PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(89) (2004), 87–94

HÖLDER SPACES OF QUASICONFORMAL MAPPINGS

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Abstract. We prove that a K-quasiconformal mapping belongs to the little Hölder space $c^{0,1/K}$ if and only if its local modulus of continuity has an appropriate order of vanishing at every point. No such characterization is possible for Hölder spaces with exponent greater than $1/K$.

1. Introduction

Let Ω denote a domain in $\mathbb{C}$, and let $f:\Omega\to\mathbb{C}$ be a continuous complex-valued function. Given $E\subset\Omega$, define the modulus of continuity of $f_{|E}$ by

$$\omega_f(E,\delta)=\sup\{|f(z_1)-f(z_2)|:z_1,z_2\in E,|z_1-z_2|\leqslant\delta\}.$$

For $0<\alpha<1$ we consider the Hölder space

$$C^{0,\alpha}(E)=\{f:E\to\mathbb{C}:\sup_{\delta>0}\delta^{-\alpha}\omega_f(E,\delta)<\infty\},$$

with the seminorm

$$\|f\|_{E,\alpha}=\limsup_{\delta\to 0}\delta^{-\alpha}\omega_f(E,\delta).$$

This seminorm vanishes on the little Hölder space

$$c^{0,\alpha}(E)=\{f\in C^{0,\alpha}(E):\|f\|_{E,\alpha}=0\}.$$

Furthermore, define $C^{0,\alpha}_{\mathrm{loc}}(\Omega)=\bigcap_{E\Subset\Omega}C^{0,\alpha}(E)$ and similarly for $c^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

We can also consider the local modulus of continuity at a point $z\in\Omega$:

$$\omega_f(z,\delta)=\sup\{|f(\zeta)-f(z)|:\zeta\in\Omega,|\zeta-z|\leqslant\delta\}.$$

If U is a neighborhood of z in Ω, then $\omega_f(z,\delta)\leqslant\omega_f(U,\delta)$ for all sufficiently small $\delta>0$. In particular,

$$\limsup_{\delta\to 0}\delta^{-\alpha}\omega_f(z,\delta)\leqslant\|f\|_{U,\alpha}. \tag{1.1}$$

2000 *Mathematics Subject Classification*: Primary 30C62; Secondary 26B35.
Key words and phrases: Quasiconformal mappings, Hölder spaces, linear dilatation, modulus of continuity.

Inequality (1.1) provides a simple necessary condition for a continuous mapping $f : \Omega \to \mathbb{C}$ to be in the class $c^{0,\alpha}_{\mathrm{loc}}(\Omega)$; namely,

$$f \in c^{0,\alpha}_{\mathrm{loc}}(\Omega) \implies \limsup_{\delta\to 0} \delta^{-\alpha}\omega_f(z,\delta) = 0 \quad \forall z \in \Omega. \tag{1.2}$$

This condition can be helpful because it is often easier to estimate $\omega_f(z,\delta)$ for $z \in \Omega$ than to estimate $\omega_f(E,\delta)$ for all $E \Subset \Omega$. Unfortunately, the implication in (1.2) cannot be reversed in general.

The present paper deals with the following question: is the reverse implication in (1.2) true under the additional assumption that f is a K-quasiconformal mapping from Ω to $\mathbb{C}$? It is well-known that under this assumption f belongs to $C^{0,1/K}_{\mathrm{loc}}(\Omega)$ [**1, 3, 7**], but not necessarily to $c^{0,1/K}_{\mathrm{loc}}(\Omega)$ (for example, $f(z) = |z|^{1/K-1}z$ is K-quasiconformal in $\mathbb{C}$, but $f \notin c^{0,1/K}_{\mathrm{loc}}(\mathbb{C})$). Therefore, our question is nontrivial only when $1/K \leqslant \alpha < 1$.

The answer turns out to be affirmative in the case $\alpha = 1/K$ (Theorem 2.1) and negative in the case $1/K < \alpha < 1$ (Proposition 2.1).

2. Main results

We start by showing that in general one cannot determine the degree of Hölder continuity of a quasiconformal mapping from its local behavior. More precisely, the following proposition exhibits a K-quasiconformal mapping which has linear local modulus of continuity at every point, yet does not belong to $c^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with α arbitrarily close to $1/K$. We use notations $\mathbb{D}(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ and $\mathbb{D} = \mathbb{D}(0,1)$.

PROPOSITION 2.1. *Given $K > 1$ and $1/K < \alpha < 1$, there exists a K-quasiconformal automorphism $f : \mathbb{D} \to \mathbb{D}$ such that $f \notin c^{0,\alpha}_{\mathrm{loc}}(\mathbb{D})$, but*

$$\limsup_{\zeta\to z} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} < \infty \tag{2.1}$$

for every $z \in \mathbb{D}$.

PROOF. Choose $\varepsilon > 0$ so that $(\alpha-\varepsilon)/(1-\varepsilon) = 1/K$. Consider two sequences of open disks $D_n = \mathbb{D}(2^{-n}, 2^{-(n+2)})$ and $D'_n = \mathbb{D}(2^{-n}, 2^{-(n+2)/\varepsilon})$, $n \geqslant 1$. We will define f separately on D'_n, $D_n \setminus D'_n$ and $\mathbb{D} \setminus \bigcup_{n=1}^{\infty} D_n$. Each disk D'_n is stretched under f by the factor of $2^{(1-\alpha)(n+2)/\varepsilon}$:

$$f(2^{-n} + re^{i\varphi}) = 2^{-n} + 2^{(1-\alpha)(n+2)/\varepsilon} re^{i\varphi}, \quad 0 \leqslant r < 2^{-(n+2)/\varepsilon},\ \varphi \in \mathbb{R}.$$

Thus $f(D'_n)$ is a disk that is concentric with D_n and has the radius $2^{-\alpha(n+2)/\varepsilon} < 2^{-(n+2)}$. Hence $f(D'_n) \subset D_n$. Next, f maps the annulus $D_n \setminus D'_n$ onto $D_n \setminus f(D'_n)$ by means of the "extremal K-quasiconformal stretch mapping" (cf. [**5**, p.63]).

$$f(2^{-n} + re^{i\varphi}) = 2^{-n} + 2^{(n+2)(1/K-1)} r^{1/K} e^{i\varphi}, \quad 2^{-(n+2)/\varepsilon} \leqslant r < 2^{-n-2},\ \varphi \in \mathbb{R}.$$

Finally, let $f(z) = z$ for $z \notin \bigcup_{n=1}^{\infty} D_n$. It is easy to see that f is continuous and thus K-quasiconformal in $\mathbb{D}$. It is also evident that f is locally Lipschitz in $\mathbb{D} \setminus \{0\}$, which implies that (2.1) holds for $z \in \mathbb{D} \setminus \{0\}$. To verify (2.1) for $z = 0$, observe that f maps each disk D_n onto itself. Hence for every $\zeta \in D_n$ we have

$$\frac{|f(\zeta)|}{|\zeta|} \leqslant \frac{2^{-n} + 2^{-n-2}}{2^{-n} - 2^{-n-2}} = \frac{5}{3}.$$

Thus (2.1) holds for all $z \in \mathbb{D}$.

Now let $a_n = 2^{-n} + 2^{-(n+2)/\varepsilon}$ and $b_n = 2^{-n}$, $n \geqslant 1$. By the definition of f we have

$$\begin{aligned} f(a_n) &= 2^{-n} + 2^{(n+2)(1/K-1)} \left(2^{-(n+2)/\varepsilon}\right)^{1/K} = 2^{-n} + 2^{(n+2)((\varepsilon-1)/\varepsilon K-1)} \\ &= 2^{-n} + 2^{-\alpha(n+2)/\varepsilon} \end{aligned}$$

and $f(b_n) = 2^{-n}$. Since

$$\frac{|f(a_n) - f(b_n)|}{|a_n - b_n|^\alpha} = \frac{2^{-\alpha(n+2)/\varepsilon}}{2^{-\alpha(n+2)/\varepsilon}} = 1,$$

it follows that for every $r > 0$ the mapping f fails to be in $c^{0,\alpha}(\mathbb{D}(0,r))$. □

Surprisingly, the situation is different for the critical Hölder exponent $1/K$. According to the following theorem, one can determine if a K-quasiconformal mapping belongs to $c^{0,1/K}$ just by looking at its local modulus of continuity. Its proof uses some ideas from [**4**].

THEOREM 2.1. *Let $f : \Omega \to \mathbb{C}$ be a K-quasiconformal mapping, and let E be a compact subset of Ω. Then $f \in c^{0,1/K}(E)$ if and only if for every $z \in E$*

$$\lim_{\substack{\zeta \to z \\ \zeta \in E}} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^{1/K}} = 0. \tag{2.2}$$

PROOF. If $f \in c^{0,1/K}(E)$, then(2.2) follows immediately from the definition of $c^{0,1/K}(E)$. Conversely, suppose that $f \notin c^{0,1/K}(E)$, i.e. $\|f\|_{E,1/K} > 0$. Our goal is to prove that (2.2) fails for some $z \in E$.

By the definition of $\|f\|_{E,1/K}$ there exists a sequence $\delta_j \to 0$ and points $a_j, b_j \in E$ such that $|a_j - b_j| = \delta_j$ and

$$|f(a_j) - f(b_j)| = \|f\|_{E,1/K}\delta_j^{1/K}(1 + o(1)), \quad j \to \infty. \tag{2.3}$$

Without loss of generality we may assume that $a_j \to 0 \in E$, $\overline{\mathbb{D}}(a_j, \delta_j) \subset \mathbb{D}$ for every j, $\overline{\mathbb{D}} \subset \Omega$, and $f(0) = 0$. Since f is continuous in $\overline{\mathbb{D}}$, the domain $\Omega' = f(\mathbb{D})$ is bounded. Let $R = \operatorname{diam} \Omega'$ be its diameter.

The set $F_j = f(\overline{\mathbb{D}}(a_j, \delta_j))$ is connected and its diameter is controlled by (2.3). We are going to use this information to estimate its capacity from below. On the other hand, the quasiconformality of f will lead to an upper bound for the capacity of F_j. Comparison of the two estimates will show that f satisfies the hypotheses of [**2**, Thm.1], which in turn implies that (2.2) fails for $z = 0$.

Let us begin by defining the conformal capacity of a compact set E with respect to a domain $\Omega \supset E$.

$$\operatorname{cap}(\Omega, E) = \inf \left\{ \int_\Omega |\nabla u(z)|^2 \, d\mathcal{L}^2(z) : u \in C_0^\infty(\Omega) \text{ and } u \geqslant 1 \text{ on } E \right\}, \tag{2.4}$$

where $\mathcal{L}^2$ is the 2-dimensional Lebesgue measure. Since $\Omega' \subset \mathbb{D}(f(a_j), R)$, it follows from (2.4) that $\operatorname{cap}(\Omega', F_j) \geqslant \operatorname{cap}(\mathbb{D}(f(a_j), R), F_j)$. Observe that $\mathbb{D}(f(a_j), R) \setminus F_j$ is a doubly-connected domain. There is another well-known conformal invariant associated with such objects, namely, the ring module [**7**, 5.49]. It can be defined as follows: $M(\mathbb{D}(f(a_j), R) \setminus F_j) = \log(r_2/r_1)$ if $\mathbb{D}(f(a_j), R) \setminus F_j$ is conformally equivalent to the circular ring $\{z : r_1 < |z| < r_2\}$. The relation between capacity and module is given by

$$\operatorname{cap}(\mathbb{D}(f(a_j), R), F_j) = \frac{2\pi}{M(\mathbb{D}(f(a_j), R) \setminus F_j)}$$

(compare [**7**, 7.8] with [**7**, 5.49]).

Since F_j is connected and contains both $f(a_j)$ and $f(b_j)$, the Grötzsch module theorem [**5**, p.54] and the estimate (2.10) in [**5**, p.61] imply

$$M(\mathbb{D}(f(a_j), R) \setminus F_j) \leqslant \log(4R/|f(a_j) - f(b_j)|).$$

Hence

$$\operatorname{cap}(\Omega', F_j) \geqslant \operatorname{cap}(\mathbb{D}(f(a_j), R), F_j) \geqslant \frac{2\pi}{\log(4R/|f(a_j) - f(b_j)|)}. \tag{2.5}$$

Now plug (2.3) into (2.5) to obtain

$$\begin{aligned}
\operatorname{cap}(\Omega', F_j) &\geqslant \frac{2\pi}{\log(4R/\|f\|_{E,1/K}) + K^{-1}\log(1/\delta_j) + o(1)} \\
&= \frac{2\pi K}{\log(1/\delta_j)} \left(1 + K\frac{\log(4R/\|f\|_{E,1/K})}{\log(1/\delta_j)} + o\left(\frac{1}{\log(1/\delta_j)}\right)\right)^{-1} \\
&= \frac{2\pi K}{\log(1/\delta_j)} \left(1 - K\frac{\log(4R/\|f\|_{E,1/K})}{\log(1/\delta_j)} + o\left(\frac{1}{\log(1/\delta_j)}\right)\right).
\end{aligned}$$

Let $C = 2\pi K^2 \log(4R/\|f\|_{E,1/K}) + 1$; then for all sufficiently large j we have

$$\operatorname{cap}(\Omega', F_j) \geqslant \frac{2\pi K}{\log(1/\delta_j)} - \frac{C}{(\log(1/\delta_j))^2}. \tag{2.6}$$

To obtain an upper bound for $\operatorname{cap}(\Omega', F_j)$, we proceed as follows. Let $g : \Omega' \to \mathbb{D}$ be the inverse of f and define

$$u(w) = \frac{\log^+\{(1 - |a_j|)/|g(w) - a_j|\}}{\log\{(1 - |a_j|)/\delta_j\}}$$

for $w \in \Omega'$. (Here $\log^+ t = \max\{\log t, 0\}$.) It is easy to see that the function u is Hölder continuous in $\Omega' \setminus F_j$, $\min\{u, 1\} \in W_0^{1,2}(\Omega')$, and $u_{|F_j} \geqslant 1$. Therefore,

$$\begin{aligned}\operatorname{cap}(\Omega', F_j) &\leqslant \int_{\Omega' \setminus F_j} |\nabla u(w)|^2 \, d\mathcal{L}^2(w) \\ &\leqslant (\log\{(1 - |a_j|)/\delta_j\})^{-2} \int_{\Omega' \setminus F_j} |\nabla \log |g(w) - a_j||^2 \, d\mathcal{L}^2(w).\end{aligned} \tag{2.7}$$

At the points where $\log |g - a_j|$ is differentiable, its gradient can be written in terms of the complex differential operators ∂ and $\bar{\partial}$.

$$\begin{aligned}|\nabla \log |g - a_j||^2 = 4|\partial \log |g - a_j||^2 &= |\partial \log(g - a_j) + \partial \log \overline{(g - a_j)}|^2 \\ &= \left| \frac{\partial g}{g - a_j} + \overline{\left(\frac{\bar{\partial} g}{g - a_j} \right)} \right|^2 .\end{aligned}$$

Since $\partial g(w)_{|w=f(z)} = \overline{\partial f(z)} J_f(z)^{-1}$ and $\bar{\partial} g(w)_{|w=f(z)} = -\bar{\partial} f(z) J_f(z)^{-1}$, we can express the last integral in (2.7) in terms of the complex dilatation $\mu = \bar{\partial} f / \partial f$. Indeed, using notation $\varphi_j = \arg(z - a_j)$, we have

$$\begin{aligned}&\int_{\Omega' \setminus F_j} |\nabla \log |g(w) - a_j||^2 \, d\mathcal{L}^2(w) \\ &\qquad = \int_{\mathbb{D} \setminus \overline{\mathbb{D}}(a_j, \delta_j)} \left| \frac{\overline{\partial f(z)}}{(z - a_j) J_f(z)} - \overline{\left(\frac{\bar{\partial} f(z)}{(z - a_j) J_f(z)} \right)} \right|^2 J_f(z) \, d\mathcal{L}^2(z) \\ &\qquad = \int_{\mathbb{D} \setminus \overline{\mathbb{D}}(a_j, \delta_j)} \frac{|\partial f(z) - e^{-2i\varphi_j} \bar{\partial} f(z)|^2}{|\partial f(z)|^2 - |\bar{\partial} f(z)|^2} |z - a_j|^{-2} \, d\mathcal{L}^2(z) \\ &\qquad = \int_{\mathbb{D} \setminus \overline{\mathbb{D}}(a_j, \delta_j)} \frac{|1 - e^{-2i\varphi_j} \mu(z)|^2}{1 - |\mu(z)|^2} |z - a_j|^{-2} \, d\mathcal{L}^2(z).\end{aligned}$$

This, together with (2.6) and (2.7), yields

$$\begin{aligned}&\int_{\mathbb{D} \setminus \overline{\mathbb{D}}(a_j, r_j)} \frac{|1 - e^{-2i\varphi_j} \mu(z)|^2}{1 - |\mu(z)|^2} |z - a_j|^{-2} \, d\mathcal{L}^2(z) \\ &\qquad\qquad \geqslant (\log\{(1 - |a_j|)/\delta_j\})^2 \left(\frac{2\pi K}{\log(1/\delta_j)} - \frac{C}{(\log(1/\delta_j))^2} \right)\end{aligned} \tag{2.8}$$

for large j. Since $a_j \to 0$, it follows that

$$(\log\{(1 - |a_j|)/\delta_j\})^2 = (\log(1/\delta_j))^2 + o(\log(1/\delta_j)), \quad j \to \infty$$

Hence the right-hand side of (2.8) is bounded from below by

$$2\pi K \log(1/\delta_j) - C + o(1), \quad j \to \infty.$$

For all sufficiently large j we have

$$\int_{\mathbb{D} \setminus \overline{\mathbb{D}}(a_j, r_j)} \frac{|1 - e^{-2i\varphi_j} \mu(z)|^2}{1 - |\mu(z)|^2} |z - a_j|^{-2} \, d\mathcal{L}^2(z) \geqslant 2\pi K \log(1/\delta_j) - C_1,$$

where $C_1 = C + 1$. Since

$$\int_{\mathbb{D}\setminus\overline{\mathbb{D}}(a_j,\delta_j)} \frac{d\mathcal{L}^2(z)}{|z-a_j|^2} \leqslant \int_{\mathbb{D}(a_j,2)\setminus\overline{\mathbb{D}}(a_j,\delta_j)} \frac{d\mathcal{L}^2(z)}{|z-a_j|^2} = 2\pi\log(2/\delta_j),$$

it follows that

$$\int_{\mathbb{D}\setminus\overline{\mathbb{D}}(a_j,r_j)} \left(K - \frac{|1-e^{-2i\varphi_j}\mu(z)|^2}{1-|\mu(z)|^2}\right) |z-a_j|^{-2}\, d\mathcal{L}^2(z) \leqslant C_1 + 2\pi K\log 2.$$

Note that the integrand is non-negative because $|\mu| \leqslant (K-1)/(K+1)$ for K-quasiconformal mappings. (See also Proposition 2.2 below.) This allows us to pass to the limit $j\to\infty$ using Fatou's lemma, thus obtaining

$$\int_{\mathbb{D}} \left| K - \frac{|1-e^{-2i\varphi}\mu(z)|^2}{1-|\mu(z)|^2}\right| |z|^{-2}\, d\mathcal{L}^2(z) < \infty, \tag{2.9}$$

where $\varphi = \arg z$. By (2.9) and Proposition 2.2

$$\int_{\mathbb{D}} \left| K^{-1} - \frac{|1+e^{-2i\varphi}\mu(z)|^2}{1-|\mu(z)|^2}\right| |z|^{-2}\, d\mathcal{L}^2(z) < \infty. \tag{2.10}$$

By virtue of (2.9) and (2.10) we can apply Theorem 1 of [**2**] which asserts that there exists $A>0$ such that $|f(z)|/|z|^{1/K} \to A$ as $z\to 0$. This leads to the conclusion that (2.2) does not hold at the point $z=0$, because 0 is a non-isolated point of the set E. □

PROPOSITION 2.2. *If $\nu\in\mathbb{C}$ and $K\geqslant 1$ are such that $|\nu|\leqslant (K-1)/(K+1)$, then*

$$0 \leqslant \frac{|1+\nu|^2}{1-|\nu|^2} - \frac{1}{K} \leqslant K - \frac{|1-\nu|^2}{1-|\nu|^2}.$$

PROOF. The first inequality follows from

$$\frac{|1+\nu|^2}{1-|\nu|^2} \geqslant \frac{(1-|\nu|)^2}{1-|\nu|^2} = \frac{1-|\nu|}{1+|\nu|} \geqslant \frac{1}{K},$$

while the second one follows from

$$\frac{|1+\nu|^2}{1-|\nu|^2} + \frac{|1-\nu|^2}{1-|\nu|^2} = 2\frac{1+|\nu|^2}{1-|\nu|^2} \leqslant 2\frac{(K+1)^2+(K-1)^2}{(K+1)^2-(K-1)^2} = K + \frac{1}{K}.$$

□

It was recently proved [**4**] that for a K-quasiconformal mapping f the limit $\lim_{\zeta\to z}|f(\zeta)-f(z)|/|\zeta-z|^{1/K}$ exists at every point z in its domain of definition. At the points where this limit is positive, the linear dilatation of f

$$H_f(z) = \limsup_{r\to 0}\sup_{z_1,z_2}\left\{\frac{|f(z_1)-f(z)|}{|f(z_2)-f(z)|} : |z_1-z| = r = |z_2-z|\right\}.$$

is evidently equal to 1. Thus we arrive at the following corollary.

COROLLARY 2.1. *For a K-quasiconformal mapping $f:\Omega\to\mathbb{C}$, one of the following statements is true:* (a) $f\in c^{0,1/K}_{\mathrm{loc}}(\Omega)$; (b) $H_f(z)=1$ *for some* $z\in\Omega$.

It is likely that the following quantitative version of Theorem 2.1 is true.

CONJECTURE 2.1. *Let $f : \Omega \to \mathbb{C}$ be a K-quasiconformal mapping, and let E be a compact subset of Ω. Then*

$$\|f\|_{E,1/K} = \sup_{z\in E} \lim_{\substack{\zeta\to z\\ \zeta\in E}} \frac{|f(\zeta)-f(z)|}{|\zeta-z|^{1/K}}. \tag{2.11}$$

It is obvious that the right-hand side of (2.11) does not exceed $\|f\|_{E,1/K}$, but the reverse inequality seems much harder to prove.

3. Concluding remarks

As Corollary 2.1 indicates, there is a tight connection between the modulus of continuity of a quasiconformal mapping and its linear dilatation. Recall that the linear dilatation H_f of a K-quasiconformal mapping f can exceed K (see [**5**] or [**6**], where the sharp upper bound for H_f is found). On the other hand, $H_f(z) \leqslant K$ if f has a non-zero derivative at z [**6**]. Also, $H_f(z) = 1$ if the upper limit

$$\limsup_{\zeta\to z} \frac{|f(\zeta)-f(z)|}{|\zeta-z|^{1/K}}$$

is strictly positive [**4**]. This naturally leads to the following question: what is the exact value of

$$H(\alpha) = \sup\Big\{H_f(z):\ f \text{ is } K\text{-qc and } \limsup_{\zeta\to z} \frac{|f(\zeta)-f(z)|}{|\zeta-z|^{\alpha}} > 0\Big\}$$

for α between $1/K$ and K? The function H increases from $H(1/K) = 1$ to $H(K) = \lambda(K)$ (as defined in [**5**, II(6.4)] or [**6**, (11)]). Apparently, none of its intermediate values are known, although it seems likely that $H(1) = K$.

Note that the authors of [**6**] use a symmetrization argument to show that

$$\sup_f H_f(z) = \sup_f \limsup_{\zeta\to 0} \frac{|f(z+\zeta)-f(z)|}{|f(z-\zeta)-f(z)|}, \tag{3.1}$$

where the supremum is taken over all K-quasiconformal mappings of the plane. The identity (3.1) is one of the crucial points in [**6**], and it is not clear if it still holds when the supremum on both sides is taken only over those K-quasiconformal mappings for which

$$\limsup_{\zeta\to z} \frac{|f(\zeta)-f(z)|}{|\zeta-z|^{\alpha}} > 0.$$

Acknowledgements

The author is grateful to Albert Baernstein and David Opěla for several helpful discussions. The referee's comments helped to improve the paper considerably.

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(Received 07 09 2003)