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DIRICHLET'S PRINCIPLE, DISTORTION AND RELATED PROBLEMS FOR HARMONIC MAPPINGS

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Abstract. We give a short review of some known and new results related to harmonic maps. In particular, we generalize classical Dirichlet's principle, area theorem, theorems of uniqueness of harmonic maps, Bloch theorem, estimates for the modulus of the derivatives of harmonic univalent mappings, etc. We use different tools: Dirichlet's principle, minimizing sequences, different versions of Reich–Strebel inequality, area theorem, etc. Also, comprehensive outlines of proofs for most of new results are given.

0. Introduction

The main purpose of this paper is to give a short review of some results related to harmonic maps, communicated by the author and the other members of the Seminar, at the University of Belgrade, during several last years. Recall in this paper we give a review of known and new results in direction suggested by the abstract, which reflects author personal research interest. We have attempted to partially compensate this by extensive bibliography covering generalizations and other important directions. Comprehensive outlines of proofs for most of new results are given but a few new results are only announced.

The paper consists of 3 sections. Shortly, in Section 1, we outlined the proofs of some properties of harmonic maps concerning connections between: Dirichlet's principle, the area theorem, extremal metrics and modulus, different versions of Reich–Strebel inequality, etc. Sections 1 has subsections A (items A1–A4), B (items B1–B4), C (items C1–C4) and D (items D1–D5).

In subsection A we consider basic properties: the Euler–Lagrange equation for the energy functional, properties of harmonic maps related to natural parameter, uniqueness property and the symmetry property.

The subsection B is devoted to Dirichlet's principle, different generalization of the area theorem, extremal metrics and modules and related problems. In B1, proof of Dirichlet's principle is given. Generalizations of the area theorem and connections with Lehto–Kühnau theorem and Dirichlet's principle, are presented

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in B2. In particular, we discuss some generalizations of the area theorem, which has independent interest, obtained by the author several years ago (which are yet not published). The item B3, is devoted to extremal metrics and modulus. We give a proof of a Beurling result, which is a modification of the proof in [Ah1]. Also, we outline a new proof of the Beurling result, using minimizing sequences. Our approach is influenced by Courant's book (see [C]) and Gehring's work in $\mathbb{R}^3$ space (see [Ge1] and [Ge2]). Concerning further generalizations in this direction we refer the interested reader to the literatures (see B3) suggested in [As]. In B4, we state a formula for the energy density which explains connection between Dirichlet principle for harmonic maps (in general sense) and via the Main Inequality with Grötzsch principle (an integral version of this formula appears in [ReS2], see also [We], [M3] and [Sh]). We expect further applications of the Main Inequality in this direction.

In C, using a new version of Reich–Strebel inequality, an outline of proof of the uniqueness property of harmonic mappings are given, which includes non-negative curved target (see [MM 3]). In this subsection, we also shortly discuss the global uniqueness theorem of Al'ber, Hartman, Jäger–Kaul and Coron–Helein theorem which roughly states that any smooth harmonic diffeomorphism between (M, h) and (N, g) is minimizing in its homotopy class.

In D, a short presentation related to Schoen conjecture and extremal quasiconformal mappings is given.

In Section 2, the content of author paper [M6] is presented. We sketch a proof of the result which roughly states that the maximal dilatation of a proper harmonic mapping of the unit disc onto a convex domain is bounded from bellow by a positive constant and which is a generalization of Heinz theorem [**H**] and some recent results (see [Ka1] and [Ka2].

In Section 3, the content of author's paper [M5] is presented. Using a version of Bloch theorem (see Lemma 3.1) we give a short proof of a Dyakonov's theorem. Also we show that Lemma 3.1 holds for quasiregular harmonic functions (see Theorem 3.1).

Recall the most part of the paper consists of the lectures communicated by the author and the other members of the Seminar, at the University of Belgrade during several last years. The author also talked extensively about this subject in a number of places. During the last several years there has been important progress in characterizing the condition under which unique extremality occurs. We refer the interested reader to the recent excellent survey of Reich [Re9].

1. Dirichlet's principle, uniqueness of harmonic maps and related problems

A. Some basic properties.

A1. Let M and N be two Riemann surfaces with local conformal metrics $\sigma(z)|dz|^2$ and $\rho(z)|dw|^2$ and let $f : M \mapsto N$. It is convenient for us to use notation in local coordinates $df = (\partial f)dz + (\bar{\partial} f)d\bar{z}$ and $p = \partial f$, $q = \bar{\partial} f$. The energy integral of f is

$$E(f,\rho)=\int_M e(f)\sigma\,dx\,dy,$$

where $e(f)$ is the energy density defined by

$$e(f)(z)=(|p|^2+|q|^2)\frac{\rho\circ f(z)}{\sigma(z)}.$$

A critical point of the energy functional is called harmonic mapping. The Euler–Lagrange equation for the energy functional is:

$$f_{z\bar z}+(\partial(\log\rho))\circ f pq=0. \tag{A1}$$

Thus harmonic maps arise from a geometric variational problem and as far as we know, one can not study solutions of this equation, using classical theory of elliptic equations. In this paper we will give an outline of the proofs of some properties of harmonic maps, using different tools: Dirichlet's principle, minimizing sequences, different versions of Reich–Strebel inequality, etc. For general properties of harmonic maps we refer the interested reader to Eells and Lemaire [EL1], [EL2], Jost [J], Schoen [Sc], Schoen and Yau [SY] and further references there. In order to explain our ideas and results it is convenient to suppose that M and N are the domains in $\mathbb{C}$. Let Δ denote the unit disc. If $f:M\to N$ is harmonic map then $\varphi=\rho\circ f p\bar q$ is a holomorphic function. For the sake of the reader, we will sketch a proof of this result in the case when $M=\Delta$ and N is a domain in $\mathbb{C}$, with the metric $\rho(w)|dw|$. Let λ be a complex valued function of class C^1 with compact support in Δ and let $\Phi_\epsilon(z)=z+\epsilon\lambda(z)$. Then,

$$\nu_\epsilon=\operatorname{Belt}[\Phi_\epsilon]=\frac{\epsilon\lambda_{\bar z}}{1+\epsilon\lambda_z}.$$

If f is a stationary point of the energy integral, using an expression (see [ReS2]) for $E(f\circ\Phi_\epsilon^{-1},\rho)-E(f,\rho)$, we conclude that

$$\iint_\Delta \bar\partial\lambda(z)\varphi(z)\,dx\,dy=0.$$

Since φ is integrable function on Δ, it follows that φ is an analytic function on Δ, by Weyl's lemma.

Now, we will state some simple, but useful, properties of harmonic maps.

A2. Properties of harmonic maps related to natural parameter. Again, we suppose, as at the beginning, that f is harmonic mapping between Riemann surfaces M and N. Then $\varphi(z)\,dz^2$ is a holomorphic quadratic differential on M, where $\varphi=\rho\circ f p\bar q$ in a local coordinate. Let P be a regular point for $\varphi(z)\,dz^2$ on M and let ζ be a natural parameter centered at P. If we compute p and q with respect to natural parameter then we have important formula

$$\rho\circ f p\bar q=1 \tag{A2}$$

Now, easy computation gives:

$$p\bar q=\frac14\left(|f_\xi|^2-|f_\eta|^2-i2\operatorname{Re}\bar f_\xi f_\eta\right)$$

Combining this formula with (A2), we find that f_ξ and f_η are orthogonal (if we consider them as vectors). Also, we can show that Jacobian $J = |p|^2 - |q|^2 = 0$ if and only if $f_\eta = 0$.

A3. Using Aronszajn's generalization of Carleman's result we can prove the following uniqueness property:

THEOREM S. *If f is a harmonic mapping of an open connected set $D \subset M$ and $f = 0$ on an open subset of D, then $f = 0$ throughout D.*

General version of this result, which is concerned with the case when M and N are Riemannian manifolds, is known as Sampson's Unique Continuation Theorem (see [Sa] and [EL2]).

A4. The symmetry property. We have the following theorem.

THEOREM RP (The reflection principle). *Suppose L is a segment of the real axis, Ω^+ is a region in $H^+ = \{z : \operatorname{Im} z > 0\}$, and every $t \in L$ is the center of an open disc B_t such that $H^+ \cap B_t$ lies in Ω^+. Let Ω^- be the reflection of Ω^+: $\Omega^- = \{\bar{z} : z \in \Omega^+\}$. Suppose u is harmonic in Ω^+ and $\lim_{n\to\infty} u(z_n) = 0$ for every sequence $\{z_n\}$ in Ω^+ which converges to a point on L. Then there is a function U, harmonic in $\Omega = \Omega^+ \cup L \cup \Omega^-$ such that $U = u$ in Ω^+. This function U satisfies the relation $U(z) = -U(\bar{z})$, $z \in \Omega$.*

PROOF. We extend u to Ω by defining $U(z) = 0$, for $z \in L$, and $U(z) = -U(\bar{z})$, for $z \in \Omega^-$. □

EXAMPLE 1. It is not difficult to verify that function $f(z) = 2x + i\cos y$ is harmonic mapping from C into C with respect to the corresponding metric. This function is periodic with respect to y. The next result shows that this periodicity is typical.

THEOREM M1. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a harmonic mapping, given w.r.t. natural parameter and that Jacobian of f equals zero on the real axis. Then $f(z) = f(\bar{z})$.*

The proof of this result is based on Theorem S.

B. Dirichlet's principle, area theorem and related problems

B1. If the metric density $\rho \equiv 1$ on N then the equation (1) reduces to $f_{z\bar{z}} = 0$. In this case we say that f is a harmonic function and write $D[f]$ instead of $E(f, 1)$ for the energy integral.

Recall that Δ denote the unit disc. Also we will use the notation $D[\phi, \psi] = \iint_\Delta (\phi_x \psi_x + \phi_y \psi_y)\, dx\, dy$. The following lemma is crucial in the proof of Dirichlet's principle.

LEMMA DP. *Suppose that*

(a) *u and h are continuous on $\overline{\Delta}$ and $h \equiv 0$ on $\partial\Delta$*

(b) *u is harmonic on Δ and h has the continuous partial derivatives of the first order on Δ*
(c) *u and h have the finite Dirichlet's integral on Δ.*

Then $D[u,h]=0$.

First we will state the Dirichlet's principle for harmonic function.

THEOREM DP (Dirichlet's principle). *Suppose that*

(a) *g is continuous function on $\overline{\Delta}$.*
(b) *g has the first partial derivatives which are continuous on Δ*
(c) *the energy integral of g is finite.*

If u is continuous on $\overline{\Delta}$, harmonic on Δ and if $u=g$ on the boundary of Δ, then $D(g)\geqslant D(u)$, where the inequality equals if and only if $u=g$ on Δ.

PROOF. If $h=g-u$, then Lemma DP shows that,

$$D[g]=D[u]+2D[u,h]+D[h]=D[u]+D[h]>D[u],$$

unless $D[h]=0$, i.e., h has the constant value zero. □

Now, we are going to discuss some results related to Dirichlet's principle. In an unpublished manuscript [M1] we gave a proof of Theorem M2 (see bellow) based on Dirichlet's principle. Before we state this result we need some definitions and we will state the area theorem and a result of Lehto–Kühnau, which motivated us.

B2. An area theorem of Lehto–Kühnau type for harmonic maps. First, we are going to prove the area theorem, which is an important tool in theory of univalent functions.

THEOREM A (The area theorem). *Let $w=f(z)=z+\dfrac{a_1}{z}+\cdots+\dfrac{a_n}{z^n}+\cdots$ be an univalent analytic function on $E=\{z:|z|>1\}$ and let $G=\mathbb{C}\setminus f(E)$ be the omitted set. Then*

$$\pi\left(1-\sum_{k=1}^{\infty}k|a_k|^2\right)=\operatorname{area}(G).$$

PROOF. Let K_ρ be the circle $|z|=\rho>1$, with the positive orientation, and set

$$I_\rho=I_\rho(f)=\frac{i}{2}\int_{K_\rho}f\,d\bar{f}.$$

If $f=u+iv$ and if γ_ρ denotes the image curve of K_ρ, we have

$$I_\rho=\int_{\gamma_\rho}u\,dv$$

and by elementary calculus this represent the area enclosed by γ_ρ. Hence $I_\rho>0$.

Direct calculation gives

$$\begin{aligned} I_\rho &= \frac{i}{2}\int_{K_\rho}\left(z+\sum_{k=1}^{\infty}\frac{a_k}{z^k}\right)\left(1-\sum_{k=1}^{\infty}k\bar a_k\bar z^{-k-1}\right)d\bar z\\ &= \frac{1}{2}\int_{K_\rho}\left(z+\sum_{k=1}^{\infty}\frac{a_k}{z^k}\right)\left(\bar z-\sum_{k=1}^{\infty}k\bar a_k\bar z^{-k}\right)d\theta\\ &= \pi\left[\rho^2-\sum_{k=1}^{\infty}k|a_k|^2\rho^{-2k}\right]. \end{aligned}$$

Thus $\sum_{k=1}^{\infty} k|a_k|^2\rho^{-2k} < \rho^2$, and theorem follows for $\rho \to 1$. □

Let us consider conformal mapping h which belongs to class Σ, i.e., h is univalent in $E = \{z : |z| > 1\}$ and has a power series expansion of the form $h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ in E. If h has a quasiconformal extension to the plane with complex dilatation μ, satisfying the inequality $\|\mu\|_\infty = k < 1$, we say that h belongs to the subclass Σ_k of Σ. Lehto (see [L1] and [L2]) and Kühnau (see [K]) established the area theorem for Σ_k.

Theorem LK (Lehto–Kühnau). *Let $h \in \Sigma_k$. Then $\sum_{n=1}^{\infty} n|a_n|^2 \leqslant k^2$. The estimate is sharp.*

If we denote by P the area of the omitted set of $h(E)$, then Theorem LK states that $P \geqslant \pi(1-k^2)$. Before we state the Theorem M2, which is a generalization of Theorem LK to univalent harmonic mappings, we need some definitions. Let Σ' be the set of all harmonic, orientation-preserving, univalent mappings $h(z) = z + f(z) + \overline{g(z)} + A\log|z|$ on E, where $f(z) = \sum_{n=1}^{\infty} a_n z^{-n}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^{-n}$ are analytic on E and $A \in C$. Let Σ'_k denote the set of all homeomorphisms h of C onto itself such that

(a) the restriction of h on E belongs to Σ' and

(b) the restriction of h on the unit disk $U = \{z : |z| < 1\}$ is a quasiconformal mapping with complex dilatation μ satisfying $\|\mu\|_\infty \leqslant k < 1$.

The Area theorem can be established for Σ'_k. Recall, that P denote the area of the omitted set of $h(E)$. Also, it is convenient to use notations $\tau = \sum_{n=1}^{\infty} n|a_n|^2$ and $s = 1 + 2\operatorname{Re} b_1 + l$, where $l = \sum_{n=1}^{\infty} n|b_n|^2$.

The following result was proved by the author about 15 years ago, but it has not been published yet.

Theorem M2. *Let $h \in \Sigma'_k$. Then*

(a) $$P \geqslant \pi(1-k^2)s.$$

The equality holds in (a) *if and only if*

$$h(z) = z + cz^{-1} + cg(z) + \overline{g(z)} + A\log|z|,$$

where $g(z) = \sum_{n=1}^{\infty} b_n z^{-n}$ is analytic on E; and $|c| = k$, $A \in \mathbb{C}$.

After writing the final version of this paper the author was kindly informed by Professor R. Kühnau that he and the others proved several results of this type.

PROOF OF THEOREM M2. In order to prove the inequality (a) we need two lemmas. Let

$$\alpha_r = \iint\limits_{U_r} |\partial h|^2 dx\, dy \quad \text{and} \quad \beta_r = \iint\limits_{U_r} |\bar\partial h|^2 dx\, dy, \quad \text{where } U_r\{z : |z| < r\}.$$

Also, we will write α and β respectively, instead of α_1 and β_1.

LEMMA 1.1. *If $h \in \Sigma_k'$, then α is finite and $P \geqslant (1-k^2)\alpha$.*

PROOF. Since h has locally integrable L^2-derivatives, using that $|\bar\partial h| \leqslant k|\partial h|$, we obtain $P_r = \alpha_r - \beta_r \geqslant (1-k^2)\alpha_r$ for $0 < r < 1$. This gives Lemma 1.1 when r approaches 1. If $h = u + iv$, then it is convenient to use notation $|\nabla h|^2 = |\nabla u|^2 + |\nabla v|^2$. Now, the desired result follows by integration over U. □

For $h \in \Sigma_k'$ and $z \in \bar U$, let $\varphi(z) = f(1/z)$ and $\psi(z) = \overline{g(1/\bar z)}$. Thus φ and ψ are continuous function on $\bar U$ and $\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$, $\psi(z) = \sum_{n=1}^{\infty} \overline{b_n} z^n$ for $z \in U$.

In order to apply Dirichlet's principle, let us consider the function H defined by $H(z) = z + \varphi(\bar z) + \psi(z)$, $z \in \bar U$. It is easy to check that $H(z) = h(z)$ for $z \in \partial U$.

LEMMA 1.2. *Dirichlet's integral of H over U is finite and*

$$D[H] = \iint\limits_{U} |\nabla H|^2 dx\, dy = 2\pi(s+\tau)$$

PROOF. Since $\alpha = \frac14 D[h] + \frac12 P$ we conclude that Dirichlet's integral $D[h]$ of h is finite and therefore, using Dirichlet's principle that Dirichlet's integral of H is finite. Using

$$H_x(z) = 1 + \varphi'(\bar z) + \psi'(z), \quad H_y(z) = i - i\varphi'(\bar z) + i\psi'(z),$$

for $z \in U$, we obtain $|\nabla H|^2 = 2(|1+\psi'(z)|^2 + |\varphi'(z)|^2)$. □

PROOF OF THE INEQUALITY (a). Using Dirichlet's principle, we conclude that

$$\alpha + \beta \geqslant \pi(s+\tau). \tag{1.1}$$

On the other hand the area of the omitted set is

$$P = \lim_{R\to 1_+} \frac{1}{2i} \int\limits_{\gamma_R} \bar h\, dh = \pi(s-\tau),$$

where γ_R is the curve defined by $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Since $P = \alpha - \beta$, we obtain

$$\alpha - \beta = \pi(s-\tau). \tag{1.2}$$

It follows from (1.1) and (1.2) that $\alpha \geqslant \pi s$. This inequality together with Lemma 1.1 gives the inequality (a). □

THE EQUALITY IN THEOREM M2. If equality holds in (1.1), using Dirichlet's principle we conclude that $h = H$ on U. An inspection of the proof of Lemma 1.1 gives $k|1 + \psi'(z)| = |\varphi'(\bar{z})|$, $z \in U$. Next application of the maximum principle shows that h has the form given by the part (b) of Theorem M2. □

Since $P = \pi(s - \tau)$ the next result follows immediately from Theorem M2.

COROLLARY M1. *If $h \in \Sigma_k'$, then $\tau \leqslant k^2 s$.*

Using Dirichlet's principle as above we can give further generalization of this result. Suppose that

a) h is continuous function on $\bar{U}$ and k quasiregular, $0 < k < 1$, on U.
b) the curve γ defined by $w = h(e^{i\theta}), 0 \leqslant \theta \leqslant 2\pi$, is of bounded variation.
c) $h(e^{i\theta}) \sim \sum_{n=1}^{\infty} a_n e^{-in\theta} + \sum_{n=0}^{\infty} B_n e^{in\theta}$

THEOREM M3. *With notation and hypothesis just stated we have*

$$A = \sum_{n=1}^{\infty} n|a_n|^2 \leqslant k^2 B, \quad \textit{where} \quad B = \sum_{n=1}^{\infty} n|B_n|^2.$$

Finally we state a generalization of the area theorem to analytic functions.

THEOREM A1. *Let $w = f(z) = \lambda z + \dfrac{a_1}{z} + \cdots + \dfrac{a_n}{z^n} + \cdots$ be an analytic function on $E = \{z : |z| > 1\}$ and let $G = \mathbb{C} \setminus f(E)$ be the omitted set. Then*

$$\pi\left(|\lambda|^2 - \sum_{k=1}^{\infty} k|a_k|^2\right) \leqslant \operatorname{area}(G). \tag{B1}$$

Equality holds if and only if f is a univalent function on E.

PROOF. Let K_ρ be the circle $|z| = \rho$ with positive orientation and let γ_ρ be the curve defined by the equation $w = f_\rho(e^{it}) = f(\rho e^{it})$, $0 \leqslant t \leqslant 2\pi$. For given $w \neq \infty$ let $n(w)$ be the number of roots of $f(z) = w$ in $|z| > \rho$. Assume that $f \neq w$ on K_ρ and $\lambda \neq 0$. Since f has a pole of order 1 at ∞, we have $f(z) \neq w$ in $|z| \geqslant r$ for a large r and consequently, by the argument principle,

$$n(w) = \frac{1}{2\pi i} \int_{K_r - K_\rho} \frac{f'(z)}{f(z) - w} dz = 1 - \chi(\gamma_\rho, w), \tag{B2}$$

where $\chi = \chi(\gamma_\rho, w)$ is the winding number (or index) of the curve γ_ρ with respect to the point w. By the analytic Green's theorem (see, for example [Po]), the area

$$I_\rho = \frac{1}{2\pi i} \int_{\gamma_\rho} \bar{w}\, dw = \frac{1}{\pi} \iint_{\mathbb{R}^2} \chi(\gamma_\rho, w)\, du\, dv. \tag{B3}$$

Let G_ρ be the set omitted by f on $E_\rho = \{|z| > \rho\}$. By (B2) $w \in G_\rho$ if and only if $\chi(\gamma_\rho, w) = 1$. Also, it follows from (B2) that $\chi(\gamma_\rho, w)$ is an integer less than or equal to zero if $w \notin \bar{G}_\rho$. This together with (B3) gives

$$\pi I_\rho \leqslant \operatorname{area}(G_\rho). \tag{B4}$$

Direct calculation as in the proof of area theorem gives (B1). For the case of equality see [M].

We announce the following generalization of Theorems A, A1 (see Theorem M4 below). A proof of Theorem M4 can be based on Theorem A1 and Wirtinger's inequality among the other things.

THEOREM M4. *Let* $w = f(z) = \lambda z + \dfrac{a_1}{z} + \cdots + \dfrac{a_n}{z^n} + \cdots$ *and* $w = g(z) = \mu z + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} + \cdots$ *be analytic functions on* $E = \{z : |z| > 1\}$, $H = (f, g)$ *and let* G *be a surface such that* $G \cup \overline{H(E)}$ *be the simple connected set. Then*

$$\pi\left(|\lambda|^2 + |\mu|^2 - \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2)\right) \leqslant \operatorname{area}(G). \tag{B5}$$

The reader can consider when the equality holds in (B5).

A version of the Area theorem can be established for harmonic non-univalent functions. We announce the following generalization of Theorems A, A1 (see Theorem M5 below). One can use properties of index as in the proof of Theorem A1, among the other things, to give a proof of Theorem M5. Suppose

(a) $h(z) = z + f(z) + \overline{g(z)}$ on E, where $f(z) = \sum\limits_{n=1}^{\infty} a_n z^{-n}$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^{-n}$ are analytic on E and
(b) $J_h \geqslant 0$ on E

Let P denote the area of the omitted set of $h(E)$. Also, it is convenient to use notations $\tau = \sum_{n=1}^{\infty} n|a_n|^2$ and $s = 1 + 2\operatorname{Re} b_1 + l$, where $l = \sum_{n=1}^{\infty} n|b_n|^2$.

THEOREM M5. *Suppose the above conditions and notations. Then*

$$P \geqslant \pi(s - \tau).$$

B3. Extremal metrics and modulus. In this item we are going to give a proof of a Beurling result, which is a modification of the proof in [Ah1]. Also, we outline a new proof of the Beurling result, using minimizing sequences. Our approach is influenced by Courant's book (see [C]) and Gehring's work in $\mathbb{R}^3$ space (see [Ge1] and [Ge2]). Some generalizations of Gehring's results are presented in [AMŠ].

In unpublished work Beurling has given the following elegant and useful criterion. Before we state Beurling result we need a few definitions. Let Ω be a region in the plane, Γ be family of curves and let $\rho(z) \geqslant 0$ be Borel measurable function defined in the z-plane. We say that ρ is admissible for Γ, if for every rectifiable $\gamma \in \Gamma$, $\int_\gamma \rho|dz|$ exists and $\infty \geqslant \int_\gamma \rho|dz| \geqslant 1$. In these circumstances every rectifiable arc γ has a well defined ρ-length $L(\gamma, \rho) = \int_\gamma \rho|dz|$, which may be infinite, and the open set Ω has a ρ-area $A = A_\rho = A(\rho, \Omega)$.

The modulus of Γ, $M = M_\Omega(\Gamma)$, with respect to Ω, is defined as $\inf A(\Omega, \rho)$ for admissible ρ. The extremal length of Γ in Ω is defined as the reciprocal of the modulus. The extremal length is denoted by $\lambda = \lambda_\Omega(\Gamma)$.

THEOREM B1 (Beurling's theorem). *The metric ρ_0 is extremal for Γ if Γ contains a subfamily Γ_0 with the following properties:*

(a) $\int_\gamma \rho_0 |dz| = 1$, *for all* $\gamma \in \Gamma_0$*;*

(b) *for real-valued h in Ω the conditions $\int_\gamma h\,|dz| > 0$ for all $\gamma \in \Gamma_0$ imply* $\iint_\Omega h\rho_0\,dx\,dy \geqslant 0$,

Let Ω be an open set and let E_1, E_2 be two sets in the closure of Ω. Take Γ to be the set of connected arcs in Ω which join E_1 and E_2. The extremal length $\lambda(\Gamma)$ is called the extremal distance of E_1 and E_2 in Ω, and we denote it by $d_\Omega(E_1, E_2)$.

EXAMPLE 1. The extremal distance between vertical sides of a rectangle $R = \{z = x + iy : a < x < b,\ c < y < d\}$ is $\lambda = (b-a)/(d-c)$.

PROOF. Let $\Lambda_y = [a+iy, b+iy]$ and Γ_0 is the family of curves $\{\Lambda_y : c \leqslant y \leqslant d\}$. If we take $\rho_0 = 1$, then Beurling's criterion is satisfied and $\rho_0 = 1$ is extremal metric.

EXAMPLE 2. Let A be the ring $A = A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$. If Γ is the family of arcs, which join circles $K_{r_1} = \{z : |z| = r_1\}$ and $K_{r_2} = \{z : |z| = r_2\}$, then

$$L(\Gamma) = \frac{1}{2\pi}\ln\frac{r_2}{r_1}. \tag{B6}$$

PROOF. Let $A' = A \setminus (r_1, r_2)$ and $R = \{w : \ln r_1 < u < \ln r_2,\ 0 < v < 2\pi\}$. Since exp maps conformally R onto A', using the Example 1 we get (B6).

Now, we state a result of Beurling, which express the Dirichlet's integral by means of extremal distance (see [Ah1]).

THEOREM B2 (Beurling's theorem). *Let Ω be a region in the complex plane bounded by a finite number of analytic Jordan curves, let E_0 and E_1 be disjoint and consist of finite number of closed arcs or curves in the boundary of Ω. Then the extremal distance $d_\Omega(E_0, E_1)$ is the reciprocal of the Dirichlet integral*

$$D(u) = \iint_\Omega (u_x^2 + u_y^2)\,dx\,dy,$$

where u satisfies:

(i) *u is bounded and harmonic in Ω*

(ii) *u has a continuous extension to $\Omega \cup E_0^\circ \cup E_1^\circ$, and $u = 0$ on E_0 and $u = 1$ on E_1, where E_0°, E_1° are the relative interiors of E_0, E_1 as subsets of $\partial\Omega$, respectively.*

(iii) *the normal derivative $\partial u/\partial n$ exists and vanishes on C_0 (C denotes the full boundary of Ω, $C_0 = C \setminus (E_0 \cup E_1)$).*

The proof of this theorem in [Ah1] is based on two important ingredients:
1) existence of solution of a mixed Dirichlet–Neuman problem (we denote it by u)
2) decomposition of a domain on rings and quadrilateral subdomains using, in fact, the orthogonal and vertical trajectories of quadratic differential defined by u.

For the theory of trajectories of holomorphic quadratic differentials see [Ga] and [S2].

Proof of Theorem B2. Let A be the set of the endpoints of E_1 and E_2 as subsets of C. The reflection principle shows that u has a harmonic extension across $\partial\Omega \setminus A$. Let $z_0 \in A$, for example, $z_0 \in E_1$. We can chose a local conjugate v in Ω near z_0 such that, on the boundary, $u = 0$ on one side of z_0 and $v = 0$ on the other side of z_0. Then, by the reflection principle, there exists neighborhood V of z_0 and an analytic function φ in $V \setminus \{z_0\}$ such that $\varphi = (u + iv)^2$ in $\Omega \cap V$. Hence, φ is an analytic function on V and has a simple zero at z_0. Therefore, $u_x - iu_y$ must tend to ∞, and the number of critical points in $\overline{\Omega} \setminus A$ is finite.

Locally, for every $z_0 \in \partial\Omega \setminus A$ there exists a neighborhood V of z_0 and an analytic function f on V such that $\operatorname{Re} f = u$ on V. Hence, we can define horizontal trajectories with respect to $w = f(z)$. The part of noncritical horizontal trajectory γ which is in $\overline{\Omega}$ can be parameterized with parameter interval $I = [0, 1]$ such that:

1. γ join E_1 and E_2 in Ω (more precisely $\gamma(0, 1) \subset \Omega$, $\gamma(0) \in E_1$ and $\gamma(1) \in E_2$).
2. $\operatorname{Re}\gamma$ is strictly increasing function on I and $\operatorname{Re}\gamma(0) = 0$, $\operatorname{Re}\gamma(1) = 1$.

Hence, we conclude that up to a set of Lebesgue 2-dimensional measure zero there exists finite number of disjoint quadrilateral Σ_k, $k = 1, 2, \ldots, n$, such that:

1. $\Omega = \bigcup_{k=1}^n \Sigma_k$
2. Each Σ_k is swept out with noncritical horizontal trajectories
3. There exists rectangles R_k of width 1 and height m_k and conformal (univalent) mapping $\Phi = \Phi_k$ of Σ_k onto R_k such that $\operatorname{Re}\Phi_k = u$ on Σ_k. Hence,

$$m_k = \iint_{\Sigma_k} |\Phi'|^2 dx\, dy \quad \text{and} \quad m = \sum_{k=1}^n m_k = D(u).$$

Together rectangles R_k fill out a rectangle with sides 1 and $D(u)$. After appropriate identification we obtain a model of Ω with E_1 and E_2 as vertical sides.

From this model and Beurling theorem (Theorem B1) it is immediately clear that the euclidean metric is extremal, and we conclude that $d_\Omega(E_1, E_2) = 1/D(u)$.

Our first purpose was to give more elementary proof of this result (that is, with no use of these two subjects), using a minimizing sequence (see, for example Courant's book [C]), and to derive some equalities not contained in the proof of Beurling's theorem. During our work on this problem we become aware of Gehring's works (see [Ge1] and [Ge2]), which strongly influenced our research.

In [Ge1] and [Ge2] Gehring proved that essentially Väisälä's definition of extremal distance between E_0 and E_1 in Ω is equivalent to the Dirichlet's integral definition due to Loewner (see [Lo]) if Ω is a ring domain in $\mathbb{R}^3$, and E_0 and E_1 are boundary components of Ω. Gehring used this result to study quasiconformal mappings in space.

We generalize this result to the setting of smooth domains in $\mathbb{R}^n$. An application of this result gives a short proof of Beurling's Theorem. As we understand, there are additional technical difficulties if we work with general domains instead of ring domains. Before we state the result we need a few definitions.

Definition B1. Let Ω be an open set in $\mathbb{R}^n$ and Γ a set whose elements γ are rectifiable arcs in Ω. Let ρ be a nonnegative Borel measurable function in Ω (such ρ we will call metric). We can define the ρ-length of γ by $L(\gamma, \rho) = \int_\gamma \rho |dx|$ the

ρ-volume of Ω as $V(\Omega,\rho) = \int_\Omega \rho^n dV(x)$ where dV is the n-dimensional Lebesgue measure in $\mathbb{R}^n$, and the minimum length of Γ by $L(\Gamma,\rho) = \inf_{\gamma\in\Gamma} L(\gamma,\rho)$. The modulus of Γ in Ω is defined by

$$\mathrm{mod}_\Omega(\Gamma) = \inf_\rho \frac{V(\Omega,\rho)}{L(\Gamma,\rho)^n},$$

where ρ is subject to the condition $0 < V(\Omega,\rho) < \infty$. The extremal length of Γ in Ω is defined as $\Lambda_\Omega(\Gamma) = \mathrm{mod}_\Omega(\Gamma)^{1/(1-n)}$.

Definition B2. Let Ω be an open set in $\mathbb{R}^n$, and let E_0, E_1 be two sets in the closure of Ω. Take Γ to be the set of connected arcs in Ω which join E_0 and E_1, i.e., each $\gamma \in \Gamma$ has one endpoint in E_0 and the other in E_1 and all other points of γ are in Ω. The extremal length $\Lambda(\Gamma)$ is called the extremal distance of E_0 and E_1 in Ω, and we denote it by $d_\Omega(E_0,E_1)$.

Now, let Ω be a bounded region whose boundary consists of a finite number of C^1 hypersurfaces. If E_0 and E_1 are disjoint, and each is a finite union of closed hypersurfaces contained in the boundary of Ω, then we define the conformal n-capacity of Ω as

$$C[\Omega,E_0,E_1] = \inf_u \int_\Omega |\nabla u|^n dV(x),$$

where infimum is taken over all functions $u:\Omega\to\mathbb{R}$ which are differentiable in Ω, continuous in $\overline{\Omega}$ and have boundary values 0 on E_0 and 1 on E_1.

The proof of the following theorem is given in [AMŠ].

Theorem AMŠ. *If Ω is a bounded domain, whose boundary consists of a finite number of C^1 hypersurfaces, and if E_0 and E_1 are disjoint sets of the boundary of Ω consisting of finite number of closed hypersurfaces, then we have*

$$\mathrm{mod}_\Omega(\Gamma) = \inf_f \frac{V(\Omega,f)}{L(\Gamma,f)^n} = C[\Omega,E_0,E_1],$$

where f is any metric in Ω and Γ is the family of all Jordan arcs joining E_0 and E_1 inside Ω.

The case $n=2$ of previous theorem enables us to give a short proof of Beurling theorem (Theorem B2). In fact, the proof immediately follows from Theorem 1.3 (see [C]), which gives a solution of a mixed Dirichlet–Neuman problem.

The proof of Theorem 1.3 in Courant's book [C], is based on using minimizing sequences. We believe that we can use minimizing sequences as Gehring in [Ge1] to show existence of the extremal admissible function $u \in E(\Omega,E_0,E_1)$ such that

$$C[\Omega,E_0,E_1] = \int_\Omega |\nabla u|^n dV.$$

After we had published [AMŠ] paper, we received some comments on it. Aseev (the letter [As]) wrote: "Indeed the problem, related to Theorem AMŠ, was initiated by L. Ahlfors and F. W. Gehring, but since 1970 year it has been thoroughly investigated and successfully resolved in a number of papers. In the case where consender has compact plates in a domain $\Omega \subset \mathbb{R}^n$ the equality was proved in 1975 by J. Hesse in Ark. Mat. (1975, 13, pp. 131–144) and independently by M. Ohtusuka in his manuscript on precise functions. In more general case where the plates of consender are compact in $\overline{\Omega}$ the equality was established in 1993 by V. M. Shlyk in Siberian Math. J. (1993, 34, pp. 216–221). Finally, the problem mentioned above has been recently considered in metric spaces with dubbling measure by J. Heinonen – P. Koskela (Acta Math., 1998, 181, pp. 1–61) and S. Kallunki – N. Shanmugalingam (Ann. Acad. Sci. Fenn., 2001, 26, pp. 455–464)."

Notice here that we could prove Theorem AMŠ in more general settings, but our emphasis was on the method of the proof. Recall, the case $n = 2$ of Theorem AMŠ enables us to give a short proof of Beurling theorem (Theorem B2). In fact, the proof immediately follows from Theorem 1.3 of [C], which gives a solution of a mixed Dirichlet–Neuman problem.

B4. Dirichlet's principle for harmonic mappings. *The Main Inequality and Dirichlet's principle.* Now we will state a formula for the energy density which explains connection between Dirichlet principle for harmonic maps (in general sense) and via the Main Inequality with Grötzsch principle (an integral version of this formula appears in [ReS2], see also [We], [M3] and [Sh].

Suppose that ρ is a metric density on Δ, f is C^1 function on $\overline{\Delta}$ and let h be a diffeomorphism of $\overline{\Delta}$ onto itself which is the identity on the boundary of Δ. If $\nu = \mathrm{Belt}[h]$, then

$$e(f \circ h^{-1}) = \left[\frac{1+|\nu|^2}{1-|\nu|^2} e(f) - 4\,\mathrm{Re}\,\frac{\nu}{1-|\nu|^2}\varphi\right],$$

where $\varphi = \varphi(f) = \rho \circ f p\bar{q}$. Hence, $e(f \circ h^{-1}) - e(f) = 2\left(|\varphi| T_\nu \varphi - |\varphi|\right) + r(h)$, where

$$r(h) = r(h, f) = \frac{2|\nu|^2}{1-|\nu|^2}\left(e(f) - 2|\varphi|\right) \geqslant 0.$$

If f is a harmonic mapping (with respect to ρ), then $\varphi = \varphi(f)$ is a holomorphic function on Δ. Hence, using the Main Inequality we obtain a version of Dirichlet principle for harmonic mappings (in general sense): $E(f \circ h^{-1}) \geqslant E(f)$. We expect further applications of the Main Inequality in this direction.

In order to illustrate this we will outline a short proof of Dirichlet's principle (Theorem M6, below) for the harmonic mappings from the unit disk into Riemannian n-dimensional manifold. Let N be complete Riemannian manifold of dimension n and let its metric in local coordinates be given by (g_{ik}), with Christoffel symbols Γ^i_{kl}. For $f \in H^{1,2}(M, N)$ we define the energy density

$$e(f)(z) = \frac{1}{\sigma^2} \sum g_{ik}(f)\left(f^i_x f^k_x + f^i_y f^k_y\right),$$

and the energy as

$$E(f) = \frac{1}{2}\int_M e(f)\sigma^2 dx\, dy = \frac{1}{2}\int_M \sum g_{ik}(f)\left(f^i_x f^k_x + f^i_y f^k_y\right)\, dx\, dy,$$

where we write $f=(f^1,f^2,\dots,f^n)$ in local coordinates. A solution of the corresponding Euler-Lagrange equation $\nabla f^i+\Gamma^i_{kl}\left(f^k_x f^l_x+f^k_y f^l_y\right)=0$, $i=1,2,\dots,n$, is called harmonic map.

THEOREM M6 (Dirichlet's principle for harmonic mappings). *Let N be Riemannian n-dimensional manifold and $f:\overline{\Delta}\to N$ be a harmonic mapping. If Φ is diffeomorphism of $\overline{\Delta}$ onto itself, which is identity on $\partial\Delta$, then $E(f\circ\Phi)\geqslant E(f)$,*

C. Uniqueness of harmonic maps

Our further discussion is concerned mainly with the case when M and N are domains in complex plane $\mathbb{C}$. Recall, that the following result enables us to use theory of trajectory of holomorphic quadratic differentials.

C1. If f is a harmonic mapping between Riemann surfaces M and N with local conformal metrics $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$, respectively, then $\varphi=\rho p\bar{q}dz^2$ is a holomorphic quadratic differential. For example if M and N are subset of the complex plane C, this simply means that the function $\rho p\bar{q}$ is a holomorphic function. This enables us to use the techniques and results from the theory of holomorphic functions.

C2. Marković and the author, using a version of Reich–Strebel inequality, proved the following uniqueness property.

THEOREM MM. *Suppose that*

(a) *f and g are harmonic difeomorphisms of Δ onto itself*
b) *f and g are continuous on $\overline{\Delta}$*
(c) *$f=g$ on $\partial\Delta$.*

If, in addition, we suppose that the energy integrals of f and g are finite, then they are identical.

This result was communicated on our Seminar at Belgrade University in 1996 and at Nevannlina Colloquium, Switzerland 1997. The proof is based on the next lemma if f and g are diffeomorphisms of $\overline{\Delta}$ onto itself and on a new version of Reich–Strebel inequality in general case.

LEMMA MM. *Suppose that f and g are diffeomorphisms of $\bar{\Delta}$ onto $\bar{\Delta}$ and that f is harmonic with respect to conformal metric $ds=\rho(w)|dw|$ on $\bar{\Delta}$. If we suppose in addition, that $E(f)<+\infty$ and that $f=g$ on $\partial\Delta$, then*

$$\int_\Delta \tilde{\rho}(\zeta)\,d\xi\,d\eta\leqslant\int_\Delta\tilde{\rho}(\zeta)\frac{1-|\tilde{\mu}(\zeta)|}{1+|\tilde{\mu}(\zeta)|}\frac{\left|1+\frac{\tilde{\chi}(\zeta)}{\tilde{\mu}(\zeta)}|\tilde{\mu}(\zeta)|\right|^2}{1-|\tilde{\chi}(\zeta)|^2}d\xi\,d\eta,$$

where $\tilde{\mu}=\mathrm{Belt}(f^{-1})$, $\tilde{\chi}=\mathrm{Belt}(g^{-1})$ *and* $\tilde{\rho}(\zeta)=\rho(\zeta)\dfrac{|\tilde{\mu}(\zeta)|}{1-|\tilde{\mu}(\zeta)|^2}$

We will outline a proof of Theorem MM in the case that $f=$Id on $\partial\Delta$ and that f is diffeomorphism of $\overline{\Delta}$ onto itself. For the proof it is useful to observe that if f is harmonic, then Beltrami dilatation μ of f has the form $\mu(z)=s(z)|\varphi(z)|/\varphi(z)$, where s is non-negative measurable function and $\varphi=\rho\circ f p\bar{q}$ is an analytic function.

Thus we have that expression $\mu\varphi/|\varphi|$, which appears in Reich–Strebel inequality equals $|\mu|$ and we get

$$\int_\Delta |\varphi|\,dx\,dy \leqslant \int_\Delta |\varphi|\frac{1-|\mu|}{1+|\mu|}\,dx\,dy.$$

If φ is not identically zero we get $\mu = 0$ a.e. Hence we conclude that f is conformal mapping. Since $f = \mathrm{Id}$ on $\partial\Delta$, we get that $f = \mathrm{Id}$ on Δ. In general, we need a version of main inequality which holds for the mapping whose maximal dilatation can be 1.

C3. Marković and the author have proved that $f = g$ under weaker condition then in Theorem MM. The following two results will appear in [MM3].

THEOREM MM1. *Suppose that*

(a) *f is homeomorphism of $\overline{\Delta}$ onto itself*
(b) *f has the first generalized derivatives on Δ*
(c) *f is identity on $\partial\Delta$*
(d) *f is harmonic w. r. t. some metric density ρ on Δ*
(e) *Hopf differential of f is integrable on Δ.*

Then f is the identity on Δ.

THEOREM MM2 (The uniqueness property). *Suppose that*

(a) *f and g are homeomorphisms of $\overline{\Delta}$ onto itself and $f = g$ on $\partial\Delta$*
(b) *f and g are loc. q.c. on Δ*
(c) *f and g are harmonic w.r.t. some metric density ρ on Δ*
(d) *Hopf differentials of f and g are integrable on Δ.*

Then f and g are identical.

Also, we might add that we have a generalization of this result if instead of the unit disk, we consider Riemann surfaces. Recall, if the metric $\rho \equiv 1$ on N, which is open subset of complex plane $\mathbb{C}$ (euclidean case), we will say harmonic function instead of harmonic mapping. Thus in euclidean case this result says that the solution of classical Dirichlet problem is unique.

The proof of Theorem MM2 is based on a new version of Riech–Strebel inequality. Note that if f and g are harmonic property (A) says that function $\varphi = \rho\circ f p\bar{q}$ and $\psi = \rho\circ g A\bar{B}$ are holomorphic functions on the unit disk, where we use notation $A = \partial g$, $B = \bar{\partial} g$. The idea of the proof is to apply a new version of Reich–Strebel inequality to functions φ and ψ.

In the next item we are going to give a short discussion of known result related to uniqueness of harmonic maps.

C4. We refer the interested reader to [J] for the global uniqueness theorem of Al'ber and Hartman, for the result of Jäger and Kaul and for further references.

THEOREM AH (Al'ber and Hartman). *Let $u : M \to N$ be a harmonic map between compact Riemannian manifolds (without boundary). Suppose N has negative* *sectional curvature. Then u is unique harmonic map in its homotopy class unless $u(M)$ is a point or a closed geodesic.*

If the sectional curvature of N is non-positive, then for any two homotopic harmonic $u_0, u_1 : M \to N$, there exist a family $u_t : M \to N$ of harmonic maps, with the property that the curves $u_t(x)$, for fixed $x \in M$, $t \in [0,1]$ varying, constitute a family of parallel geodesics, parameterized proportionally to arc length. In particular, all maps u_t have the same energy.

THEOREM JK (Jäger and Kaul). *Suppose that $u_i : \overline{\Omega} \to N$ $(i = 1, 2)$ are harmonic maps of class $C^0(\overline{\Omega}, N) \cap C^2(\Omega, N)$, Ω is a bounded domain in some Riemannian manifold, and $u_i(\overline{\Omega}) \subset B(p,\rho)$, where $B(p,\rho)$ is a geodesic ball in N, disjoint to the cut locus of p and with radius $\rho < \pi/2\kappa$ where κ^2 is an upper bound for the sectional curvature of $B(p,\rho)$. If $u_1 = u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$.*

We refer the interested reader to the Schoen–Yau book [SY] for uniqueness theorems concerning harmonic maps into non-positive curved metric spaces and further references.

After writing the previous version Reich pointed out to us that Wei [We] studied uniqueness property of harmonic mappings. Also, we became aware of the Coron–Helein paper [CH].

Wei, using the formula for the energy of variation of a mapping (see [ReS2]) and Reich–Strebel inequality, proved a weaker version of Theorem MM2 concerning q.c. mapping. Namely, he proved Theorem MM2 under additional hypotheses that

(c) f and g are q.c. mappings on the unit disk Δ onto itself

(d) the metric density ρ is an integrable function on Δ.

Note that the hypotheses (c) and (d) provide that the energy integral of f and g are finite.

In [CH], Coron and Helein used completely different approach then Wei in [We] to study minimizing harmonic mappings. Their approach was based on decomposition of given metric g on Δ as the sum of two metrics c and h such that c is conformal metric of the euclidean metric e, h has non-positive Gaussian curvature and Id is harmonic map between (Δ, e) and (Δ, h).

THEOREM CH (Coron and Helein). *Let (M,h) and (N,g) be two Riemannian compact surfaces of class C^∞ possibly with boundary. Then any smooth harmonic diffeomorphism between (M,h) and (N,g) is minimizing in its homotopy class. Moreover, if ∂M is nonempty or if the genus of M is strictly larger then one, then such a diffeomorphism is the unique minimizing map in its homotopy class.*

D. Related results

First, we will give an application of Theorem MM2 in the case when the energy integral is infinite.

D1. Suppose that

(a) f and g are harmonic diffeomorphisms from the Δ onto itself w.r.t. Poincaré metric.

(b) Hopf differentials φ=Hopf (f) and $\psi = \text{Hopf}(g)$ are integrable on Δ.

Since φ and ψ belong to Bers space (see, for example, [Ah2], [W] and [AMM] for definition and properties of Bers space) a result of Wau [W] shows that f and g are q.c. mappings of Δ onto itself. If, in addition, we suppose that $f = g$ on the boundary of the unit disk, an application of Theorem MM2 shows that f and g are identical.

Note that every harmonic diffeomorphism of Δ onto itself w.r.t. Poincaré metric has infinite energy integral.

The following example shows that without assumption that Hopf differentials are integrable Theorem MM1 is not valid.

D2. Let φ be the conformal mapping of the unit disk Δ onto upper half-plane H and let $\rho(w) = |\varphi'(w)|$. Next, let $g = \psi \circ h \circ \varphi$, where ψ is the inverse function of φ and h is given by $h(z) = x + iky$, $k > 0$. We leave to the reader to verify that g is q.c. harmonic mapping (w.r.t. ρ) of the unit disk Δ onto itself and that g =Id on the boundary of Δ.

Although, the metric defined by the density ρ is flat on the complex plane C except at one point, Theorem MM1 is not valid.

D3. In connection with the parts (D1) and (D2) of this section, we will give a short discussion (we follow Schoen [Sc]). There is an interesting conjecture which is due to Schoen (see also [Sc]).

Conjecture. The q.c. harmonic homeomorphisms from the unit disk Δ onto itself, w.r.t. Poincaré metric, are parameterized by the boundary values of q.c. maps of the disk.

This is a question which involves proving both an existence and a uniqueness theorem. The existence result for this ideal boundary value problem has been shown by Li and Tam [LT1] under the additional hypothesis that boundary map be sufficiently differentiable. They have also obtained counterexamples to uniqueness without the quasi-conformal hypothesis (but with continuity) and then proved the uniqueness part of Schoen's conjecture (see [LT2]).

A result of Wan [W] gives a parametrization of the q.c. harmonic homeomorphisms of Δ in terms of bounded holomorphic quadratic differentials on Δ. Wan has shown that if f is q.c. mapping then Hopf differential of f is bounded w.r.t. the Poincaré metric on Δ. Conversely, he has shown that for any bounded holomorphic quadratic differential Φ on Δ there is a unique q.c. harmonic homeomorphism $f : \Delta \to \Delta$ such that $\text{Hopf}(f) = \Phi$.

D4. Theorem MM2 remains valid if the condition (b) (in the hypotheses of Theorem MM2 is replaced by the following.

(e) f, g and their inverse mapping have L^2-derivatives.

The idea of the proof is as follows. If the condition (e) holds, then one can get that $f \circ g^{-1}$ and $g \circ f^{-1}$ have L^1-derivatives and its partial derivatives satisfy the chain rule (for a details see [LV, Lemma 6.4, p. 151]).

It is well known that the condition (b) implies the condition (e) (see, for example, [LV]).

For a development of theory of harmonic mappings by means of Sobolev spaces, we refer to Schoen–Yau book [SY].

D5. Harmonic maps and extremal QC mapping

Before we state the results, we need some notations. Suppose that f is quasiconformal mapping of the unit disk Δ onto itself. Let $k[f] = \operatorname{ess\,sup}\{|\mu_f(z)| : z \in \Delta\}$ and let $Q(f)$ denote the collection of all q.c. mappings of Δ whose pointwise boundary values on $\partial\Delta$ agree with those of f. We call f extremal (in its Teichmüller class) if $k[f] \leqslant k[g]$ for every $g \in Q(f)$. An extremal q.c. mapping f is uniquely extremal (in its Teichmüller class) if $k[f] < k[g]$ for every other g in $Q(f)$.

Theorem M7 (The first removable singularity theorem). *Suppose that*

(a) *f is q.c. mapping from Δ onto Δ*
(b) *f is a harmonic function with respect to the metric density ρ on $\Delta \setminus K$, where K is compact subset of Δ*
(c) *f is extremal in its Teichmüller class*
(d) *there are two positive constant m and M such that $m \leqslant |\varphi(z)| \leqslant M$ for each $z \in \Delta \setminus K$, where φ is Hopf differential of f.*

Then φ has an analytic extension $\tilde\varphi$ from $\Delta \setminus K$ to Δ and $\mu(z) = k|\tilde\varphi(z)|/\tilde\varphi(z)$ a.e. in Δ, where k is a constant.

Theorem M8 (The second removable singularity theorem). *Suppose that*

(a) *f is uniquely extremal q.c. mapping, in its class, from Δ onto Δ*
(b) *f is a harmonic function with respect to the metric density ρ on $\Delta \setminus K$, where K is compact subset of Δ.*

Then we have the same conclusion as in the previous theorem.

During my work with Božin and Markovć on the problems related to uniquely extremal q.c. mapping (see [BMM]), we also obtained some results of this type.

2. Estimates for the modulus of the derivatives of harmonic univalent mappings

In this section we follow closely author paper [M6]. Let U denote the unit disc and $T = \partial U$ denote the unit circle.

Theorem 2A. *Suppose that:*

(a) *h is an euclidean harmonic mapping from an open set D which contains $\overline{U}$ into $\mathbb{C}$*
(b) *$h(\overline{U})$ is a convex set in $\mathbb{C}$*
(c) *$h(U)$ contains a disc $B(a;R)$, $h(0) = a$ and $h(T)$ belongs to the boundary of $h(\overline{U})$.*
Then
(d) $|h_r(e^{i\varphi})| \geqslant R/2$, $0 \leqslant \varphi \leqslant 2\pi$.

Also, we can generalize this result to several variables.

Proposition 1. *Suppose that:*

(a′) *h is an euclidean harmonic orientation preserving univalent mapping from an open set D which contains $\overline{U}$ into $\mathbb{C}$*

(b′) *$h(\overline{U})$ is a convex set in $\mathbb{C}$*

(c′) *$h(U)$ contains a disc $B(a;R)$ and $h(0)=a$.*
Then

(d′) *$|\partial h(z)| \geqslant R/4$, $z \in U$.*

Proof of Theorem 2A. Without loss of generality we can suppose that $h(0)=0$. Let $0 \leqslant \varphi \leqslant 2\pi$ be arbitrary. Since $h(U)$ is a bounded convex set in $\mathbb{C}$, there exists $\tau \in [0,2\pi]$ such that harmonic function u, defined by $u = \mathrm{Re}H$, where $H(z) = e^{i\tau}h(z)$, has a maximum on $\overline{U}$ at $e^{i\varphi}$. Since Poisson kernel for U satisfies $P_r(\theta) \geqslant (1-r)/(1+r)$, using Poisson integral representation of the function $u(e^{i\varphi}) - u(z)$, $z \in U$, we obtain

$$u(e^{i\varphi}) - u(re^{i\varphi}) \geqslant \frac{1-r}{1+r}(u(e^{i\varphi}) - u(0)),$$

and hence (d).

Applying Maximum Principle to the analytic function ∂h, we obtain Proposition 1. As a corollary of it we obtain

Theorem 2B. *Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc onto convex domain Ω. If Ω contains a disc $B(a;R)$ and $h(0)=a$, then $|\partial h(z)| \geqslant R/4$, $z \in U$.*

As a corollary of Theorem 2B we obtain

Proposition 2. *Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc into $\mathbb{C}$ such that $f(U)$ contains a disc $B_R = B(a;R)$ and $h(0)=a$. Then*

$$|\partial h(0)| \geqslant R/4. \tag{2.1}$$

Proof. Let $V = V_R = h^{-1}(B_R)$ and φ be a conformal mapping of the unit disc U onto V such that $\varphi(0)=0$ and let $h_R = h \circ \varphi$. By Schwarz lemma

$$|\varphi'(0)| \leqslant 1. \tag{2.2}$$

Since $\partial h_R(0) = \partial h(0)\varphi'(0)$, by Proposition 2 we get $|\partial h_R(0)| = |\partial h(0)||\varphi'(0)| \geqslant R/4$. Hence, using (2.2) we get (2.1). □

Also as a corollary of Theorem 2B we obtain

Theorem 2C. (see [Ka1] and [Ka2]) *Let h be an euclidean harmonic diffeomorphism of the unit disc onto convex domain Ω. If Ω contains a disc $B(a;R)$ and $h(0)=a$ then $D(h)(z) \geqslant R^2/16$, $z \in U$, where $D(h)(z) = |\partial h(z)|^2 + |\overline{\partial} h(z)|^2$.*

The following example shows that Theorem 2A, Proposition 1, Theorem 2B, Proposition 2 and Theorem 2C are not true if we omit the condition $h(0) = a$.

EXAMPLE. The mapping $\varphi_b(z) = \dfrac{z-b}{1-\bar{b}z}$, $|b| < 1$, is a conformal automorphism of the unit disc onto itself and

$$|\varphi_b'(z)| = \frac{1-|b|^2}{|1-\bar{b}z|^2}, \; z \in U.$$

In particular $\varphi_b'(0) = 1 - |b|^2$.

Heinz proved (see [H]) that if h is a harmonic diffeomorphism of the unit disc onto itself such that $h(0) = 0$, then $D(h)(z) \geqslant 1/\pi^2$, $z \in U$.

Using Proposition 2 we can prove Heinz theorem:

THEOREM 2D (Heinz). *There exists no euclidean harmonic diffeomorphism from the unit disc U onto $\mathbb{C}$.*

Note that this result was a key step in his proof of the Bernstein theorem for minimal surfaces in $\mathbb{R}^3$.

Schoen obtained a nonlinear generalization of Proposition 2 by replacing the target by complete surface of nonnegative curvature (see Proposition 2.4 of [Sc]) and using this result he proved

THEOREM 2E (Schoen). *There exists no harmonic diffeomorphism from the unit disc onto a complete surface (S, ρ) of nonnegative curvature $K_\rho \geqslant 0$.*

Let f be a harmonic diffeomorphism from B_r to (S, ρ) and $\operatorname{dist}(f(0), \partial(f(B_r))$ $\geqslant R$. Then it suffices to show that $|df|^2(0) \geqslant CR^2/r^2$, where C is a universal constant. By hypothesis, we have $|\partial f| > |\overline{\partial} f| \geqslant 0$ and

$$\Delta \ln |\partial f| = -K_\rho J_f \leqslant 0.$$

$\lambda = |\partial f|^2 |dz|^2$. Therefore $\operatorname{dist}(0, \partial(B_r) \geqslant \frac{1}{2} \operatorname{dist}(f(0), \partial(f(B_r)) \geqslant \frac{1}{2} R$.

LEMMA 1. *If σ is a metric density of nonnegative curvature $K_\sigma \geqslant 0$ on B_r and $d = \operatorname{dist}_\sigma(0, T_r)$, then $\sigma(0) \geqslant Cd^2/r^2$, where C is a universal constant.*

A proof can be given by means the estimate of harmonic function in terms of curvature (Cheng–Yau, CPAM 28 (1975), 333–354). We apply this lemma to metric density $\lambda = |\partial f|^2$. By the above estimate we have $|\partial f|^2(0) \geqslant CR^2/r^2$. This proves the theorem.

QUESTION. Can one prove Lemma 1 elementarily? Note that $\ln \sigma$ is superharmonic function. Therefore $\ln \frac{1}{\sigma}$ and $\frac{1}{\sigma}$ are subharmonic functions.

3. A version of Bloch theorem

Besides above mentioned, for further results related to the subject of this paper and in particular to this section we refer the interested reader to author's papers [**M5**], [**M7**] and [**M6**] (see also author's review papers [**M9**] and [**M8**]). For example, in [**M5**], using a version of Bloch theorem (see Lemma 3.1 below) we give a short proof of a Dyakonov's theorem [**Dyk**]. Also we show that Lemma 3.1 holds for quasiregular harmonic functions (see Theorem 3.1 below).

Let U denote the unit disc in the complex plane. If z and w are complex numbers by $\Lambda(z,w)$ we denote the half-line $\Lambda(z,w)=\{z+\rho(w-z):\rho\geqslant 0\}$ and $\Lambda(w)=\Lambda(0,w)$.

LEMMA 3.1. *Suppose that f is an analytic function on the unit disc U, $f(0)=0$ and $|f'(0)|\geqslant 1$. Then there is an absolute constant s such that for every $\theta\in\mathbb{R}$ there exists a point w on the half-line $\Lambda_\theta=\Lambda(0,e^{i\theta})=\{\rho e^{i\theta}:\rho\geqslant 0\}$, which belongs to $f(U)$, such that $|w|\geqslant 2s$.*

Since an analytic function is a quasiregular harmonic mapping, Lemma 3.1 is a special case of Theorem 3.1 (the main result).

The example $f_n(z)=e^{nz}/n$ shows that under condition of Lemma 1 there is no absolute constant s such that the disc $B(0,s)$ belongs to $f(U)$.

If $0<\alpha<1$ and f is a complex function defined on a domain Ω we say that f belongs $\operatorname{Lip}\alpha$ if $|f(z)-f(w)|\leqslant c|z-w|^\alpha$ for some $c=c_f<\infty$ and for all $z,w\in\Omega$.

LEMMA 3.2. *Let f be an analytic function on U. Then*

$$(1-|z|)|f'(z)|\leqslant K\omega_{|f|}(1-|z|),\ z\in U, \tag{3.1}$$

where $K=1/s$ is the absolute constant and $w_{|f|}$ is the modulus of continuity of $|f|$.

PROOF. Let $z\in U$, $r=(1-|z|)/2$, $w=f(z)$ and $\tilde{B}=f(B)$. By Lemma 3.1 there is a point w_1, belonging to $\tilde{B}\cap\Lambda(w)$, such that

$$|w_1-w|>2s|f'(z)|r. \tag{3.2}$$

Let z_1 be preimage of w_1. Since $|w_1-w|=||w_1|-|w||=|f(z_1)-f(z)|$ and $|z_1-z|\leqslant r$, then (3.1) follows from (3.2). □

Using Lemma 3.2 and known result (see for example [**Rud**], Lemma 6.4.8) we get Dyakonov's theorem: If f is an analytic function on U then f belongs $\operatorname{Lip}\alpha$ if and only if $|f|$ belongs $\operatorname{Lip}\alpha$. Using Schwarz lemma, Pavlović [**Pa**] found simple proof of Dyakonov's theorem. In addition, a very elementary proof of Dyakonov's theorem in some special cases has been given by Kaljaj [**Ka**]. We can show that Lemma 3.1 (and therefore Dyakonov's theorem) is true for classes of functions which include pseudo-holomorphic functions, real harmonic functions of several variables, holomorphic functions of several variables, quasiregular harmonic mappings, etc.

Now we will show that Lemma 3.1 holds for quasiregular harmonic functions. For basic definitions and results we refer to [**LV**] book. First, we need to prove Lemma 3.3. For definition of φ_K see [**LV**] and for definition of quasiregular function see [**AMM**].

LEMMA 3.3. *Let f be a K-quasiregular mapping from the unit disc U into hyperbolic domain G. Then*

$$\tanh\rho_G(f(z_1),f(z_2))\leqslant\varphi_K(\tanh\rho_U(z_1,z_2)),\ z_1,z_2\in U. \tag{3.3}$$

PROOF. Since $f = F \circ g$, where F is an analytic function from U into G and g is K-quasiconformal mapping from U onto itself, (3.3) follows from inequality 3.13 of [**LV**]. $\square$

THEOREM 3.1. *Suppose that f is a K-quasiregular harmonic mapping on the unit disc U, $f(0) = 0$ and $|\operatorname{grad} f(0)| \geqslant 1$. Then, there exists an absolute constant α such that for every $\theta \in \mathbb{R}$ there exists a point w on the half-line $\Lambda_\theta = \Lambda(0, e^{i\theta}) = \{\rho e^{i\theta} : \rho \geqslant 0\}$, which belongs to $f(U)$, such that $|w| \geqslant 2\alpha$.*

PROOF. If we suppose that this result is not true, then there is a sequence of positive numbers a_n, converging to zero, and a sequence of K-quasiregular functions f_n, such that $f_n(U)$ does not intersect $[a_n, +\infty)$, $n \geqslant 1$. Next, the functions $g_n = f_n/a_n$ map U into $G = \mathbb{C} \setminus [1, +\infty)$ and hence, by inequality (3.3), the sequence g_n is equicontinuous and therefore forms normal family. Thus, there is a subsequence, which we denote again by g_n, which converges uniformly on compact subset of U to quasiregular harmonic function g, together with partial derivatives. Since $\operatorname{grad} g_n(0)$ converges to $\operatorname{grad} g(0)$ and $|\operatorname{grad} g_n(0)| = |\operatorname{grad} f_n(0)|/a_n$ converges to infinity, we have a contradiction. $\square$

After writing this paper we have found very simple proof of the following result which seems as an appropriate generalization of Koebe one-quarter theorem (with the best constant 1/4) .

THEOREM 3.2. *Suppose that f is an analytic function on the unit disc $\overline{U}$, $f(0) = 0$ and $|f'(0)| \geqslant 1$. Then for every $\theta \in \mathbb{R}$ there exists a point w on the half-line $\Lambda_\theta = \Lambda(0, e^{i\theta}) = \{\rho e^{i\theta} : \rho \geqslant 0\}$, which belongs to $f(\overline{U})$, such that $|w| \geqslant 1/4$.*

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References

[Ah1] L. V. Ahlfors, *Conformal Invariants*, McGraw-Hill, 1973.

[Ah2] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Princeton, 1966.

[AMM] I. Anić, V. Marković, M. Mateljević, *Uniformly Bounded Maximal Φ-disks, Bers Spaces and Harmonic Maps*, Proc. Amer. Math. Soc. **128** (2000), 2947–2956.

[AMŠ] I. Anić, M. Mateljević, D. Šarić, *Extremal metrics and modules*, Czech. Math. J. (2000), 52(127), No 2, pp. 225-235.

[As] V. V. Aseev, *Remark on the paper by Anić, I., Mateljević, M. and Šarić, D.*, A letter.

[BA] L. Beurling, L. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.

[BLMM] V. Božin, N. Lakić, V. Marković, M. Mateljević, *The unique extremality*, J. Anal. Math. **75** (1998), 299–338.

[BMM] V. Božin, V. Marković, M. Mateljević, *The unique extremality in the tangent space of Teichmüller space*, Proc. Internat. Conf. "Generalized Functions – Linear and Nonlinear Problems", Novi Sad, Aug. 31 – Sep. 04, 1966; Integral Transforms and Special Functions **6** (1997), 223–227.

[C] R. Courant, *Dirichlet's principle, conformal mappings and minimal surfaces*, Interscience, New York, 1950.

[CH] J. M. Coron, and F. Helein, *Harmonic diffeomorphisms, minimizing harmonic maps and rotational symetry*, Composito Math. **69** (1989), 175–228.

[Dyk] K. M. Dykonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. **178** (1997), 143–167.

[EL] C. Earle, J. Li Zhong, *Extremal quasiconformal mappings in plane domains*, Quasiconformal Mappings and Analysis, A Collection of Papers Honoring F. W. Gehring, Peter Duren, et all. Eds., Springer-Verlag, 1998, pp. 141–157.

[EL1] J. Eells, L. Lemaire, *A report of harmonic maps*, Bull. London Math. Soc. **10** (1978), 109–160.

[EL2] J. Eells, L. Lemaire, *Selected topics in harmonic maps*, AMS Conference Board **50** (1980).

[Ga] F. P. Gardiner, *Teichmüller theory and quadratic differentials*, Wiley–Interscience, New York, 1987.

[Ge1] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 383–393.

[Ge2] F. W. Gehring, *Quasiconformal mappings in space*, Bull. Amer. Math. Soc. **69** (1963), ???.

[Ha] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967), 673–687.

[H] E. Heinz, *On one-to-one harmonic mappings*, Pac. J. Math. **9** (1959), 101–105.

[J] J. Jost, *Two Dimensional Geometric Variational Problems*, Wiley, 1991.

[Ka] D. Kaljaj, *On Dyakonov's paper*, unpublished manuscript, 2002.

[Ka1] D. Kaljaj, *On harmonic diffeomorphisms of the unit disc onto a convex domain*, to appear in Complex Variables Theory and Applications.

[Ka2] D. Kaljaj, *Harmonic and quasiconformal functions between convex domains*, Doctoral Thesis, 2001.

[K] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typfür quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77–105.

[L1] O. Lehto, *Schlicht functions with a quasiconformal extension*, Ann. Acad. Sci. Fenn. AI Math. **500** (1971), 1–10.

[L2] O. Lehto, *Univalent Functions and Teichmüler Spaces*, Springer-Verlag, New York, 1987.

[Lo] C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414.

[LT1] P. Li, L. F. Tam, *The heat equation and harmonic maps of complete manifolds*, Inv. Math. **105** (1991), 1–46.

[LT2] P. Li, L. F. Tam, *Uniqueness and regularity of proper harmonic maps II*, Indiana U. Math. J. **42** (1993), 593–635.

[LV] O. Lehto, K. I. Virtanen, *Quasiconformal Mapping*, Springer-Verlag, Berlin and New York, 1965.

[M] M. Mateljević, *An extension of the Area Theorem*, Complex Variables **15**, (1990), 155–157.

[M1] M. Mateljević, *An area theorem of Lehto-Kühnau type for harmonic mappings*, (unpublished manuscript 1990).

[M2] M. Mateljević, *Some geometric question related to harmonic map*, Abstract, Symposium Contemporary Mathematics, Univ. Belgrade, 18–20 Dec. 1998.

[M3] M. Mateljević, *Dirichlet's principle, uniqueness of harmonic maps and related problems*, in: *Proc. Symp. Contemporary Math.*, Univ. Belgrade, 1998, 251–267.

[M4] M. Mateljević, *Dirichlet's principle, uniqueness of harmonic maps and extremal qc mappings*, (unpublished manuscript, lectures notes, 2001); Lectures on quasiconformal mappings, Scoala Normala Superioara Buchurest, (SNSB), 2003-2004.

[M5] M. Mateljević, *A version of Bloch theorem for quasiregular harmonic mappings*, Proceedings of International Conference on Complex Analysis and Related Topics (IXth Romanian-Finnish Seminar, 2001), Rev. Roum. Math. Pures Appl. 47 (2002), 5-6, pp. 705–707.

[M6] M. Mateljević, *Estimates for the modulus of the derivatives of harmonic univalent mappings*, Proceedings of International Conference on Complex Analysis and Related Topics

(IXth Romanian-Finnish Seminar, 2001), Rev. Roum. Math. Pures Appl. 47 (2002), 5-6, pp. 709-711.

[M7] M. Mateljević, *Ahlfors–Schwarz lemma and curvature*, Kragujevac J. Math. (Zbornik radova PMF), **25** (2003), 155–164.

[M8] M. Mateljević, *The unique extremality II*, Math. Reports **2(52)**:4 (2000), 503–525.

[M9] M. Mateljević, *Dirichlet's principle, uniqueness of harmonic maps and extremal QC mappings*, in: *Three topics from contemporary mathematics*, Matematički institut, Beograd, 2004

[MM1] M. Mateljević, V. Marković, *The unique extremal QC mapping and uniqueness of Hahn-Banach extensions*, Mat. Vesnik **48** (1996), 107-112.

[MM2] V. Marković, M. Mateljević, *New version of Grötzsch principle and Reich–Strebel inequality*, Mat. Vesnik **49** (1997), 235–239.

[MM3] V. Marković, M. Mateljević, *New version of the main inequality and uniqueness of harmonic maps*, J. Anal. Math. **79** (1999).

[MM4] M. Mateljević, V. Marković, *Reich–Strebel inequality, harmonic mappings and uniquely extremal q.c. mapping*, Abstract, Nevanlinna Colloquium, Lausanne, 1997.

[Pa] M. Pavlović, *On Dyakonov's paper "Equivalent norms on Lipschitz-type spaces of holomorphic functions"*, Acta Math. (Djursholm) **183** (1999), 141–143.

[Po] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[Re1] E. Reich, *A generalized Dirichlet integral*, J. Anal. Math. **30** (1976), 456–463.

[Re2] E. Reich, *On the variational principle of Gestenhaber and Rauch*, Ann. Acad. Sci. Fenn. **10** (1985), 469–475.

[Re3] E. Reich, *The unique extremality counterexample*, J. Anal. Math. **75** (1998), 339–347.

[Re4] E. Reich, *Uniqueness of Hahn-Banach extensions from certain spaces of analytic functions*, Math. Z. **167** (1979), 81–89.

[Re5] E. Reich, *A criterion for unique extremality of Teichmüller mappings*, Indiana Univ. Math. J. **30** (1981), 441–447.

[Re6] E. Reich, *On criteria for unique extremality for Teichmüller mappings*, Ann. Acad. Sci. Fenn **6** (1981), 289–301.

[Re7] E. Reich, *On the uniqueness question for Hahn–Banach extensions from the space of L^1 analytic functions*, Proc. Amer. Math. Soc. **88** (1983), 305–310.

[Re8] E. Reich, *Extremal extensions from the circle to the disc*, Quasiconformal Mappings And Analysis, A Collection Of Papers Honoring F. W. Gehring, Peter Duren et all. Eds., Springer-Verlag, 1998, 321–335.

[Re9] E. Reich, *Extremal Quasiconformal Mapping of the Disk*, Chapter 3 in *Geometric Function Theory, Volume 1*, Edited by R. Kühnau, Elsevier, 2002.

[ReS1] E. Reich, K. Strebel, *On quasiconformal mapping which keep the boundary points fixed*, Trans. Amer. Math. Soc. **136** (1969), 211–222.

[ReS2] E. Reich, K. Strebel, *On the Gerstenhaber–Rauch principle*, Israel J. Math. **57** (1987), 89–100.

[Ru] W. Rudin, *Real and complex analysis*, McGraw-Hill, 1966.

[Rud] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[Sa] J.K. Sampson, *Some properties and application of harmonic mappings*, Ann. École Norm. Sup. **11** (1978), 211–228.

[Sc] R. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Lecture Notes in Pure and Appl. Math. **143** (1993), 179–200.

[SY] R. Schoen, S. T. Yau, *Lectures on Harmonic Maps*, Conf. Proc. and Lect. Not. in Geometry and Topology, Vol. II, Inter. Press, 1997.

[S1] K. Strebel, *Zur Frage der Eidentigkeit extremaler quasikonformer Abbildungen des Einheitskreiss II*, Coment. Math. Helv. **39** (1964), 77–89.

[S2] K. Strebel, *Quadratic Differential*, Springer-Verlag, 1984.

[S3] K. Strebel, *Eine Abschätzung der Länge gewisser Kurven bei quasikonformer Abbildung*, Ann. Acad. Sci. Fenn. **243** (1957), 1–19.

[S4] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Anal. Math. **30** (1976), 464–480.

[S5] K. Strebel, *On quasiconformal mappings of open Riemann surfaces*, Comment. Math. Helv. **53** (1978), 301–321

[S6] K. Strebel, *Extremal quasiconformal mappings*, Resultate der Mathematik **10** (1986), 168–210.

[Sh] Y. L. Shen, *Extremal Problems for Quasiconformal Mappings*, J. Math. Anal. Appl. **247**, (2000), 27-44.

[W] T. Wan, *Constant mean curvature surfaces, harmonic maps and universal Teichmüller space*, J. Diff. Geom. **35** (1992), 643–657.

[W1] T. Wan, *Harmonic diffeomorphisms between complete noncompact surfaces*, unpublished manuscript (Lecture notes), July 1999.

[We] H. B. Wei, *On the uniqueness problem of harmonic mappings*, Proc. Amer. Math. Soc., **124** (1996), 2337–2341.

[LV] O. Lehto, and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973.

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