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DISCRETE VALUE – DISTRIBUTION OF L-FUNCTIONS OF ELLIPTIC CURVES

Virginija Garbaliauskienė and Antanas Laurinčikas

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Abstract. A discrete universality theorem in the Voronin sense for L-functions of elliptic curves is proved.

1. Introduction

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{C}$ denote the sets of all positive integers, integers, rational and complex numbers, respectively. Consider an elliptic curve E given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}.$$

Suppose that the discriminant $\Delta = -16(4a^3 + 27b^2)$ of the curve E is non-zero; then E is non-singular.

For each prime p, let $\nu(p)$ be the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b \ (\mathrm{mod} p).$$

Denote $\lambda(p) = p - \nu(p)$. H. Hasse proved that $|\lambda(p)| \leqslant 2\sqrt{p}$. For the investigations of value-distribution of the numbers $\lambda(p)$ H. Hasse and H. Weil introduced the L-function attached to the curve E. Let $s = \sigma + it$ be a complex variable. Then the L-function $L_E(s)$ of E is defined, for $\sigma > 3/2$, by

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

The Hasse conjecture on analytic continuation and the functional equation of $L_E(s)$ became true after proving the Shimura–Taniyama–Weil conjecture [1]. Therefore, the function $L_E(s)$ is analytically continuable to an entire function and satisfies the following functional equation

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$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s)L_E(s) = \eta \left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s)L_E(2-s).$$

Here q is a positive integer composed from prime factors of Δ, $\eta = \pm 1$ is the root number, and $\Gamma(s)$ denotes the gamma-function.

In [6] the universality of the function $L_E(s)$ was obtained. Let $\text{meas}\{A\}$ denote the Lebesque measure of the set $A \subset \mathbb{R}$. Then we have the following assertion [6].

THEOREM 1.1. *Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : 1 < \sigma < \frac{3}{2}\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |L_E(s+i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

In [2] the assertion of Theorem 1.1 was extended to powers $L_E^k(s)$, $k \in \mathbb{N}$. If $L_E(s) \neq 0$ on D, then the function $L_E^{-k}(s)$, $k \in \mathbb{N}$, is also universal in the above sense.

Note that the universality of the Riemann zeta-function was discovered by S. M. Voronin [10]. Later A. Reich, S. M. Gonek, B. Bagchi, K. Matsumoto, J. Steuding, Y. Mishou, H. Bauer, A. Laurinčikas, R. Garunkštis and others obtained the universality of other classical zeta-functions and of some classes of Dirichlet series. The Linnik–Ibragimov conjecture asserts that all functions given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some growth conditions are universal.

Our aim here is to obtain the discrete universality for the function $L_E(s)$. Let, for $N \in \mathbb{N}$,

$$\mu_N(\ldots) = \frac{1}{N+1} \sharp \{0 \leqslant m \leqslant N : \ldots\},$$

where in place of dots a condition satisfied by m is to be written. In discrete theorems instead of the translations $L_E(s+i\tau)$, $\tau \in [0,T]$, the translations $L_E(s+imh)$, $m = 0, 1, \ldots, N$, where $h > 0$ is a fixed number, are considered.

THEOREM 1.2. *Suppose that* $\exp\{2\pi k/h\}$ *is an irrational number for all $k \in \mathbb{Z} \setminus \{0\}$. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \mu_N\Big(\sup_{s\in K} |L_E(s+imh) - f(s)| < \varepsilon\Big) > 0.$$

Theorem 1.2 shows that the set $\{mh, m = 0, 1, \ldots\}$ such that $L_E(s+imh)$ approximates a given analytic function is sufficiently rich: it has a positive lower density. Since by the Hermite–Lindemann theorem $\exp\{a\}$ is irrational with an algebraic number $a \neq 0$, we can take, for example, $h = 2\pi$. On the other hand, Theorems 1.1 and 1.2 are non-effective in the sense that it is impossible to indicate τ or m with approximation properties.

In the sequel we suppose that $\exp\{2\pi k/h\}$, $k \in \mathbb{Z} \setminus \{0\}$, is an irrational number.

2. A limit theorem

For the proof of Theorem 1.2 we need a discrete limit theorem in the sense of the weak convergence of probability measures in the space of analytic fundations for the function $L_E(s)$. Theorems of such a kind were obtained in [3] for the Matsumoto zeta-function which was introduced in [7]. The Matsumoto zeta-function $\varphi(s)$ is defined by

$$\varphi(s)=\prod_{m=1}^{\infty}A_m^{-1}\left(p_m^{-s}\right),$$

where

$$A_m(x)=\prod_{j=1}^{g(m)}\left(1-a_m^{(j)}x^{f(j,m)}\right)$$

is a polynomial of degree $f(1,m)+\cdots+f(g(m),m)$, $g(m)\in\mathbb{N}$, $a_m^{(j)}\in\mathbb{C}$, $f(j,m)\in\mathbb{N}$, $j=1,\ldots,g(m)$, and p_m denotes the mth prime number. If

$$g(m)\leqslant c_1p_m^{\alpha},\quad |a_m^{(j)}|\leqslant c_2p_m^{\beta} \tag{2.1}$$

with some positive constants c_1,c_2 and non-negative α and β, then the infinite product for $\varphi(s)$ converges absolutely in the half-plane $\sigma>\alpha+\beta+1$, and defines there an analytic function with no zeros. Suppose that the function $\varphi(s)$ is analytically continuable to the region $D_1=\{s\in\mathbb{C}:\sigma>\rho\}$ where $\alpha+\beta+\frac{1}{2}\leqslant\rho<\alpha+\beta+1$, and, for $\sigma>\rho$,

$$\varphi(\sigma+it)=B|t|^{c_3},\quad c_3\geqslant 0, \tag{2.2}$$

and

$$\int_0^T|\varphi(\sigma+it)|^2dt=BT,\quad T\to\infty. \tag{2.3}$$

Here and in the sequel B denotes a quantity bounded by a constant.

Let G be a region on the complex plane, and let $H(G)$ stand for the space of analytic on G functions equipped with the topology of uniform convergence on compacta. To state a limit theorem in the space $H(D_1)$ for the function $\varphi(s)$ we need the following topological structure. Let, for all $m\in\mathbb{N}$, $\gamma_{p_m}=\gamma=\{s\in\mathbb{C}:|s|=1\}$, and

$$\Omega=\prod_{m=1}^{\infty}\gamma_{p_m}.$$

With product topology and pointwise multiplication Ω is a compact topological Abelian group. Denoting by $\mathcal{B}(S)$ the class of Borel sets of the space S, then we have that the probability Haar measure m_H on $(\Omega,\mathcal{B}(\Omega))$ exists, and this leads to a probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Let $\omega(p_m)$ be the projection of $\omega\in\Omega$ to the coordinate space γ_{p_m}, and on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$ define an

$H(D_1)$-valued random element $\varphi(s,\omega)$ by

$$\varphi(s,\omega) = \prod_{m=1}^{\infty}\prod_{j=1}^{g(m)}\left(1 - \frac{\omega^{f(j,m)}(p_m)a_m^{(j)}}{p_m^{sf(j,m)}}\right)^{-1}.$$

Let P_ξ stand for the distribution of a random element ξ.

Lemma 2.1. *Suppose that conditions* (2.1)–(2.3) *are satisfied. Then the probability measure*

$$\mu_N(\varphi(s+imh) \in A), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to P_φ as $N \to \infty$.

Proof. The lemma is a particular case of the theorem from [3], where a limit theorem in the space of meromorphic functions for the function $\varphi(s)$ was proved. $\square$

Let $V > 0$ and $D_V = \{s \in \mathbb{C} : 1 < \sigma < 3/2,\ |t| < V\}$. Later we will use a more convenient notation $\Omega = \prod_p \gamma_p$, and $\omega(p)$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D_V)$-valued random element $L_E(s,\omega)$ by

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Lemma 2.2. *The probability measure*

$$\mu_N(L_E(s+imh) \in A), \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the measure P_{L_E} as $N \to \infty$.

Proof. Clearly, for $\sigma > 3/2$,

$$L_E(s) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta}\left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1}$$

where $\lambda(p) = \alpha(p) + \beta(p)$, and $|\alpha(p)| \leqslant \sqrt{p}$, $|\beta(p)| \leqslant \sqrt{p}$. Therefore, (2.1) is valid with $\alpha = 0$ and $\beta = 1/2$. Moreover, $L_E(s)$ is an entire function and satisfies (2.2). From the validity of the Shimura–Taniyama–Weil conjecture and [8] we have that (2.3) is satisfied, too. Consequently, in view of Lemma 2.1 the probability measure

$$\mu_N(L_E(s+imh) \in A), \quad A \in \mathcal{B}(H(\hat{D})),$$

$\hat{D} = \{s \in \mathbb{C} : \sigma > 1\}$, converges weakly to the distribution of the random element $L_E(s,\omega)$, $s \in \hat{D}$. Since the function $F : H(\hat{D}) \to H_1(D_V)$ given by the formula $F(f) = f\big|_{s\in D_V}$, $f \in H(\hat{D})$, is continuous, hence the lemma follows. $\square$

3. The support of the measure P_{L_E}

Let P be a probability measure on $(S, \mathcal{B}(S))$, where S is a separable metric space. We recall that the support of P is a minimal closed set $S_P \subset S$ such that for every neighbourhood G of each $x \in S_P$ we have $P(G) > 0$. Let

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

LEMMA 3.1. *The support of the measure P_{L_E} is the set S_V.*

PROOF. The proof of the lemma is similar to that of Lemma 8 in [6], therefore we will give only a sketch of the proof. Let, for $a_p \in \gamma$ and $s \in D_V$,

$$g_p(s, a_p) = \begin{cases} -\log\left(1 - \dfrac{\lambda(p)a_p}{p^s} + \dfrac{a_p^2}{p^{2s-1}}\right), & \text{if } p \nmid \Delta, \\ -\log\left(1 - \dfrac{\lambda(p)a_p}{p^s}\right), & \text{if } p \mid \Delta. \end{cases}$$

Then it is proved that the set of all convergent series $\sum_p g_p(s, a_p)$ is dense in $H(D_V)$. For this some properties of functions of exponential type are applied, see, for example, [5].

The sequence $\{\omega(p)\}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. The support of each $\omega(p)$ is the unit circle γ. Therefore, $\{g_p(s, \omega(p))\}$ is a sequence of independent $H(D_V)$-valued random elements, and the support of the random element $g_p(s, \omega(p))$ is the set

$$\{g \in H(D_V) : g(s) = g_p(s, a) \ \text{ with } |a| = 1\}.$$

Hence, in view of Theorem 1.7.10 of [5] the support of the random element

$$\log L_E(s, \omega) = \sum_p g_p(s, \omega(p))$$

is the closure of the set of all convergent series $\sum_p g_p(s, a_p)$ with $a_p \in \gamma$. However, as we have seen above, the later set is dense in $H(D_V)$. Let $h : H(D_V) \to H(D_V)$ be given by the formula $h(g) = \exp\{g\}$, $g \in H(D_V)$. Then clearly, h is a continuous function sending $\log L_E(s, \omega)$ to $L_E(s, \omega)$, and $H(D_V)$ to $S_V \setminus \{0\}$. This shows that the support S_{L_E} of the random element $L_E(s, \omega)$ contains the set $S_V \setminus \{0\}$. However, the support is a closed set, therefore in view of the Hurwitz theorem [9] we obtain that $\overline{S_V \setminus \{0\}} = S_V$. Hence $S_V \subseteq S_{L_E}$. On the other hand, $L_E(s, \omega)$ is an almost surely convergent product of non-vanishing factors, and the Hurwitz theorem again shows that $L_E(s, \omega) \in S_V$. Thus, $S_{L_E} \subseteq S_V$, and the lemma is proved. □

4. Proof of Theorem 1.2

Let K be a compact subset of the strip D with connected complement, and suppose that $V > 0$ is such that $K \subset D_V$. First let the function $f(s)$ have a non-vanishing analytic continuation to the region D_V. Denote by G the set of functions $g \in H(D_V)$ satisfying

$$\sup_{s \in K} |g(s) - f(s)| < \varepsilon.$$

The set G is open, and Lemma 3.1 implies that $G \subset S_V$. Now properties of the weak convergence of probability measures and of support together with Lemma 2.2 yield

$$\liminf_{N \to \infty} \mu_N \left(L_E(s + imh) \in G\right) \geqslant P_{L_E}(G) > 0. \tag{4.1}$$

Now let $f(s)$ satisfy the hypotheses of Theorem 1.2. Then by the Mergelyan theorem, see, for example, [11], there exists a polynomial $p(s)$ which has no zeros on K and such that

$$\sup_{s\in K}|f(s)-p(s)|<\varepsilon/4. \tag{4.2}$$

Similarly, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K}|p(s)-\exp\{q(s)\}|<\varepsilon/4.$$

This and (4.2) show that

$$\sup_{s\in K}|f(s)-\exp\{q(s)\}|<\varepsilon/2, \tag{4.3}$$

and $\exp\{q(s)\}\neq 0$. Therefore, by (4.1)

$$\liminf_{N\to\infty}\mu_N\left(\sup_{s\in K}|L_E(s+imh)-\exp\{q(s)\}|<\varepsilon\right)>0.$$

Hence in virtue of (4.3) the theorem follows.

5. Concluding remarks

Theorem 1.2 discusses the case when $\exp\{2\pi k/h\}$ is an irrational number for all integers $k\neq 0$. Also, a more complicated case when $\exp\{2\pi k/h\}$ is rational for some k is possible. In this case a limit theorem in the sense of the weak convergence of probability measures on the complex plane for the Matsumoto zeta-function was obtained in [4]. A similar limit theorem can be proved also in the space of analytic functions. However, the limit measure in such a theorem is different from P_{L_E}, therefore for such h we need some additional considerations.

Reasoning similarly to [2], we can obtain that the assertion of Theorem 1.2 remains true for the function $L_E^k(s)$, $k\in\mathbb{N}$. If $L_E(s)\neq 0$ for $\sigma>1$, then the function $L_E^{-k}(s)$, $k\in\mathbb{N}$, also is universal in the sense of Theorem 1.2.

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Department of Physics and Mathematics
Šiauliai University
Višinskio 19
77156 Šiauliai
Lithuania
mat.kat@fm.su.lt

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Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
03225 Vilnius
Lithuania
antanas.laurincikas@maf.vu.lt