

THREE-SPACE-PROBLEM FOR INDUCTIVELY (SEMI)-REFLEXIVE LOCALLY CONVEX SPACES

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ABSTRACT. Three-space-stability of inductively (semi)-reflexive and some related classes of locally convex spaces is considered. It is shown that inductively (semi)-reflexive spaces behave more regularly than (semi)-reflexive spaces in that sense.

Let (E, t) be a Hausdorff locally convex space (l.c.s.) with the topological dual E' ; there exist several topologies on E' (the weak topology $\sigma(E', E)$, the topology $\kappa(E', E)$ of uniform convergence on compact and absolutely convex sets, Mackey topology $\tau(E', E)$, the strong topology $b(E', E)$ and others). The so-called inductive topology TE' on E' was introduced in [3] and [5] as the inductive-limit topology of the Banach spaces E'_{V° , where V runs through a zero-neighborhood basis of (E, t) formed by closed and absolutely convex sets. Here $E'_{V^\circ} = \bigcup_{n \in \mathbb{N}} nV^\circ$ is equipped with the norm having V° as the unit ball. The zero-neighborhood basis of TE' is formed by all absolutely convex subsets of E' that absorb all t -equicontinuous subsets. This topology is the strongest locally convex topology on E' for which all t -equicontinuous subsets are bounded. Particularly, it is finer than the strong topology $b(E', E)$.

Obviously, (E', TE') is an ultrabornological l.c.s. While the weak, Mackey and strong topologies depend only on the dual pair $\langle E, E' \rangle$, topology TE' depends on the topology t . E.g., the topology corresponded in this way to the weak topology $\sigma(E, E')$ is the strongest locally convex topology $\tau(E', E'^*)$, i.e., $T(E, \sigma(E, E'))' = \tau(E', E'^*)$.

It was defined in [3], resp. [5], resp. [1] that an l.c.s. (E, t) is *inductively semi-reflexive* (resp. *b-reflexive*, resp. *with the property HC*) if the topology TE' is compatible with the duality $\langle E, E' \rangle$, i.e., if $TE' = \tau(E', E)$; in other words if $(E', TE')' = E$ (algebraically). If, moreover, $T(TE')' = t$, then (E, t) is called *inductively reflexive*.

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By the previous remark, $(E, \sigma(E, E'))$ is inductively semi-reflexive if and only if E is finite-dimensional.

W. Roelcke and S. Dierolf showed in [10, Ex. 1.5] that neither of the properties “being semi-reflexive” and “being reflexive” of l.c.s.’s is three-space-stable, i.e., there exists a non-semi-reflexive space E having a closed subspace F such that both F and E/F are reflexive. We shall prove here that inductively (semi)-reflexive spaces behave more regularly, i.e., that the properties “being inductively semi-reflexive” and “being inductively reflexive” are three-space-stable. This will also be a result better than the one obtained in [7, Prop. 3.2].

Terminology that is not defined here explicitly is taken from [9].

THEOREM 1. *If the outer terms F and E/F of the short exact sequence*

$$(1) \quad 0 \rightarrow F \xrightarrow{i} E \xrightarrow{q} E/F \rightarrow 0$$

of l.c.s.’s are inductively semi-reflexive, then the middle term E is inductively semi-reflexive, too.

In order to prove the theorem we state two lemmas which may be of interest on their own.

LEMMA 1. *If F is a closed subspace of an l.c.s. (E, t) , then the quotient topology TE'/F° of the topology TE' is equal to the topology TF' , i.e., $TE'/F^\circ = TF'$.*

PROOF. First we prove that $TF' \geq TE'/F^\circ$. TF' is the strongest locally convex topology on F' such that all $t|F$ -equicontinuous subsets of F' are bounded. So, it is enough to prove that all $t|F$ -equicontinuous subsets of F' are TE'/F° -bounded. Let $A \subset F'$ be a $t|F$ -equicontinuous subset, i.e., $A = i'(B)$, where $B \subset E'$ is t -equicontinuous. Then, A is TE'/F° -bounded since B is TE' -bounded. So, $TF' \geq TE'/F^\circ$.

Conversely, let us prove that $TF' \leq TE'/F^\circ$. Let W be a TF' -neighborhood of zero, so that W absorbs all $t|F$ -equicontinuous subsets of F' . Then $(i')^{-1}(W)$ absorbs all t -equicontinuous subsets of E' , and so $(i')^{-1}(W)$ is a TE' -neighborhood of zero. Hence, W is a TE'/F° -neighborhood of zero and so $TF' \leq TE'/F^\circ$ is proved. \square

Note that the strong topology $b(E', E)$ does not possess the mentioned property of topology TE' .

LEMMA 2. *If F is a closed subspace of an l.c.s. (E, t) , then $TE'|F^\circ \leq TF^\circ$ on $F^\circ \simeq (E/F)'$.*

PROOF. Let V be a $TE'|F^\circ$ -neighborhood of zero. Then there exists a TE' -neighborhood of zero U such that $V \supset U \cap F^\circ$. Since each $t|F$ -equicontinuous subset of F° is also a t -equicontinuous subset of E' (indeed, if $A \subset F^\circ$ is $t|F$ -equicontinuous, then $A \subset (U + F)^\circ \subset U^\circ \cap F^\circ \subset U^\circ$, for some t -neighborhood of zero U), we have that $U \cap F^\circ$ absorbs all $t|F$ -equicontinuous subsets of F° . This means that $V \cap F^\circ$ absorbs all $t|F$ -equicontinuous subsets of F° and so it is a TF° -neighborhood of zero. Thus, $TE'|F^\circ \leq TF^\circ$ is proved. \square

Note that there exist examples when $TE'|F^\circ < TF^\circ$. E.g., let (E, t) be an ultrabornological space and $(F, t|F)$ its subspace that is not ultrabornological (such examples exist). Then $TE = t$ and $TF > TE|F = t|F$, where TE and TF are the associated ultrabornological topologies on E, F , respectively.

PROOF OF THEOREM 1. The following relations among topologies in the space E'/F° are valid:

$$\tau(E'/F^\circ, F) = b(E'/F^\circ, F) = TF' = TE'/F^\circ \geq b(E', E)/F^\circ \geq b(E'/F^\circ, F).$$

The first and the second equality follow from the inductive semi-reflexivity of the subspace F ; the third follows from Lemma 1; the last two inequalities are obvious. In the subspace F° we have:

$$\tau(F^\circ, E/F) = b(F^\circ, E/F) = TF^\circ \geq TE'|F^\circ \geq \sigma(E', E)|F^\circ = \sigma(F^\circ, E/F)$$

by inductive semi-reflexivity of the quotient E/F and Lemma 2. Hence the following sequence

$$0 \rightarrow (F^\circ, TE'|F^\circ) \xrightarrow{q'} (E', TE') \xrightarrow{i'} (F', TF' = TE'/F^\circ) \rightarrow 0$$

is exact (both algebraically and topologically). Denote by E'_1 the topological dual of the space (E', TE') . Then the sequence

$$0 \rightarrow F \rightarrow E'_1 \rightarrow E/F \rightarrow 0$$

is algebraically exact. It remains to prove the inclusion $E'_1 \subset E$.

Let $x'' \in E'_1 = (E', TE)'$. The restriction $x''|F^\circ$ to the subspace F° is $TE'|F^\circ$ -continuous by Lemma 2, hence $x''|F^\circ \in E/F$ (the space E/F is inductively semi-reflexive). So, there exists $x_1 \in E$ such that

$$x''(x') = x'(x_1) \quad \text{for each } x' \in F^\circ.$$

Hence, $x'' - x_1$ is a continuous linear form on the space (E', TE') which vanishes on F° , and so $x'' - x_1 \in U^\circ$ for a TE' -neighborhood of zero U . Further, this means that $x'' - x_1$ is a bounded linear form on $U + F^\circ$ (and so, by Lemma 1, on a TF' -neighborhood of zero in the space (F', TF')). So, there exists $x_2 \in F$ such that $(x'' - x_1)(x') = x_2(x')$ for each $x' \in F'$, i.e., $x'' = x_1 + x_2 \in E + F \subset E + E = E$, which finishes the proof. \square

Following [3], we shall call an l.c.s. (E, t) *strongly distinguished* if each $\sigma(\cdot, E')$ -bounded subset A of $(E', TE)'$ is contained in the $\sigma(\cdot, E')$ -closure of a t -bounded subset B of E (here, $\sigma(\cdot, E')$ stands for the weak topology in $(E', TE)'$). Using the associated Schwartz topology, it was proved in [3, Prop. 3.2] that the space (E, t) is strongly distinguished if and only if $b(E', E) = TE'$. We give a direct proof.

PROPOSITION 1. *An l.c.s. (E, t) is strongly distinguished if and only if $b(E', E) = TE'$.*

PROOF. Since the dual space E' with the topology TE' is ultrabornological, and so barrelled, the equality $b(E', E) = TE'$ implies that the space (E, t) is distinguished in the classical (Grothendieck) sense. Hence, the bidual E'' of the space E is equal to the topological dual $(E', TE)'$ of the space (E', TE) and so for each

$\sigma(E'', E')$ -bounded subset A of E'' there exists a t -bounded subset B of E such that A is contained in the $\sigma(E'', E')$ -closure of B . By the definition, it means that (E, t) is strongly distinguished.

Conversely, let (E, t) be a strongly distinguished space and let V be a closed and absolutely convex TE' -neighborhood of zero. Then the polar V° (corresponding to the duality $\langle E', (E', TE')' = E_1'' \rangle$) is a $\sigma(E_1'', E')$ -bounded, closed and absolutely convex subsets of E_1'' . By the assumption, there exists a t -bounded subset B of E such that A is contained in the weak closure $B^{\circ\circ}$ of B . It follows that $V = V^{\circ\circ} \supset B^\circ$. Hence, V is a neighborhood of zero in the space (E', TE') , and so $b(E', E) = TE'$. \square

In the sequel we prove propositions on the three-space-stability of strongly distinguished and inductively reflexive spaces. First we state a dual property of inductively reflexive spaces.

PROPOSITION 2. *Let (E, t) be an l.c.s. and consider the following properties:*

- (a) *(E, t) is inductively reflexive (i.e., inductively semi-reflexive and ultrabornological);*
- (b) *$(E, \tau(E, E'))$ is inductively reflexive;*
- (c) *$(E', \tau(E', E))$ is inductively reflexive.*

Then, (a) implies (b) and (b) is equivalent to (c).

PROOF. Proof can be deduced from the following observations. If an l.c.s. (E, t) is inductively semi-reflexive (with $\sigma(E, E') \leq t \leq \tau(E, E')$), then $(E', \tau(E', E))$ is an ultrabornological space; conversely, if the space $(E', \tau(E', E))$ is ultrabornological, then $(E, \tau(E, E'))$ is inductively semi-reflexive. Dually, if (E', t') is inductively semi-reflexive (with $\sigma(E', E) \leq t' \leq \tau(E', E)$), then $(E, \tau(E, E'))$ is ultrabornological; conversely, if $(E, \tau(E, E'))$ is ultrabornological, then $(E', \tau(E', E))$ is inductively semi-reflexive. \square

THEOREM 2. *If the quotient map $q : E \rightarrow E/F$ lifts bounded sets with closure and if the closed subspace F and the corresponding quotient E/F are strongly distinguished, then the space E has the same property.*

PROOF. Recall that the mapping q is said to lift bounded sets with closure if for each bounded set $B \subset E/F$ there exists a bounded set $A \subset E$ such that $B \subset \overline{q(A)}$. We shall prove that under this assumption the topologies $b(E', E)$ and TE' coincide both on the subspace F° and on the quotient E/F ; according to [6, Lemma 1] it will follow that they coincide on E' , i.e., that the space E is strongly distinguished.

On the space F° we have that:

$$b(F^\circ, E/F) = b(E', E)|_{F^\circ} \leq TE'|_{F^\circ} \leq TF^\circ = b(F^\circ, E/F).$$

The first equality follows from the assumption about lifting of bounded sets, and last one because the space E/F is strongly distinguished. The first inequality is obvious and the second follows from Lemma 2. Therefore, $b(E', E)|_{F^\circ} = TE'|_{F^\circ}$.

On the space F' we have that:

$$b(F', F) = TF' = TE'/F^\circ \geq b(E', E)/F^\circ \geq b(F', F),$$

hence $TE'/F^\circ = b(E', E)/F^\circ$. The first equality follows since the space F is strongly distinguished, and the second from Lemma 1. The last two inequalities are clear. \square

Since the notions of “distinguished” and “strongly distinguished” spaces coincide for Fréchet spaces, the example from [4] shows that the “lifting” condition cannot be omitted in the previous Theorem. In other words, without the lifting assumption the property of “being strongly distinguished” is not three-space-stable.

By an old result from [8], “being a reflexive space” is a three-space-stable property in the class of Banach spaces. This is no longer the case for arbitrary locally convex spaces as the mentioned example 1.5 from [10] shows. However, for inductively reflexive spaces we have

THEOREM 3. *If the outer terms F and E/F of the short exact sequence (1) of l.c.s.'s are inductively reflexive, then the middle term E is inductively reflexive, too.*

PROOF. According to Theorem 1, the space E is inductively semi-reflexive; it is also barrelled (barrelledness is three-space-stable by [10, Th. 2.6]). We have to prove that E is bornological, i.e. ultrabornological since it is complete [3, Th. 1.7].

Note that each topology ξ on the dual E' of an l.c.s. (E, t) satisfying $\kappa(E', E) \leq \xi \leq \tau(E', E)$ gives in E the same topology TE and this topology is not weaker than t . Particularly, the space (E, t) is ultrabornological if and only if $t = TE$.

On the other hand, by [9, Lemma 24.21], if (E, t) is a complete l.c.s., then $\kappa(E', E)$ is the finest locally convex topology on E' which coincides with the weak topology $\sigma(E', E)$ on t -equicontinuous subsets of E' . Consequently, $\kappa(E', E)|_{F^\circ} = \kappa(F^\circ, E/F)$.

Consider now the sequence

$$(2) \quad 0 \rightarrow (F^\circ, \kappa(E', E)|_{F^\circ}) \xrightarrow{i'} (E', \kappa(E', E)) \xrightarrow{j'} (F', \kappa(E', E)/F^\circ) \rightarrow 0.$$

By the previous remark, outer terms in the sequence (2) are strongly distinguished, and since the transposed mapping i' lifts bounded sets with closure (can be checked directly), according to Theorem 2 the middle term $(E', \kappa(E', E))$ is strongly distinguished, too. This means that $T(E', \kappa(E', E))' = TE = b(E, E')$ and since the topology TE on E is ultrabornological, we obtain that the space E is inductively reflexive. \square

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