this, special, case the comma category $(* \downarrow i)$ of objects i-under $*$ consists of an object $(*, m)$ for each morphism m in $\mathbb{Z}$, and of a morphism $n : (*, m) \to (*, k)$ for each m and k in $\mathbb{Z}$ and n in $\mathbb{N}$ such that $n \circ m = k$ (in the category $\mathbb{Z}$), i.e., $n + m = k$ (in $(\mathbb{Z}, +)$). So, for each m and k in $\mathbb{Z}$, there is a morphism from $(*, m)$ to $(*, k)$ if and only if $m \leqslant k$. Also, it is unique: namely, it is $k - m$. Therefore, the comma category $(* \downarrow i)$ is isomorphic to the category of the directed set, (filtering), $(\mathbb{Z}, \leqslant)$, and, so, it is contractible [Q, Corollary 2]. It follows by Quillen's Theorem A that $i : \mathbb{N} \hookrightarrow \mathbb{Z}$ is a homotopy equivalence. □

Since $B(\mathbb{Z}) = K(\mathbb{Z}, 1) \simeq S^1$, we have the following corollary:

Corollary 11. *$B(\mathbb{N}) \simeq S^1$.*

The same argument used in the proof of Proposition 10 can be applied in the case in which we have $\mathbb{N}^2$ and $\mathbb{Z}^2$ instead of $\mathbb{N}$ and $\mathbb{Z}$, the addition on $\mathbb{Z}^2$ is the pointwise addition, and the order relation on $\mathbb{Z}^2$ is the double product of the order relation on $\mathbb{Z}$.

In that way we get: $B(\mathbb{N}^2) \cong B(\mathbb{Z}^2) \cong B(\mathbb{Z}) \times B(\mathbb{Z}) \cong S^1 \times S^1 = T^2$.

Analogously, $B(\mathbb{N}^n) \cong T^n$.

These results are special cases of a result about conical submonoids of $\mathbb{Z}^n$.

Let us note that $\mathbb{Z}^n$ and $\mathbb{N}^n$ are subsets of the real vector space $\mathbb{R}^n$. and that $\mathbb{N}^n = K \cap \mathbb{Z}^n$, where K is a polyhedral, convex cone of all $x \in V$ such that $x \geqslant 0$. This result can be extended to the case of a cone K which has 0 as the apex, meaning that K does not contain any nontrivial linear subspace in $\mathbb{R}^n$. Moreover, we assume that K has a nonempty interior.

The following lemma can be easily proved.

Lemma 12. *Let $\mathbb{R}^n$ be the n-dimensional real vector space. Let $K \subseteq \mathbb{R}^n$ be a polyhedral, convex cone, with apex 0 and with a nonempty interior. Define on $\mathbb{R}^n$ an order relation $\leqslant_K$ by:*

$$\textit{for} \quad x, y \in \mathbb{R}^n, \quad x \leqslant_K y \quad \textit{iff} \quad y - x \in K.$$

Then:

a) *$\leqslant_K$ is an order relation which agrees with $+$ and $\cdot$ on $\mathbb{R}^n$.*

b) *Both $(\mathbb{R}^n, \leqslant_K)$ and $(\mathbb{Z}^n, \leqslant_K)$ are filterings.*

c) *$K \cap \mathbb{Z}^n$ is a submonoid of the monoid $\mathbb{Z}^n$ with respect to the addition inherited from $\mathbb{R}^n$, and $\mathbb{Z}^n$ is a group.*

Using the parts b) and c) of the lemma, and the arguments used in Proposition 10, we obtain the following proposition:

Proposition 13. *If K is a polyhedral, convex cone with the apex 0 and a nonempty interior in the real vector space $\mathbb{R}^n$, then $B(K \cap \mathbb{Z}^n) \cong T^n$.*

Proof. The proof follows the pattern as the proof of Proposition 10. □

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Matematički fakultet
11000 Beograd, p.p. 550
Serbia and Montenegro
maja@matf.bg.ac.yu

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