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ON SOLUTIONS OF THE BELTRAMI EQUATION. II

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ABSTRACT. We study the existence of solutions of the generalized Beltrami equation $f_{\bar{z}} = \mu(z) f_z$, $\|\mu(z)\|_\infty = 1$, in a plane domain Δ, under general conditions that include previously known results.

1. Introduction

Let $\mu(z)$ be a measurable complex valued function. In our previous paper [2] we treated the question of existence and uniqueness of solutions for the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{1}$$

assuming that $|\mu(z)|$ satisfies a subexponential integrability condition. In the present paper we treat the existence problem under general conditions which include previous results.

2. Main results

Let $h(x)$ be a convex, increasing function defined on $[1, \infty)$ such that $h(x) \geqslant C_\lambda x^\lambda$ for any $\lambda > 1$ with $C_\lambda > 0$. From now on we will assume also that

$$\int_1^\infty \frac{1}{t h^{-1}(t)} dt = \infty. \tag{2}$$

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Main Theorem. *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\iint_B h\left(\frac{1}{1-|\mu|}\right) dA < \Phi_B. \tag{3}$$

Then there exists an ACL homeomorphism $f(z)$ of Δ into the plane, which satisfies the Beltrami equation a.e., with partials f_z and $f_{\bar z}$, locally in L^q, for $0 < q < 2$. The partials are also distributional derivatives. The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$, and has partials g_w and $g_{\bar w}$ locally in L^2.

THEOREM A. (the case of the plane) *If Δ is the plane and if, in addition to* (3), *$\mu(z)$ satisfies*

$$\iint_{\{|z|<R\}} \frac{1}{1-|\mu|} dA = O(R^2), \quad R \to \infty.$$

then there exists an ACL homeomorphism f which maps the plane onto itself with all the properties listed in the Main Theorem.

3. Auxiliary Results and an Equivalent Statement

Let $h(x)$ be the function defined in Section 2. Denote by $\theta(x) = \ln(h(x))$ for x greater than some constant $c \geqslant 1$, such that $h(c) > e$. $\theta(x)$ is a positive increasing function in $[\ln h(c), \infty)$. Next we show that the following conditions

$$(4) \qquad \int_{c_1}^{\infty} \frac{dx}{xh^{-1}(x)} = \infty, \qquad\qquad (5) \qquad \int_{c_2}^{\infty} \frac{\theta(x)}{x^2} dx = \infty.$$

hold simultaneously, where c_1 and c_2 are suitable constants. The result can be stated as:

LEMMA 1. *Conditions* (4) *and* (5) *are equivalent.*

Proof. Make a change of variables in $\int_{c_1}^{\infty} \frac{dx}{xh^{-1}(x)}$ by using the substitution $y = \ln(x)$. Then the last integral becomes $\int_{c^*}^{\infty} \frac{dx}{\theta^{-1}(x)}$, where $c^* = \ln c_1$. Since $\frac{1}{\theta^{-1}(x)}$ and $\theta\left(\frac{1}{x}\right)$ are inverses of each other, it follows that $\int_{c^*}^{\infty} \frac{dx}{\theta^{-1}(x)}$ is divergent iff $\int_{0}^{c_*} \theta\left(\frac{1}{x}\right) dx$ is for some suitable constant $c_* < 1$. After another substitution $y = \frac{1}{x}$ we obtain that the divergence of the last integral is equivalent to the divergence of $\int_{c_3}^{\infty} \frac{\theta(x)}{x^2} dx$, where $c_3 = \frac{1}{c_*}$.

From the auxiliary results above follows a statement equivalent to the Main Theorem.

THEOREM B. *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\iint_B \exp\left(\theta\left(\frac{1}{1-|\mu|}\right)\right) dA < \Phi_B.$$

If

$$\int_1^\infty \frac{\theta(x)}{x^2} dt = \infty,$$

there exists an ACL homeomorphism $f(z)$ of Δ into the plane, which satisfies the Beltrami equation a.e., with partials f_z and $f_{\bar z}$, locally in L^q, for $0 < q < 2$. The partials are also distributional derivatives . The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$, and has partials g_w and $g_{\bar w}$ locally in L^2.

4. Construction of the solution $f(z)$

Here we assume that $\mu(z)$ satisfies condition (3), with $h(z)$ satisfying (2). In Δ we define μ_n, $n = 1, 2, \dots$, so that

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leqslant 1 - 1/n \\ 0, & \text{if } |\mu(z)| > 1 - 1/n. \end{cases}$$

From the theory of quasiconformal mappings we know that there exist q.c. mappings f_n, $n = 1, 2, \dots$, of Δ into the plane with complex dilatations μ_n, $n = 1, 2, \dots$.

Let z_0 be a fixed point in the plane. For $r_2 > r_1 > 0$ denote by A the circular ring $A = \{z : r_1 < |z - z_0| < r_2\}$, and by $M_n(r_1, r_2)$ the module of its image under f_n.

Proposition 1. *For any point z_0 and circular ring $A = \{r_1 < |z - z_0| < r_2\}$, the module $M_n(r_1, r_2)$ of the image of A under f_n tends uniformly to ∞ as $r_1 \to 0$.*

Proof. The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation μ_n, where $\mu_n = \mu_n(z) = \mu_n(z_0 + re^{i\theta})$, as follows (see [4]):

$$M_n(r_1, r_2) \geqslant \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1 - |\mu_n|^2} d\theta} \frac{dr}{r}.$$

Using this we obtain:

$$M_n(r_1, r_2) \geqslant \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{1}{1 - |\mu|} d\theta} \frac{dr}{r}.$$

For any z_0 in a compact subset T of the plane containing the disc $|z - z_0| < r_2$

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} h\left(\frac{1}{1 - |\mu|}\right) d\theta \frac{dr}{r} \leqslant C,$$

where C depends only on the compact subset T and the choice of r_2.

Now we have

$$r^2\int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right)d\theta < \frac{2C}{\log\dfrac{r_2}{r_1}}$$

on a set E of logarithmic measure $\dfrac{1}{2}\log\dfrac{r_2}{r_1}$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right)d\theta < \frac{C}{\pi r^2\log\dfrac{r_2}{r_1}} \text{ on } E.$$

Using the convexity of $h(x)$, we have

$$h\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta\right) < \frac{C}{\pi r^2\log\dfrac{r_2}{r_1}} \quad \text{on } E$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta < h^{-1}\left(\frac{C}{\pi r^2\log\dfrac{r_2}{r_1}}\right) \quad \text{on } E.$$

From the estimates of the module and monotonicity properties of $h(x)$ we have

$$M_n(r_1,r_2)\geqslant \frac{1}{8\pi}\int_{r_1}^{r_2}\frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)}\frac{dr}{r}\geqslant \frac{1}{8\pi}\int_{r_1}^{\sqrt{r_1r_2}}\frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)}\frac{dr}{r}.$$

Now we consider a monotonically decreasing sequence $\{s_k\}_{k=1}^{\infty}$ of positive numbers tending to 0 such that each interval $[s_{k+1},s_k]$ has the same logarithmic length, where $\dfrac{s_k}{s_{k+1}}=c$. By a ring decomposition we mean a family of rings $r_1^{(j)}<|z-z_0|<r_2^{(j)}$ with $r_2^{(j+1)}\leqslant r_1^{(j)}$ and $r_1^{(j)}$ and $r_2^{(j)}\to 0$ as $j\to\infty$.

We take two ring decompositions with

$$r_1^{(j)}=s_{2j+1}, r_2^{(j)}=s_{2j-1}$$

$$\hat{r}_1^{(j)}=s_{2j+2}, \hat{r}_2^{(j)}=s_{2j}.$$

Now

$$\sum_{j=1}^{\infty}M_n\left(r_1^{(j)},r_2^{(j)}\right)\geqslant\frac{1}{8\pi}\sum_{j=1}^{\infty}\int\limits_{s_{2j+1}}^{s_{2j}}\frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2\log c}\right)}\frac{dr}{r},$$

while

$$\sum_{j=1}^{\infty}M_n\left(\hat{r}_1^{(j)},\hat{r}_2^{(j)}\right)\geqslant\frac{1}{8\pi}\sum_{j=1}^{\infty}\int\limits_{s_{2j+2}}^{s_{2j+1}}\frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2\log c}\right)}\frac{dr}{r},$$

so

$$\sum_{j=1}^{\infty} M_n\left(r_1^{(j)}, r_2^{(j)}\right) + \sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi} \int_0^{s_1} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)} \frac{dr}{r}.$$

Making the change of variables $t = \dfrac{C}{\pi r^2 \log c}$ this last term becomes equal to $\dfrac{1}{8\pi} \int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)} dt$, with a well defined lower limit. Thus at least one of the ring decompositions has module sum bounded below by $\dfrac{1}{16\pi} \int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)} dt$ and therefore approaches ∞ uniformly with respect to n and z_0. From the superadditivity property of the module it follows that $\lim_{r_1 \to 0} M_n(r_1, r_2) = \infty$, uniformly with respect to z_0 and n.

From now on we shalll assume that the quasiconformal mappings $\{f_n(z)\}$, $n = 1, 2 \ldots$ have two fixed points a_1 and a_2, with $d = |a_2 - a_1|$. The following proposition was proved in [2]:

Proposition 2. *If* $\lim_{r_1 \to 0} M_n(r_1, r_2) = \infty$, *uniformly with respect to* z_0 *and* n, *then the family of quasiconformal mappings* $\{f_n(z)\}$, $n = 1, 2, \ldots$, *is uniformly equicontinuous on each compact subset* T *of* Δ.

Thus from this proposition and the Arzela-Ascoli's theorem follows:

Proposition 3. *For the sequence* $\{f_n(z)\}$ *there exists a subsequence of functions, which converges uniformly to a function* $f(z)$ *on compact subsets.*

We follow the statements in [2] to show the properties of $f(z)$ outlined in the Main Theorem.

5. $f(z)$ is a homeomorphism

In the same manner as in [2], one can prove that:

Proposition 4. *The function* $f(z)$ *constructed in Proposition 3 is a homeomorphism of* Δ *into the plane.*

6. Differentiability properties of $f(z)$

In the same manner as in [2], one can prove that:

Proposition 5. *The function f(z) is ACL.*

Proposition 6. *The partials* f_z *and* $f_{\bar{z}}$ *of* $f(z)$ *are in* L^q *on compact subsets of* Δ *for every* $q < 2$.

Thus $f(z)$ has generalized L^q-derivatives according to the terminology introduced in [3].

7. $f(z)$ satisfies the Beltrami equation

Using the same methods as in [2], one can prove that:

Proposition 7. *The function $f(z)$ satisfies the Beltrami equation.*

8. The inverse function $g(w)$ of $f(z)$

In the same manner as in [2], one can prove that:

Proposition 8. *The function g is ACL and g_w and $g_{\bar{w}}$ are locally in L^2.*

So far we have proved the Main Theorem and Theorem B.

9. The case of mapping the plane onto itself

In the same manner as in [2], one can prove that

Proposition 9. *If*

$$\iint\limits_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2) \qquad \text{as } R \to \infty,$$

then $f_n(z)$ converges uniformly to ∞, as $z \to \infty$.

This proposition and the rest of the results imply Theorem A. This concludes the proofs of the Main Theorem, Theorem A and Theorem B.

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