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QUASICONFORMAL REFLECTION COEFFICIENT AND FREDHOLM EIGENVALUE OF AN ELLIPSE OF HYPERBOLIC GEOMETRY

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Abstract. We consider extremal quasiconformal reflections at a Jordan curve C, and related questions. Then we examine in more detail e.g., the special case of an ellipse C of hyperbolic geometry.

1. Introduction

If C is a quasicircle on the extended w-plane, then the reflection coefficient $Q_C \geqslant 1$ is the smallest dilatation bound in the class of all quasiconformal reflections at C. Here a quasiconformal reflection is a (schlicht) sense-reversing quasiconformal mapping of the extended plane onto itself for which all points of C are fixpoints. Closely related to the reflection coefficient of C is the Fredholm eigenvalue λ_C. In the elegant form of G. Schober [4] λ_C is defined by

$$\frac{1}{\lambda_C} = \sup \frac{|D_G(u) - D_{G^*}(u)|}{D_G(u) + D_{G^*}(u)}. \tag{1}$$

Here G and $G*$ are respectively the interior and exterior of C. D denotes the Dirichlet integral and the sup is taken over all functions u, which are continuous on the extended plane and harmonic on $G \cup G^*$. A classical result of L. V. Ahlfors states

$$\lambda_C \geqslant \frac{Q_C + 1}{Q_C - 1}. \tag{2}$$

We have $Q_C = 1$ or $\lambda_C = \infty$ only in the case of a circle C. Today there are several proofs of (2); cf. [9]. The question of equality in (2) was considered in [3], in [8] for analytic C more explicitly; cf. also [4].

There are several equivalent definitions of the reflection coefficient [9]:

(i) If we transform conformally both simply connected domains on the extended plane, which are bounded by C, respectively onto the interior and the exterior of the unit circle on the extended $\mathfrak{w}$-plane, we obtain on the unit circle a sewing function $\mathfrak{w}^* = \mathfrak{w}^*(\mathfrak{w})$. This sewing function is quasisymmetric exactly in the case of a

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quasicircle C. And the question for the reflection coefficient of C transforms then onto the question of the smallest dilatation bound in the class of all quasiconformal mappings of the exterior of the unit circle onto the interior, such that the boundary points transform in accordance with the sewing function. This is a classical problem of Teichmüller which gave rise to the comprehensive theory of Strebel, Reich and others [12]. Because of the invariance of Dirichlet integrals also the Fredholm eigenvalue of C transforms onto an eigenvalue for such sewing functions $\mathfrak{w}^*(\mathfrak{w})$, analogously described with Dirichlet integrals as in (1).

(ii) For the quasicircle C with the reflection coefficient Q_C, there exists for all given $Q_1 \geqslant 1$ and $Q_2 \geqslant 1$ with $Q_1 Q_2 = Q_C$ a quasiconformal mapping $Z = Z(w)$ of the extended w-plane which transforms C onto the unit circle and which is Q_1-quasiconformal in the exterior of C and Q_2-quasiconformal in the interior of C; cf. [9, p. 107] (in (29) Q_C^2 should there be replaced by Q_C). In particular (in the case $Q_1 = 1$, $Q_2 = C$) we have a Q_C-quasiconformal extension of the Riemann mapping function of one of the simply-connected domains bounded by C. And we have further (in the case $Q_1 = Q_2 = \sqrt{Q_C}$) a $\sqrt{Q_C}$-quasiconformal mapping of the extended w-plane which transforms C onto the unit circle. The inverse is also true, with a simpler proof.

This gives rise to the following new definition of the reflection coefficient Q_C.

PROPOSITION 1. *$\sqrt{Q_C}$ is the smallest dilatation bound in the class of all quasiconformal mappings of the extended plane which transforms C onto a circle.*

By [9, p. 95] the extremal $\sqrt{Q_C}$-quasiconformal mapping $Z(w)$, which transforms C onto a circle, is unique up to a following Möbius transformation if C is for example an analytic Jordan curve. Then from the Strebel–Reich theory it follows the well-known description of this extremal mapping with a quadratic differential. The dilatation has the constant value $\sqrt{Q_C}$. And, if we observe for the $\sqrt{Q_C}$-quasiconformal mapping $Z(w)$ in the Z-plane the infinitesimal ellipses which are the images of infinitesimal circles inside and outside of C, we have the

PROPOSITION 2. *The major axes of these ellipses in $|Z| > 1$ transform by reflection at $|Z| = 1$ into the minor axes of these ellipses in $|Z| < 1$.*

This means for the inverse mapping $w(Z)$ that the Beltrami coefficient has only a different sign for points Z and $1/\bar{Z}$.

The Proposition 1 yields further immediately

PROPOSITION 3. *If $w_1, w_2.w_3$ are arbitrary distinct points of the quasicircle C, if also $w_1^*, w_1^* w_3^*$ are arbitrary distinct points, given in the same order on C, if further $w_0 \notin C$ and $w_0^* \notin C$, then the following assertions are true:*

a. *There exists a sense-preserving Q_C-quasiconformal mapping of the extended plane with $C \to C$ and $w_k \to w_k^*$ for $k = 1, 2, 3$.*

b. *There exists a sense-reversing Q_C-quasiconformal mapping of the extended plane with $C \to C$ and $w_k \to w_k^*$ for $k = 1, 2, 3$.*

c. *There exists a sense-preserving Q_C-quasiconformal mapping of the extended plane with $C \to C$ and $w_1 \to w_1^*$, $w_0 \to w_0^*$ if w_0 and w_0^* are not separated by C.*

d. *There exists a sense-reversing Q_C-quasiconformal mapping of the extended plane with $C \to C$ and $w_1 \to w_1^*$, $w_0 \to w_0^*$ if w_0 and w_0^* are separated by C.*

For the proof we have only to consider a $\sqrt{Q_C}$-quasiconformal mapping of the extended plane which transforms C onto a circle. Here we have the possibility to transform by a Möbius transformation three given points of the circle into three given points (cases a, b) and then to pull back this with two $\sqrt{Q_C}$-quasiconformal mappings to the original plane containing C. The cases c and d are analogous.

In all the cases a, b, c, d it is the question: Does there always exist for this C with the reflection coefficient Q_C a Q-quasiconformal mapping with the mentioned property and with a $Q < Q_C$? The situation is clear in the cases b and d. Namely, the case of a square C [10] shows that not always $Q < Q_C$ is possible. And this example of a square C also shows that in the cases a and c a dilatation bound $Q < \sqrt{Q_C}$ is not always possible. But of course there remain several open questions.

Remarks. Without an explicit dilatation bound, to Proposition 3 related results were obtained by J. Sarvas and by T. Erkama; cf. [2, p. 45].

(iii) Now we consider the class of all pairs of quasiconformal mappings of the two simply-connected domains, which are bounded by the quasicircle C, onto the same given and fixed Jordan domain B, such that the boundary values at C are the same. (The individual boundary values are not prescribed; we require only that at every point of C both boundary values are the same.) One of these two mappings has to be (schlicht and) sense-preserving, the other (schlicht and) sense-reversing. Then we have

Proposition 4. *In the class of these pairs of quasiconformal mappings the smallest dilatation bound (observed for both mappings of the pair) is $\sqrt{Q_C}$.*

This follows immediately from Proposition 1, at first in the special case, B is the unit disk. After a conformal mapping of the unit disk we get the case of a general Jordan domain B.

In the case of an analytic quasicircle C, for the extremal pair of quasiconformal mappings always two points which exchange by the extremal quasiconformal reflection at C transforms onto the same point inside of B. And the corresponding two infinitesimal circles transform onto infinitesimal ellipses whose great axes are orthogonal to each other. This means, we have over B in one version the trajectories of a quadratic differential, in the other version of B the orthogonal trajectories, for the description of these infinitesimal ellipses.

A collection of quasicircles C with a known reflection coefficient Q_C was given in [9], [4]. Now we will add two further examples.

2. Ellipses of the hyperbolic geometry

Such ellipses are defined by the property that the sum S of the hyperbolic distances to two fixed points (foci) is a constant. We use the unit disk with the usual hyperbolic metric

$$\frac{|dw|}{1-|w|^2} \tag{3}$$

and the two foci $\pm\rho$ $(0<\rho<1)$.

By a conformal mapping $\zeta=\zeta(w)$ of the ring-domain between $|w|=1$ and the segment $-\rho\cdots+\rho$ onto an annulus of the form $1/r<|\zeta|<\sqrt{R/r}$ with some suitable r, R $(1<r<R)$ such an "ellipse" transforms onto the unit circle (cf. [5, p. 24], [1, p. 124]). We set the side condition $\zeta(1)=\sqrt{R/r}$ (cf. Figure 1). The quantities r and R can easily be calculated by ρ and S, and vice versa (for example, we obtain $\mu(\rho^2)=\log(rR)$ by considering the module μ of the corresponding Grötzsch domain [11] in the the w^2-plane).

For such an "ellipse" the following calculation of the reflection coefficient and of the Fredholm eigenvalue is much more complicated than in the case of a usual ellipse (in the euclidean sense). For such a usual ellipse with semiaxes a and $b<a$ a we simply have the reflection coefficient a/b and the Fredholm eigenvalue $(a+b)/(a-b)$; cf. [9] and references there. (In statu nascendi known to S. D. Poisson.)

Now we consider a chain of further mappings as in Figure 1.

The mapping $\eta=r\zeta$ transforms the annulus $1/r<|\zeta|<1$ onto the annulus $1<|\eta|<r$. And $\eta=R/\bar{\zeta}$ transforms the annulus $1<|\zeta|<R$ onto itself.

The next conformal mapping

$$\mathfrak{z}=\eta+1/\eta \tag{4}$$

produces sewings of the two unit circles in the arising two η-planes, such that the two transformations $\mathfrak{z}(w)$ produce two conformal mappings of the two domains outside and inside of our "ellipse" C onto two domains inside of two different usual ellipses with the foci ± 2. (We can now forget the cuts $-\rho\cdots+\rho$, $1/\rho\cdots+\infty$, $-\infty\cdots-1/\rho$ on the real axis in the w-plane.) Now we transform the first ellipse with semiaxes $r+1/r$ and $r-1/r$ by an affine mapping of the form

$$z=\alpha(\mathfrak{z}-q\bar{\mathfrak{z}})\quad(\alpha>0,\ 0<q<1) \tag{5}$$

and the second ellipse with semiaxes $R+1/R$ and $R-1/R$ by an affine mapping of the form

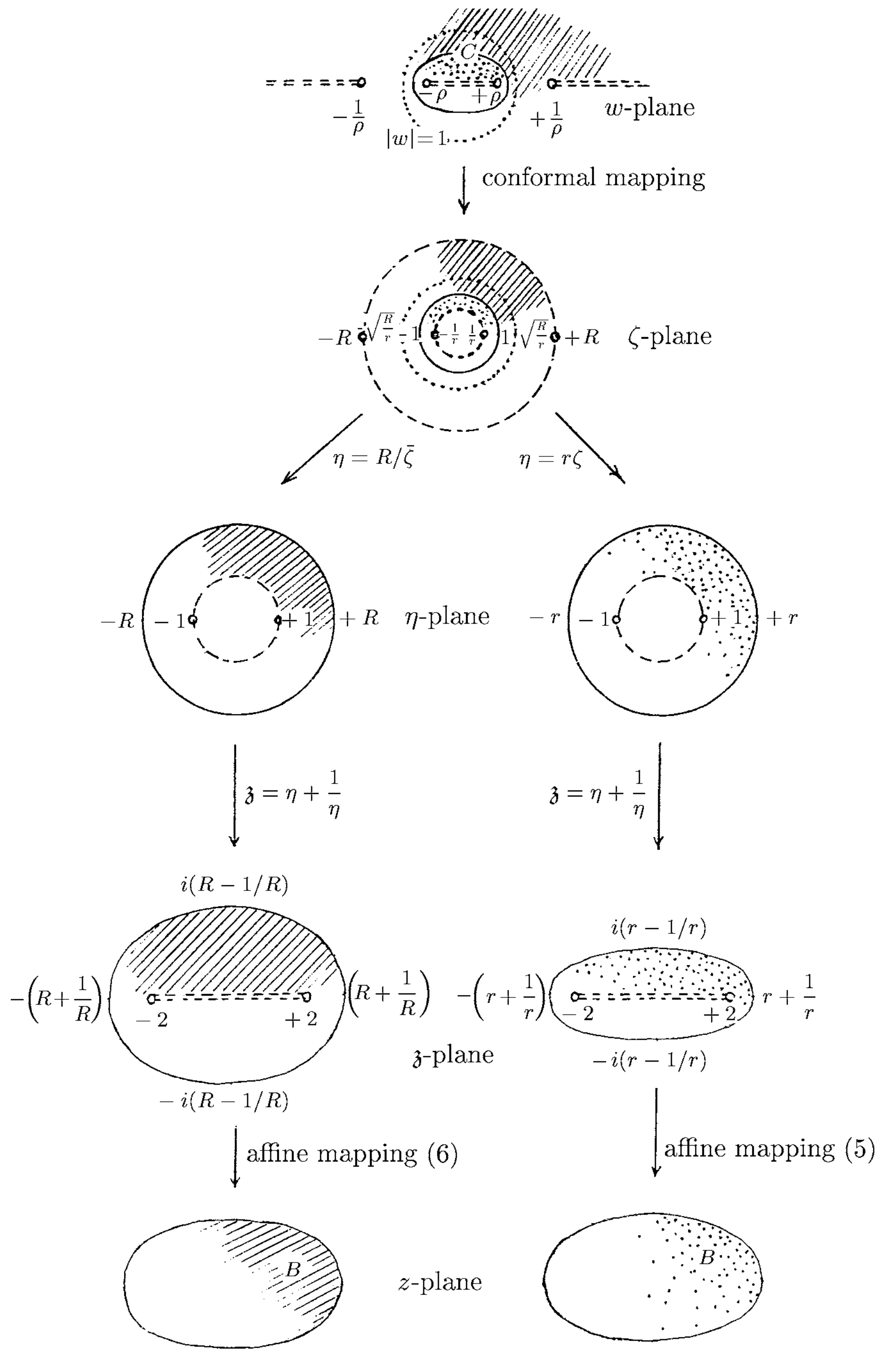
$$z=\beta(\mathfrak{z}+q\bar{\mathfrak{z}})\quad(\beta>0,\ 0<q<1) \tag{6}$$

such that finally in the z-plane the two images of C (again usual ellipses) are *pointwise* the same (because (4) is also an affine mapping on the circles with center $\eta=0$). The interior of the two usual ellipses in the z-plane corresponds now to B in Proposition 4, $z(w)$ yields the pair of mappings in the extremal case. Equality of the ratios of the semiaxes of the ellipses in the z-plane leaves us with

$$\frac{r+1/r}{r-1/r}\cdot\frac{1-q}{1+q}=\frac{R+1/R}{R-1/R}\cdot\frac{1+q}{1-q}$$

because we have for both mappings $z(w)$ the dilatation $(1+q)/(1-q)>1$ (cf. (5) and (6)). This yields by (iii)

$$Q_C=\left(\frac{1+q}{1-q}\right)^2,$$



$-\frac{1}{\rho}$
$-\rho$
$+\rho$
C
$|w|=1$
$+\frac{1}{\rho}$
w-plane
conformal mapping
$-R$
$\sqrt{\frac{R}{r}}$
-1
$-\frac{1}{r}$
$\frac{1}{r}$
1
$+R$
ζ-plane
$\eta = R/\bar{\zeta}$
$\eta = r\zeta$
$-R$
-1
$+1$
$+R$
η-plane
$-r$
-1
$+1$
$+r$
$\mathfrak{z} = \eta + \frac{1}{\eta}$
$\mathfrak{z} = \eta + \frac{1}{\eta}$
$i(R-1/R)$
$-\left(R+\frac{1}{R}\right)$
-2
$+2$
$\left(R+\frac{1}{R}\right)$
$-i(R-1/R)$
$i(r-1/r)$
$-\left(r+\frac{1}{r}\right)$
-2
$+2$
$r+\frac{1}{r}$
$-i(r-1/r)$
$\mathfrak{z}$-plane
affine mapping (6)
affine mapping (5)
B
B
z-plane

FIGURE 1

therefore

$$Q_C = \frac{r^2+1}{r^2-1} \cdot \frac{R^2-1}{R^2+1}. \tag{7}$$

The extremal quasiconformal reflection at C arises from the affine mapping with the dilatation Q_C between the two domains inside the two ellipses in the $\mathfrak{z}$-plane. So we see further that the quadratic differential in the w-plane does not contain zeros. This means by [8] (cf. also [3]) that we have equality between the reciprocal Fredholm eigenvalue λ_C and $(Q_C+1)/(Q_C-1)$. This yields

$$\lambda_C = \frac{R^2r^2-1}{R^2-r^2}. \tag{8}$$

We summarize to the

THEOREM 1. *Reflection coefficient Q_C and Fredholm eigenvalue λ_C of the ellipse of hyperbolic geometry, defined by the parameters R and r, are given by* (7) *and* (8).

We remark here that the assertion concerning the Fredholm eigenvalue also follows with the resulting mapping $z(w)$ of Figure 1 (and an additional affine mapping) by Satz 5 in [6]. Namely, after the affine mapping $z - q\bar{z}$ of both domains in the z-plane we get outside of C an analytic function, with a continuous quasiconformal extension as in [6], The same remark holds for Theorem 2.

3. Reflection coefficient and Fredholm eigenvalue of related quasicircles

The procedure of Section 2 works also in the case of the following Jordan curves C. We consider the w-plane with cuts along the rays $-\infty \cdots -1/\rho$ and $+1/\rho \cdots +\infty$, $\rho > 0$, on the real axis. We endow this simply-connected slit-domain with the corresponding hyperbolic metric and define now C as an ellipse in this metric and with foci $\pm i\rho$. If we transform conformally the ring-domain between the two rays and the segment $-i\rho \cdots + i\rho$ onto an annulus, this ellipse transforms onto a concentric circle. This is clear in connection with the ellipses of Section 2 after a conformal mapping between our slit-domain and the slitted w-plane of Section 2.

By the way, the upper half of these curves C are in the hyperbolic metric of the upper halfplane also "semihyperbolas" (cf. [5, p. 26]). We mention also that in a special case this C is a Cassinian, namely in the case, the conformal module of the ring-domain between C and the segment $-i\rho \cdots + i\rho$ equals the conformal module of the ring-domain between C and the mentioned two rays. This is clear because in the w^2-plane the image of C must be a circle because of equality of the then arising modules. The reflection coefficient and the Fredholm eigenvalue of the Cassinians were already calculated with another method in [7] (cf. also [9, (25)]). The Cassinians depend on one essential parameter, our more general curves C depend on two essential parameters (as the ellipses in Section 2).

To calculate now the reflection coefficient of our curves C, we start with a conformal mapping $\zeta = \zeta(w)$ of the ring-domain between the rays $-\infty \cdots -1/\rho$, $+1/\rho \cdots +\infty$ and the segment $-i\rho \cdots + i\rho$ on the real axis, onto an annulus of the

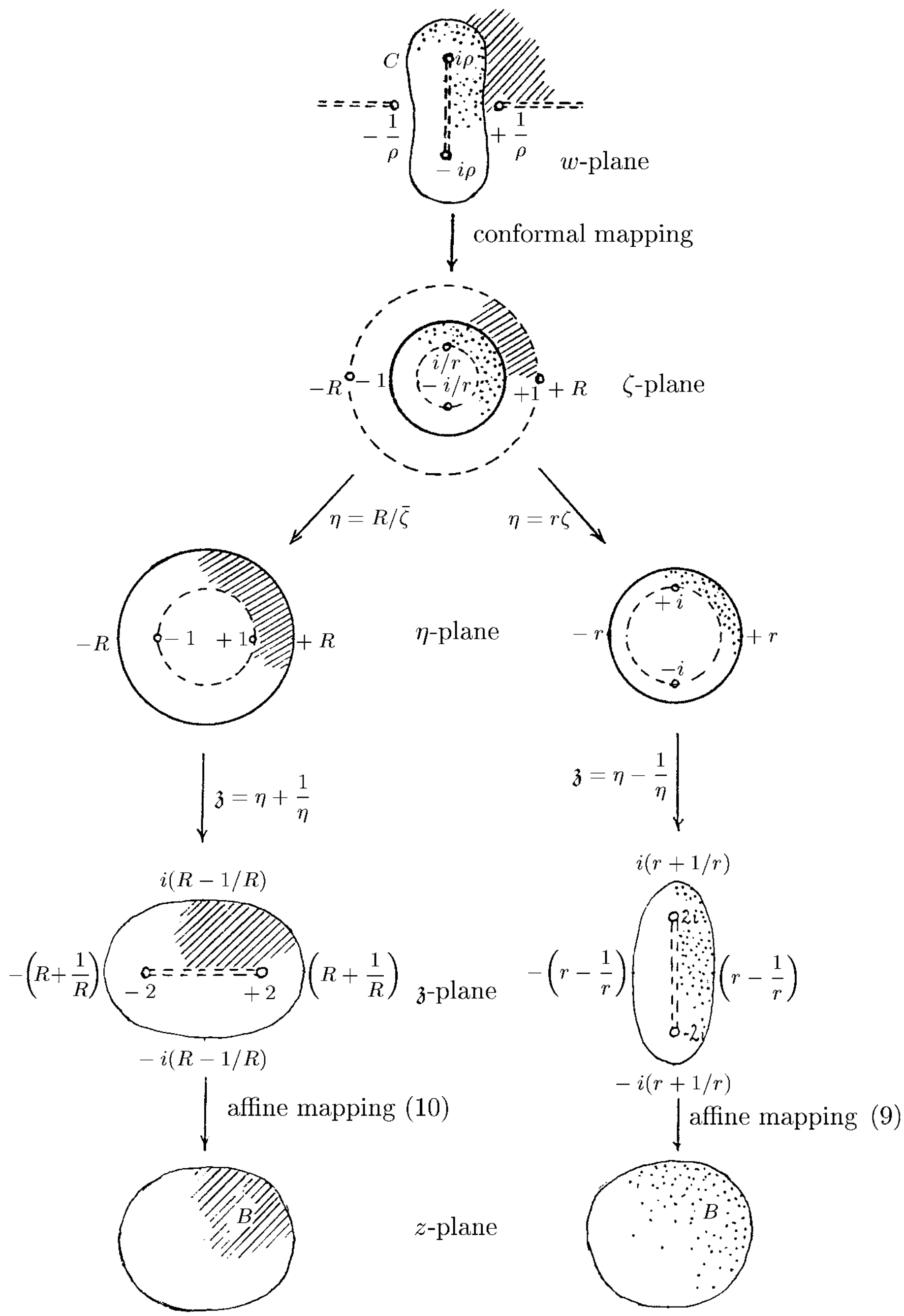

C
$i\rho$
$-\frac{1}{\rho}$
$+\frac{1}{\rho}$
$-i\rho$
w-plane
conformal mapping
$-R$ -1
i/r
$-i/r$
$+1$ $+R$
ζ-plane
$\eta = R/\bar{\zeta}$
$\eta = r\zeta$
$-R$ -1 $+1$ $+R$
η-plane
$-r$ $+i$ $-i$ $+r$
$\mathfrak{z} = \eta + \frac{1}{\eta}$
$\mathfrak{z} = \eta - \frac{1}{\eta}$
$i(R-1/R)$
$-\left(R+\frac{1}{R}\right)$ -2 $+2$ $\left(R+\frac{1}{R}\right)$
$\mathfrak{z}$-plane
$-i(R-1/R)$
$i(r+1/r)$
$2i$
$-\left(r-\frac{1}{r}\right)$ $\left(r-\frac{1}{r}\right)$
$-2i$
$-i(r+1/r)$
affine mapping (10)
affine mapping (9)
B
z-plane
B

FIGURE 2

form $1/r < |\zeta| < R$ with some suitable $r > 0$ and $R > 0$, and the side condition $\zeta(1/\rho) = R$, such that our C transforms onto the unit circle $|\zeta| = 1$. For the following compare Figure 2.

As the two mentioned parameters, which characterize our C, we can now take these values r and R.

Similarly as in Figure 1 we consider now a chain of further mappings in Figure 2. In the last affine step we take

$$z = \gamma(\mathfrak{z} + q\bar{\mathfrak{z}}) \quad (\gamma > 0,\ 0 < q < 1) \tag{9}$$

resp.

$$z = \delta(\mathfrak{z} - q\bar{\mathfrak{z}}) \quad (\delta > 0,\ 0 < q < 1) \tag{10}$$

with such a value q that both ellipses in the z-plane are the same. In particular, equality of the ratios of the semiaxes of these ellipses leaves us with

$$\frac{r - 1/r}{r + 1/r} \cdot \frac{1+q}{1-q} = \frac{R + 1/R}{R - 1/R} \cdot \frac{1-q}{1+q},$$

$$Q_C = \Big(\frac{1+q}{1-q}\Big)^2 = \frac{R^2+1}{R^2-1} \cdot \frac{r^2+1}{r^2-1}. \tag{11}$$

Again the Fredholm eigenvalue λ_C equals $\dfrac{Q_C+1}{Q_C-1}$:

$$\lambda_C = \frac{R^2 r^2 + 1}{R^2 + r^2}. \tag{12}$$

We summarize to the

Theorem 2. *Reflection coefficient Q_C and Fredholm eigenvalue λ_C of the curves C, defined by the parameters r and R, are qiven by* (11) *and* (12).

4. An addendum: "Pseudo-reflections"

Let C be a closed Jordan curve. We consider now sense-reversing quasiconformal mappings of the extended plane which transforms C onto itself and exchange the outside and the inside. But now these mappings must not be necessarily usual reflections, that means not all points of C have to be fixpoints. Let us call these mappings "pseudo-reflections" at C.

Corresponding to a fixed C, we ask for the infimum $\mathcal{Q}_C$ of the dilatation bounds ($\geqslant 1$) of these mappings. Clearly, we have

$$(1 \leqslant)\ \mathcal{Q}_C \leqslant Q_C. \tag{13}$$

Here the case $\mathcal{Q}_C = 1$ is possible, although C is not a circle. As an example we consider the Jordan curve C on the Riemann sphere which bisects the Riemann sphere into two congruent parts like the well-known curve on a tennis-ball.

Problem. For which C it holds $\mathcal{Q}_C = 1$?

But also the case $\mathcal{Q}_C > 1$ is possible. We remark as an example the case of a square C with $Q_C = 3$ [9]. Here we have $\mathcal{Q}_C \geqslant \sqrt{3}$. Namely, if we have a Q-quasiconformal pseudo-reflection at this C which transforms all corners onto corners, then it holds by [13, Satz 3] $Q \geqslant 3$ (we have to consider only neighbourhoods of the corners). And if at least one corner transforms not onto a corner, then we obtain by [13, Satz 3] or [10] $Q \geqslant \sqrt{3}$ (again we have to consider only neighbourhoods of the corners). Probably it holds $\mathcal{Q}_C = \sqrt{3}$. To prove this we have to construct a $\sqrt{3}$-quasiconformal pseudo-reflection at C.

QUESTIONS. Does there exist curves C with $\mathcal{Q}_C = Q_C$? What is the value $\mathcal{Q}_C$ in the case of an ellipse? (It is clear that there does not exist a conformal pseudo-reflection at an ellipse, because the Riemann mapping function for the exterior is an elementary function, while not for the interior.)

5. Multiply-connected domains

An interesting new type of problems arises if we consider a fixed system of a finite number of closed disjoint Jordan curves $C_1, \ldots, C_n$. In the class of all quasiconformal mappings of the extended plane which transform all C_k onto circles we ask for those which are extremal (with the smallest maximal dilatation). Even the case $n = 2$ seems not so easy. What about the limit case of curves C_k with a contact? Another type of problems arises if we consider non-schlicht quasiconformal mappings of the extended plane such that all C_k transforms onto the unit circle. Again we have the question for the corresponding extremal quasiconformal mappings.

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